

ANALYSIS OF A GUIDED DEEP BEAM OF COMPOSITE MATERIALS

Thesis

M. Sc. Engg (Mechanical)

Submitted by

M Muzibur Rahman

Roll No. 100610001



**Department of Mechanical Engineering
Bangladesh University of Engineering and Technology (BUET)
Dhaka – 1000, Bangladesh**

January 2009

ANALYSIS OF A GUIDED DEEP BEAM OF COMPOSITE MATERIALS

A thesis submitted to the Department of Mechanical Engineering
Bangladesh University of Engineering and Technology (BUET)
in partial fulfilment for the requirement of the degree of
Master of Science in Mechanical Engineering

Submitted by
M Muzibur Rahman
Roll No. 100610001
Session: October 2006



Department of Mechanical Engineering
Bangladesh University of Engineering and Technology (BUET)
Dhaka – 1000, Bangladesh
February 2009

RECOMMENDATIONS OF THE BOARD OF EXAMINERS

This thesis titled, “**Analysis of a Guided Deep Beam of Composite Materials**”, submitted by M Muzibur Rahman, Roll no: 100610001, Session: October 2006 has been accepted as satisfactory in partial fulfilment of the requirements for the Degree of Master of Science in Mechanical Engineering on 09 February 2009.

BOARD OF EXAMINERS

Dr. S. Reaz Ahmed Professor Department of Mechanical Engineering, BUET Dhaka-1000, Bangladesh	Chairman
--	----------

Dr. Abu Rayhan Md. Ali Professor and Head Department of Mechanical Engineering, BUET Dhaka-1000, Bangladesh	Member (ex-officio)
--	------------------------

Dr. M. Mahabubur Razzaque Associate Professor Department of Mechanical Engineering, BUET Dhaka-1000, Bangladesh	Member
--	--------

Dr. Sanjib Chandra Chowdhury Assistant Professor Department of Mechanical Engineering, BUET Dhaka-1000, Bangladesh	Member
---	--------

Prof. Dr. Md. Fazli Ilahi Former Vice Chancellor Islamic University of Technology House # 13, Road # 1, Sector # 11, Uttara, Dhaka-1230	Member (external)
--	----------------------

DECLARATION

This thesis work has been done by myself and does not contain any material extracted from elsewhere or from a work published by anybody else. The work of this thesis has not been presented elsewhere by the author for any degree or diploma. No other person's work has been used or included without due acknowledgement.

M Muzibur Rahman

Roll No: 100610001

Session: October 2006

ACKNOWLEDGEMENTS

First of all, I would like to express my sincere gratitude to Allah (Subhanahu wa Taala) for all the knowledge and science He has given to mankind and for bestowing me with health and patience to complete this work. I acknowledge, with deep gratitude and appreciation, my supervisor Prof. S. Reaz Ahmed for his enthusiastic and untiring guidance in learning, planning, executing and bringing out the findings in all phases of this research work. He has been found devoted to extend the helping hand towards me round the clock to get rid of any difficulty arising from any corner during the study and work. It is no way exaggeration to say that it would be difficult or might be impossible to materialize the work without the utmost support of my supervisor.

I would like to extend my gratefulness to Professor Dr. Abu Rayhan Md. Ali, Head, Department of Mechanical Engineering, BUET for his overall supportive stance in providing a very congenial academic environment in order to complete the research work. I am indebted to all other faculty members for their cordial support. Special thanks are due to the respective members of examination board for their valuable comments and suggestions. I also acknowledge the administrative support provided by the Department as a whole.

At this auspicious occasion I would like to pay full respect and gratitude to Assistant Chief of Naval Staff (Material) Commodore A K Chowdhury, (E), ndc, psc, BN and Director of Naval Engineering Captain M Fazlur Rahman, (E), psc, BN for allowing me to devote fully in order to carry out the research work and bring out results.

I am sincerely thankful to my family members for boosting up my morale and to provide me adequate time for devoted study. My special regards to my daughter who had to face a lot of difficulties during my absence from her, while her mother was also far away from her for undergoing her higher studies in abroad.

CONTENTS

	Page no
<i>Title page</i>	i
<i>Board of Examiners</i>	ii
<i>Declaration</i>	iii
<i>Acknowledgement</i>	iv
<i>Table of Contents</i>	v
<i>Abstract</i>	ix
<i>List of Symbols</i>	x
<i>List of Figures</i>	xii
<i>List of Tables</i>	xvii

Chapter One: Introduction

1.1 Preamble	1
1.2 Literature Review	2
1.3 Analysis of Deep Beam	5
1.4 Objectives	7
1.5 Study Procedure	8
1.6 Significance of Present Study	8

Chapter Two: Theoretical Outline and Formulation

2.1 Preamble	10
2.2 Equilibrium and Compatibility Conditions	10
2.3 Hooke's Law	13

2.4	Two - Dimensionalization of the Problem	17
2.5	Usual Method for Solution	19
2.6	Stress Function Approach	20
2.7	Displacement Potential Formulation	22
	2.7.1 Displacement Potential Formulation for Orthotropic Materials	26
	2.7.2 Displacement Potential Formulation for Isotropic Materials	26
2.8	Consideration of Boundary Conditions	28
2.9	Solution Procedure of Displacement Potential Approach	29

Chapter Three: Solution of Isotropic Deep Beams

3.1	Analytical Solution of Guided Isotropic Deep Beam	31
	3.1.1 Problem Articulation	31
	3.1.2 Boundary Conditions	32
	3.1.3 Analytical Solution	33
	3.1.4 Results of Displacement Potential Solution	42
	3.1.5 Comparison of Results	47
	3.1.6 Effects of Aspect Ratio on Stresses	50
	3.1.7 Effect of Distribution of Support Reaction	56
3.2	Solution of Unguided Simply Supported Beams	63
	3.2.1 Analytical Solution Using Displacement Potential	63
	3.2.2 Results of Unguided Deep Beam	66
3.3	Stress Function Solution of a Simply Supported Beam	70

3.3.1 Particular Assumptions	70
3.3.2 Boundary conditions	70
3.3.3 Analytical solution	71
3.4 Comparison of • -Solution Results with • and Numerical Solutions	77
3.5 Summary	81

Chapter Four: Analytical Solution for Orthotropic Deep Beams

4.1 Analytical Solution of Guided Composite Deep Beam	83
4.2 Results of Composite Deep Beam	93
4.3 Verification of Present Results of Composite Materials	98
4.4 Effect of Aspect Ratio on Stresses	101
4.5 Effect of Material Properties on Elastic Fields	106
4.6 Analysis of Elastic Fields in Orthotropic Slender Beams	108
4.6 Summary	110

Chapter Five: Solution of Deep Beams for Three Point Bending

5.1 Guided Composite Beam under Symmetric Three Point Bending (Case-I)	111
5.2 Analytical Solution for Guided Composite Beam under Asymmetric Three Point Bending (Case-II)	121
5.3 Verification of • -solution for Three Point Bending	128
5.4 Summary	131

Chapter Six: Conclusions and Recommendations

6.1 Conclusions 132

6.1 Recommendations 134

References 135

Appendix 138

ABSTRACT

Analytical solution scores the highest degree of importance in structural analysis. But the existing mathematical models for analytical solutions are still inadequate. Neither the beam theory nor the stress function approach can address the guided deep beam appropriately. Though, in many cases, the numerical techniques can well approximate the response, if analytical solution is possible, that remains as the highly desirable one.

An ideal mathematical model called displacement potential formulation is used to develop a new scheme for analyzing guided simply-supported deep beam of isotropic as well as orthotropic materials with different loading and support arrangements. The results are then analyzed and the effect of fiber reinforcement and beam aspect-ratio on the distributions of displacement and stresses in the beam are investigated. Besides, taking into account the effect of Saint Venant's principle, a new analytical scheme is developed to generate the solution of an unguided simply supported deep beam, in which the guided beam solution is considered as the limiting case of the unguided one.

Solutions of the guided isotropic and orthotropic deep beams are obtained satisfying all the physical conditions of the beam appropriately. In the investigation of solutions, the guided ends of the beam are identified to be the most critical section in terms of stresses. The bending stress distribution is found highly non-linear near the guides. The shearing stress distributions assume the standard parabolic pattern. The stresses at the load transition section are found to be different from other sections of the beam.

Finally, comparative studies are carried out to ascertain the reliability and credibility of the present displacement potential solutions with those of classical beam theory, standard theory of elasticity as well as numerical method. Here the numerical solution is obtained using the standard finite element method. The study reveals that the displacement potential approach is the most appropriate way to deal with the guided deep beams analytically. Results of the present analysis are claimed to be highly reliable and accurate, and thus will provide a reliable design guideline for deep composite beams with/without guides.

LIST OF SYMBOLS

x, y, z	Coordinates in Cartesian system
E_x	Elastic modulus along x-axis
E_y	Elastic modulus along y-axis
E_z	Elastic modulus along z-axis
G_{xy}	Shear modulus in xy plane
G_{yz}	Shear modulus in yz plane
G_{zx}	Shear modulus in zx plane
μ_{xy}	Poisson's ratio towards x-axis with respect to y-axis
μ_{yx}	Poisson's ratio towards y-axis with respect to x-axis
E_1	Elastic modulus towards fibre of orthotropic material
E_2	Elastic modulus perpendicular to fibre of orthotropic material
G_{12}	Shear modulus for on-axis fibre orientation plane
μ_{12}	Poisson's ratio towards fibre with respect to perpendicular axis
E	Elastic modulus of isotropic material
G	Shear modulus of isotropic material
μ	Poisson's ratio of isotropic material
u_x	Displacement component in x-direction (Axial displacement)
u_y	Displacement component in y-direction (Lateral displacement)
u_z	Displacement component in z-direction
ϵ_{xx}	Strain component in x-direction
ϵ_{yy}	Strain component in y-direction
ϵ_{zz}	Strain component in z-direction
ϵ_{xy}	Shear strain component in xy plane
ϵ_{yz}	Shear strain component in yz plane

ϵ_{zx}	Shear strain component in zx plane
σ_{xx}	Normal stress component in x-direction
σ_{yy}	Normal stress component in y-direction
σ_{zz}	Normal stress component in z-direction
σ_{xy}	Shear stress component in xy plane
σ_{yz}	Shear stress component in yz plane
σ_{zx}	Shear stress component in zx plane
q_0	Intensity of uniform loading on the beam
F_x, F_y, F_z	Components of body force in x, y and z directions, respectively
ϕ	Displacement potential function
ψ	Airy's stress function
L	Length of the beam
W	Width/thickness of the beam
D	Depth of the beam

LIST OF FIGURES

Fig. 2.1 Elementary Cubic Body

Fig. 2.2 Stress components on a plane

Fig. 2.3 Stress components on a composite plane

Fig. 3.1 Geometry and loading of a simply supported deep beam: (a) Unguided simply supported beam, (b) Guided simply supported beam

Fig. 3.2 Analytical model of a guided simply supported beam subjected to uniform loading

Fig. 3.3 Normalized axial displacement, $L/D = 3$: (a) distribution over longitudinal sections, (b) distribution over transverse sections from left guide to mid of span

Fig. 3.4 Normalized lateral displacement distribution, $L/D = 3$: (a) over longitudinal sections (b) over transverse sections from left guide to mid of span

Fig. 3.5 Deformed shape of the guided isotropic deep beam, $L/D = 3$ (magnification factor $\times 500$)

Fig. 3.6 Distribution of bending stress of the guided isotropic deep beam, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 3.7 Normalized lateral stress distribution of the guided isotropic deep beam, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 3.8 Shear stress distribution of the guided isotropic deep beam for $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 3.9 Normalized maximum principal stress contour of guided isotropic beam, $L/D=3$

Fig. 3.10 FEM meshing (20 x 20) and loading arrangement of steel beam, $L/D=3$

Fig. 3.11.1 Comparison of ψ -solution with those of beam theory and FEM: (a) lateral displacement [$y/D=0.5$], (b) bending stress [$x/L=0.0$], (c) bending stress [$x/L=0.5$], (d) bending stress [$x/L=0.1$]

Fig. 3.11.2 Comparison of ψ -solution with those of classical beam theory and FEM: (a) lateral stress [$x/L=0.1$], (b) lateral stress [$x/L=0.5$], (c) shear stress [$x/L=0.1$], (d) shear stress [$x/L=0.25$]

Fig. 3.12 Effect of beam aspect ratio on normalized bending stress at (a) $x/L=0.0$, (b) $x/L=0.1$, (c) $x/L=0.2$, (d) $x/L=0.3$, (e) $x/L=0.4$, (f) $x/L=0.5$

Fig. 3.13 Effect of beam aspect ratio on normalized lateral stress at: (a) $x/L=0.0$, (b) $x/L=0.1$, (c) $x/L=0.2$, (d) $x/L=0.3$, (e) $x/L=0.4$, (f) $x/L=0.5$

Fig. 3.14 Effect of beam aspect ratio on normalized shear stress at: (a) $x/L=0.1$, (b) $x/L=0.2$, (c) $x/L=0.3$, (d) $x/L=0.4$

Fig. 3.15 Geometry and loading of guided isotropic beam with parabolic reaction

Fig. 3.16 Axial displacements of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

Fig. 3.17 Lateral displacements of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

Fig. 3.18 Bending stress of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

Fig. 3.19 Lateral stress of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

Fig. 3.20 Shear stress of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

Fig. 3.21 Geometry and loading of an unguided simply supported beam

Fig. 3.22 Geometry and loading of model for analytical solution of unguided beam

Fig. 3.23 Axial displacement of unguided beam, $L/D=2$: (a) over longitudinal sections, (b) over transverse sections

Fig. 3.24 Lateral displacement of unguided beam, $L/D=2$: (a) over longitudinal sections, (b) over transverse sections

Fig. 3.25 Distribution of bending stress for unguided consideration, $L/D = 2$: (a) over longitudinal, (b) over transverse sections

Fig. 3.26 Distribution of lateral stress for unguided consideration, $L/D = 2$: (a) over longitudinal, (b) over transverse sections

Fig. 3.27 Shear stress distribution for unguided consideration, $L/D = 2$: (a) over longitudinal, (b) over transverse sections

Fig. 3.28 Configuration of unguided beam for stress function solution

Fig. 3.29 Comparison of bending stress amongst • -solutions with those of stress function and FEM, $L/D=2$: (a) at $x/L=0.25$, (b) at $x/L=0.5$

Fig. 3.30 Comparison of lateral stress amongst • -solutions with those of stress function and FEM, $L/D=2$: (a) at $x/L=0.25$, (b) at $x/L=0.5$

Fig. 3.31 Shear stress comparison amongst • -solutions with stress function and FEM, $L/D=2$: (a) at $x/L=0.0$, (b) at $x/L=0.25$

Fig. 3.32 Comparison of lateral displacement amongst • -solutions with the solution of stress function and FEM [$y/D=0.5$]

Fig. 4.1 Analytical model of an orthotropic guided beam

Fig. 4.2 Normalized axial displacement for the Glass-Epoxy guided beam, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 4.3 Normalized lateral displacement for the Glass-Epoxy guided beam, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 4.4 Deformed shape of the Glass-Epoxy beam, $L/D = 3$ (magnification factor $\times 500$)

Fig. 4.5 Normalized bending stress for the Glass-Epoxy guided beam, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 4.6 Normalized lateral stress for the Glass-Epoxy guided beam, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 4.7 Normalized shearing stress for the Glass-Epoxy guided beam, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 4.8 Comparison of lateral deflection at $y/D=0.5$ of Glass-Epoxy deep beam, $L/D=3$

Fig. 4.9 Comparison of bending stress of Glass-Epoxy deep beam, $L/D=3$: (a) at $x/L=0.0$ (b) at $x/L=0.5$

Fig. 4.10 Comparison of lateral stress of Glass-Epoxy deep beam, $L/D=3$: (a) at $x/L=0.0$ (b) at $x/L=0.1$

Fig. 4.11 Comparison of shear stress of Glass-Epoxy deep beam, $L/D=3$: (a) at $x/L=0.1$ (b) at $x/L=0.25$

Fig. 4.12 Bending stress distribution of the Glass-Epoxy beam for different aspect ratio: (a) at $x/L=0.0$, (b) at $x/L=0.5$

Fig. 4.13 Lateral stress distribution of the Glass-Epoxy beam for different aspect ratio: (a) at $x/L=0.0$, (b) at $x/L=0.5$

Fig. 4.14 Shearing stress distribution of glass epoxy beam for different aspect ratio, $L/D=1$ to 4: (a) at $x/L=0.1$, (b) at $x/L=0.4$

Fig. 4.15 Contour of maximum principal stress distribution of the orthotropic beam of glass epoxy material: (a) for $L/D=1$, (b) for $L/D=2$, (c) for $L/D=3$

Fig. 4.16 Displacement of the beam ($L/D=3$) for glass-epoxy, boron-epoxy and graphite-epoxy materials : (a) axial at $x/L=0.25$, (b) lateral at $x/L=0.5$

Fig. 4.17 Bending stress of the beam ($L/D=3$) for glass-epoxy, boron-epoxy and graphite-epoxy materials: (a) at $x/L=0.0$, (b) at $x/L=0.5$

Fig. 4.18 Shear stress of the beam ($L/D=3$) for glass-epoxy, boron-epoxy and graphite-epoxy materials: (a) at $x/L=0.1$, (b) at $x/L=0.25$

Fig. 4.19 Lateral displacement of the glass-epoxy beam, $L/D=3$ (at $y/D=0.5$): (a) displacement potential solution, (b) classical beam theory

Fig. 4.20 Bending stress distribution of the glass-epoxy beam, $L/D=3$ (at $x/L=0.5$): (a) displacement potential solution, (b) classical beam theory

Fig. 4.21 Shear stress distribution of the glass-epoxy beam, $L/D=3$ at $x/L=0.1$ and 0.4

Fig. 5.1 Geometry and loading (symmetric) of orthotropic beam for three point bending

Fig. 5.2 Normalized axial displacement for three point bending, $L/D = 3$: (a) over longitudinal sections, (b) transverse sections

Fig. 5.3 Normalized lateral displacement distribution for three point bending, $L/D = 3$: (a) over longitudinal sections, (b) transverse sections

Fig. 5.4 Deformation under three point bending, $L/D = 3$ (magnification factor $\times 500$)

Fig. 5.5 Distribution of bending stress of a glass epoxy beam for three point bending, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.6 Distribution of lateral stress of a glass epoxy beam for three point bending, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.7 Distribution of shearing stress of a glass epoxy beam for three point bending, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.8 Geometry and loading (asymmetric) of orthotropic beam

Fig. 5.9 Normalized axial displacement glass-epoxy beam with asymmetric point loading, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.10 Normalized lateral displacement glass-epoxy beam with asymmetric point loading, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.11 Deformed shape of the glass-epoxy beam with asymmetric three point bending, $L/D = 3$ (magnification factor $\times 500$)

Fig. 5.12 Distribution of bending stress of a glass epoxy beam under asymmetric three point loading, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.13 Distribution of lateral stress of a glass epoxy beam under asymmetric three point loading, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.14 Distribution of shearing stress of a glass epoxy beam under asymmetric three point loading, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.15 Comparison of distribution of stresses of glass-epoxy beam for three point bending, $L/D=3$: (a) bending stress at $x/L=0.5$, (b) lateral stress at $x/L=0.0$, (c) shearing stress at $x/L=0.25$

Fig. 5.16 Comparison of distribution of stresses of steel beam for three point bending, $L/D=3$: (a) bending stress at $x/L=0.5$, (b) shearing stress at $x/L=0.25$

LIST OF TABLES

Table 2.1 Number of Elastic Constants

Table 3.1 Maximum Normalized Bending Stress at $x/L=0.0$ and 0.5

Table 3.2 Maximum Normalized Lateral Stress at $x/L=0.0, 0.1$ and 0.25

Table 3.3 Maximum Normalized Shearing Stress at $x/L=0.1$ and $x/L=0.25$

Table 3.4 Bending Stress Comparison for Unguided Beam at $x/L=0.25$ and $x/L=0.5$

Table 3.5 Lateral Stress Comparison for Unguided Beam at $x/L=0.25$ and $x/L=0.5$

Table 3.6 Shear Stress Comparison for Unguided Beam at $x/L=0.0$ and $x/L=0.25$

Table 4.1 Properties of Composites Used for Study

Table 4.2 Comparison of Maximum Normalized Bending Stress for composites at $x/L=0.5$

Table 4.3 Comparison of Maximum Lateral Stress for Composites at $x/L=0.0$ and $x/L=0.5$

Table 4.4 Comparison of Shear Stress for Composites at $x/L=0.1$ and $x/L=0.25$

CHAPTER 1

INTRODUCTION

1.1 Preamble

Structural analysis comprises the set of physical laws and mathematics required to study and predicts the behavior of structures under load. It is such an engineering artifact whose integrity is judged largely based upon the ability to withstand loads on building, bridge, ship, submarine, aircraft etc. From theoretical perspective, the primary goal of structural analysis is the computation of deformations, internal forces and stresses. In practice, structural analysis can be viewed more abstractly as a method to drive the engineering design process or to prove the soundness of a design without a dependence on directly testing it. Engineering subjects which deal with the matter are namely the mechanics of materials (also known as strength of materials) and the theory of elasticity.

The classical tools of Mechanics of Materials are derived from mathematical physics and in many cases they are not sufficient to describe the stress distribution in engineering structures. They are also inadequate to give information regarding local stress near the loads and near the supports of beams. Among the great number of problems conforming this field of science, there is necessity of profound investigation of the properties of elastic materials and the construction of a mathematical theory, which would permit studying as completely as possible the internal forces occurring in an elastic body under the action of external forces as well as the deformation of a body i.e., the change of its shape. This led to the emergence of a special trend in physics, i.e. the Theory of Elasticity to apply to elastic solids [1-3]. Individual members such as beams, columns, shafts, plates, shells etc may be modeled using either of the classical or the elasticity based mathematical approach. However, each process has got some of its own advantages as well as limitations.

The equations of theory of elasticity are a system of partial differential equations. Due to the nature of the mathematics involved in solving such equations, manual solutions have been only developed for relatively simple geometries. For complex geometries, computation using a modern computer is considered more suitable for better solution.

1.2 Literature Review

The foundation of the classical theory of elastic beam was laid down in the 18th century (1705-1750) by Leonhard Paul Euler and Daniel Bernoulli. It was then Claude-Louis Navier, who gave the details of the relation between the cross-section and the bending stiffness parameter in 1819. This theory has remained as an exact solution of the equations of three-dimensional elasticity in the case of homogenous bending, i.e. for a beam with constant bending moment. Since then the elementary methods of strength of materials were the primary tools of the practicing engineers for handling the engineering problems of structural elements for long time. However these methods are often found inadequate to furnish satisfactory information regarding local stresses near the loads and near the supports of the structures. The elementary theory gives no means of investigating stresses in regions of sharp variation in cross section of beams or shafts. Stresses in screw threads, around various shapes of holes in structures, near contact points on gear teeth, rollers and balls of bearings, have all remained beyond the scope of elementary theories. It is thus obvious that, for the designers of modern machines, recourse to the more powerful methods of the Theory of Elasticity is the necessity of time.

Considerable progress could have been made in solving important practical problems of stress analysis using the methods of theory of elasticity. In cases where a rigorous solution could not be obtained, approximate methods have been developed. In other cases, where even approximate methods could not be developed, solutions have been obtained by using experimental methods. Photoelastic methods, soap-film methods, application of strain gages, moiré fringe etc. are some of these experimental methods applied in the study of stress concentration at points of sharp variation of cross-sectional dimension and at sharp fillets of re-entrant corners. These results have considerably influenced the modern design of machine parts and helped in many cases to improve the construction by eliminating weak spots through which crack may switch on and thereby propagate.

The field of elasticity deals mainly with deformation parameters and stress parameters for the solution of two dimensional problems since most of the three-dimensional problems may be resolved to a two dimensional one. If it remains beyond the extent of analytical studies anyway, the problem has to be handled experimentally as a particular case.

Although the theories of elasticity had been established long before, the solutions of practical problems started mainly after the introduction of a stress function by George Biddell Airy, a British astrologer and mathematician, in 1862 [1]. The Airy's stress function is governed by a fourth order partial differential equation and stress components are related to it through its various second order derivatives. The stress function solutions were initially sought taking polynomial expressions of various degrees and suitably adjusting their coefficients. By this way, a number of practically important problems of long rectangular strips could be solved [2-3]. But the success of this approach was very limited. Using these polynomial expressions, an elementary derivation of the effect of the shearing force on the curvature of the deflection curve of beams were made by Rankine [4] and by Grashof [5]. The problem of stresses in masonry dams is of great practical interest and has been attempted for solution using polynomial expressions for the stress functions [6-7]. But the solutions obtained do not satisfy the conditions at the bottom of the dam where it is connected with the foundation.

Thereafter the use of trigonometric series is considered more fruitful in producing results instead of polynomial expressions. The first application of trigonometric series in the solution of elastic problems using stress function method was given by Ribiere in his thesis [8]. Further progress in the application of these solutions was made by Filon [9]. Several particular examples were worked by Bleich [10].

Then the use of Fourier series has come into being for the structural analysis. Beyer [11] solved the problem of a continuous beam on equidistant supports under gravity loading using Fourier series. Stress function technique has also been used by Ribiere [12] for analyzing the stresses around a circular hole in plate, by Flamant [13] for stresses around a concentrated load on a straight boundary and by Stokes [14] for stresses around a concentrated load on a beam. A number of works are found with the stress function in this regard [15- 18]. Even tough the stress analysis problems are bearing enumerable shortcomings to be addressed yet.

The stumbling block of obtaining exact solution using stress function is the inability of managing the physical conditions imposed on them, i.e., management of boundary conditions of practical problems. Boundary restraints specified in terms of the displacement components cannot be satisfactorily imposed on the stress function. Since most of the practical problems in elasticity are of mixed boundary conditions, the approach fails to

provide any explicit understanding of the state of stresses at the critical regions of supports and loadings. Moreover, the famous Saint Venant's principle is still applied and its merit is evaluated in solving problems of solid mechanics in which full boundary effects could not be taken into account satisfactorily in the process of solution [19-21]. For complex shapes of boundary, the difficulties of obtaining analytical solutions become formidable. These difficulties were partially avoided by renovating to experimental methods, such as extensometers, strain gauges or photoelastic methods. Using Photoelasticity, Hetengi investigated the stresses in the threads of a bolt and nut fastening. Most of the experimental investigations of elastic problems are reported in the "handbook of Experimental Stress Analysis" [22] and in "Photoelasticity" by Frocht [23]. Even now, photoelastic studies are being carried out for classical problems like uniformly loaded beams on two supports mainly because the boundary effects could not be taken into account fully in the analytical method of solutions.

The drawback of Airy's stress function and the difficulties of experimental works led to works for finding ways to solve the engineering problems where boundary conditions are set in terms of displacements. Thus the displacement formulation was introduced for those problems where boundary restraints exist [24]. This process involves finding of two displacement functions simultaneously from the two second-order elliptic partial differential equations of equilibrium, which is extremely difficult and the problem becomes more serious when the boundary conditions are mixed. The difficulties involved in trying to solve practical stress problems using the existing models have been pointed out by Durelli and Ranganayakamma [25]. The complications for solution beams especially short/deep beams were also brought to light by Rehfield and Murthy [26], Murty [27], Suzuki [28], and Hardy [29].

Since neither the stress function nor the displacement formulation is suitable for solving problems of mixed-boundary conditions, a new mathematical model called displacement potential formulation is used to solve the elastic problems [30-37]. The current modelling approach reduces the two dimensional problem to the solution of a single differential equations of equilibrium and also enables the mixed mode of the boundary conditions to be managed appropriately. It is worth mentioning that a number of researchers worked on the advancement of displacement potential approach to handle the beam and column like structures with different loading and supporting conditions. Ahmed et. al. have developed

numerical solution of both ends fixed deep beams based on displacement potential formulation [30]. Ahmed et. al. [31] have carried out investigation of stresses at the fixed end of deep cantilever beams. Akanda et. al. have carried out stress analysis of gear teeth using displacement potential function and finite differences [32]. The potential of the formulation has also been investigated by Ahmed et. al. [33] to design optimum shapes of tire-treads for avoiding lateral slippage between tires and roads. Recently, Ahmed et. al. [34] have proposed a general mathematical formulation for finite-difference solution of mixed-boundary-value problems of anisotropic materials. Further, Debnath et. al. have carried out analytical solution of short stiffened flat composite bars under axial loadings [35], and stiffened orthotropic composite panels under uniaxial tensile load [37]. All these solutions are mainly applicable for rigid boundaries; none of them is applicable to a guided beam under bending. As such the solution for guided structural element is yet to be developed.

1.3 Analysis of Deep Beam

A beam may be considered as one of the most commonly used structural elements in engineering applications. A beam is said to be a deep beam when the depth is comparable to its span. Design of deep beams based on classical Euler bending theory can be seriously erroneous, since the simple theory of flexure takes no account of the effect of normal pressures on the top and bottom edges of the beam caused by the loads and reactions [16]. The effect of normal pressures on the stress distribution in deep beams is such that the distribution of bending stresses on vertical sections is not linear and the distribution of shear stresses is not parabolic. Consequently, a plane transverse section does not remain plane after bending, and the neutral axis does not lie at the mid-depth, which eventually causes the basis of classical theory to be violated. In an attempt to make up the limitation, different theories as well as methods of solution have been reported in the literature [15-17, 27-28].

However, each solution possesses certain limitations, and eventually none of the solutions are found to conform to all the physical characteristics of the problem for deep beam appropriately. Even, photoelastic studies [24], finite element analysis [29] and finite difference solutions [30-34] have also been carried out for deep beams on two supports, mainly because all the physical conditions imposed on the beam could not be fully taken into account in the analytical methods of solution. Among the existing mathematical models

of elasticity for the plane boundary-value problems, the stress function approach and the displacement formulation are noticeable. The stress function approach accepts boundary conditions in terms of loading only; boundary restraints cannot be satisfactorily imposed on it. On the other hand, the displacement formulation involves extreme difficulty especially when the boundary conditions are a mixture of restraints and stresses. As a consequence, serious attempts had hardly been made in the past for stress analysis using this formulation. As such, neither of the existing formulations is suitable for solving problems of mixed boundary conditions.

Further, the use of standard structures, like beams, columns, etc. with guides on part or full of their bounding surfaces is receiving increased importance in order to satisfy precise and strict design criteria in many of the engineering applications. Guided boundaries usually help in reducing the level of deformation in the structural elements, which eventually resist the change of the original shape of the bounding surfaces under loading. But structures with guided boundaries always remain away from the scope of analytical solutions, because the physical conditions of guided boundaries need to be mathematically modelled in terms of a mixed mode of boundary conditions.

Again the use of fibre-reinforced composites is found to increase extensively in almost all the areas of structural applications, mainly because of their specific characteristics of light-weight and high-strength. As a result, the analysis of composite structures has now become a key subject in the field of solid mechanics. These analyses are mainly handled by approximate numerical techniques, as, in most cases, the available mathematical models are found to be inadequate to provide exact analytical solutions for them.

Since the exact analytical solution of mixed-boundary-value elastic problems, especially with fibre reinforced composite materials is beyond the scope of existing mathematical models of elasticity, the use of a new mathematical formulation will be investigated to analyze the elastic behaviour of a guided deep beam of fibre reinforced composites under different loading and support arrangements. It would be worth mentioning that, as far as the reporting in the literature is concerned, the author has not come across any reliable study of the present problem, either theoretical or experimental. Therefore, the analytical solution for a guided deep beam of orthotropic composite materials is yet to be developed at various loading conditions.

1.4 Objectives

The present study is an attempt to extend the capability of the displacement potential formulation in order to address the structural analysis of orthotropic composites and isotropic materials having mixed boundary conditions. The main objectives of the present research work are summarized as follows:

- a. To develop a suitable scheme for obtaining analytical solution of the plane elastic field of a guided simply-supported deep beam of isotropic material under bending.
- b. To extend the analytical scheme to obtain solutions for the deep beam with different fibre reinforced composites as well as with different loading and support arrangements.
- c. To extend the scheme developed for the guided beam to obtain analytical solutions for the unguided simply supported deep beams.
- d. To analyze the deformed shape as well as the distributions of different displacement and stress components of interest in the perspectives of beam depth and span.
- e. To investigate the effect of fibre reinforcement and beam aspect-ratio on the distributions of displacement and stresses in the beam.
- f. To obtain numerical solution of the present composite beam and then compare with the corresponding results of analytical solutions obtained using displacement potential function.

Results of the present analysis are expected to provide a reliable design guide for deep composite beams of the present kind, which will be of significant help for their improved and economic design. More importantly, the present analytical solution will remain as a standard guide for checking reliability and accuracy of approximate solutions of the problem. The study would be particularly important for machine parts under bending, supported on two supports which are placed/inserted in a position that allows no axial deformation of the structural member.

1.5 Study Procedure

In the present study, the elastic behaviour of guided deep beams of composite materials are investigated under bending through an analytical procedure based on displacement potential formulation [30-32], which is also suitable for mixed-boundary-value elastic problems of composite materials. In the displacement potential boundary modelling approach, the plane elastic problem is formulated in terms of a single function of space variables, called as displacement potential function and defined in terms of displacements components, which has to satisfy a single fourth-order partial differential equation of equilibrium. The relevant displacement and stress components are derived into infinite series using Fourier integral with coincided boundary conditions along with the physical boundary conditions. The beam is assumed to be simply supported on two supports at the bottom, and the two opposing lateral edges are guided for which any change in the axial displacement is restrained. The fibres of the composite materials are assumed to be situated along the beam length. Both orthotropic and isotropic cases are taken into consideration for the study and the respective materials in this regard are steel and glass epoxy. However, some other materials may also be taken into consideration if required at the instance. The numerical solution of the composite beam is obtained by finite element method with the help of standard commercial software.

1.6 Significance of Present Study

The present study has the significance in regard to academic concept, design reference and manufacturing engineering. The study is going to present the application of a new concept, i.e., displacement potential approach for the analysis of stress and displacement in structures for both isotropic and orthotropic materials under mixed mode of boundary conditions. It is expected to find out some additional aspects to the theory of elasticity, which in turn may encourage academicians and researchers to explore the concept further and eliminate the lack of suitable method for dealing with mixed boundary value problems of complex geometries. In this study a number of problems of guided elements are solved by using the present analytical methods and results are presented in the form of graphs. The results may

be used as a database and may be helpful to the designers working in the industries of aerospace, shipbuilding and automobile, where numerous composite structures are used. The study of guided beam analysis on structural point of view is very pertinent for those machine parts where expansion in one dimension i.e. either longitudinal or lateral cannot be allowed.

CHAPTER 2

THEORETICAL OUTLINE AND FORMULATION

2.1 Preamble

The theory of elasticity sets forth the solution of problem in determining internal forces in a solid elastic body. The internal forces represent interaction between molecules; they insure the external forces applied to body. Under the action of external forces the body deforms, the mutual position of molecules changes and so do the distance between them. The action of external forces that produce deformation gives rise to additional internal forces causing the stress of the body.

Thus structural analysis necessitates the requirements to investigate the state of stresses, strains and displacements at any point due to given body forces and given conditions at the boundary of the body. Most of the cases the requirement is to find the stress distribution in an elastic body. In some cases, it is also required to find the strain distribution of the body. The state of stress, strains and displacement are termed as the elastic field. A complete description of the elastic fields requires specification of forces acting on the elementary body and its surface orientation.

2.2 Equilibrium and Compatibility Conditions

Let us take an infinitesimal cubic element from an elastic body with sides parallel to the coordinate axes. To ensure the equilibrium of the element, six forces will act on the six different faces of the element. The forces acting on each face may be resolved into two types i.e. one perpendicular to the plane of the face and the other parallel to the face. The stress component acting perpendicular to the face is the normal stress and the two stress components acting parallel of the face are the shearing stress as illustrated in fig. 2.1.

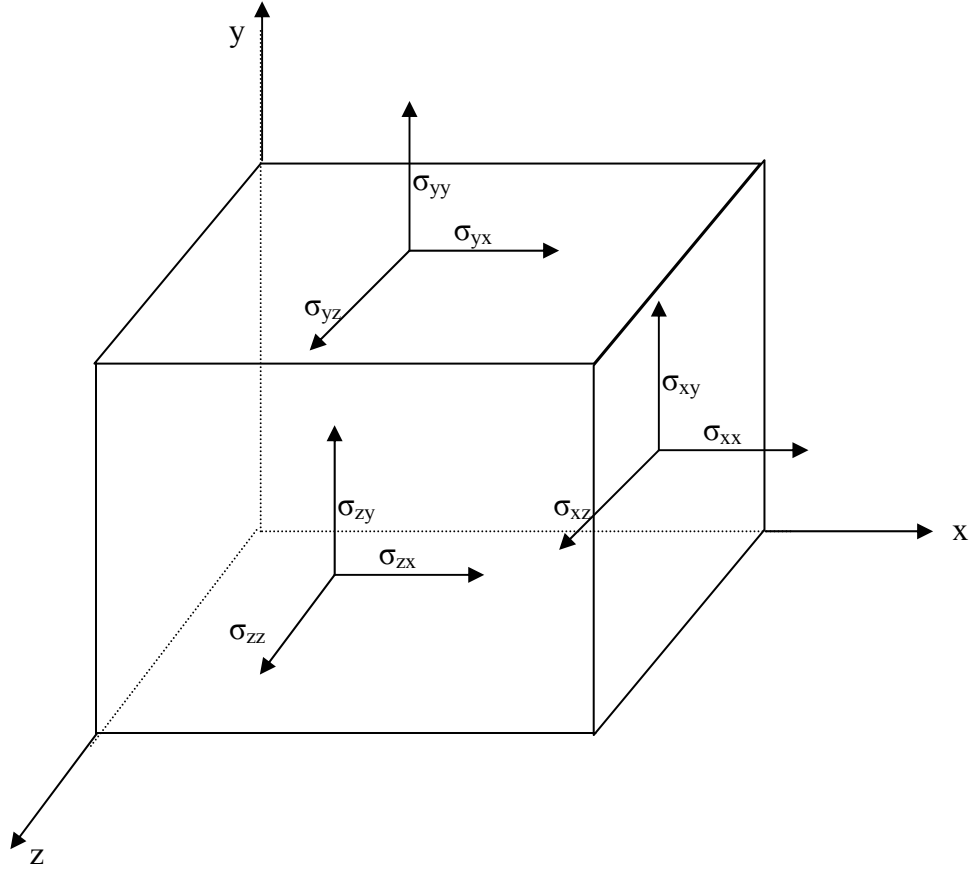


Fig. 2.1 Elementary Cubic Body.

In order to provide complete information of an elastic field, it is necessary to determine nine stress components ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yx}, \sigma_{yz}, \sigma_{zy}, \sigma_{xz}$ and σ_{zx}) and six strain components ($\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}$ and γ_{zx}). Instead of strain components, sometimes the displacement components (u_x, u_y and u_z) are determined. It is worthy to mention that the components of strain and displacement can be determined from each other and each set provide the similar information. Therefore, either the components of strain or displacement are sufficient for a particular purpose.

According to general conventions, the normal stress is taken positive when producing tension and negative when producing compression. On any side the direction of the positive shearing stress coincides with the positive direction of the axis if the outward normal of this side has the positive direction of the corresponding axis. If the outward normal has a direction opposite to positive axis the positive shearing stress will also have the opposite direction of the corresponding axis. The first subscript of the symbol indicates the direction of the normal of the plane on which the stress acting and the second subscript indicates the

direction of stress. By a simple consideration of the equilibrium of the element shown in fig. 2.1, it can be shown that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{zx} = \sigma_{xz}$ and $\sigma_{yz} = \sigma_{zy}$. Thus, the nine components of stress are reduced to six σ_{xx} , σ_{yy} , σ_{zz} , σ_{xy} , σ_{yz} and σ_{zx} [1-2].

From the consideration of an infinitesimal cubic element surrounding a given point in a body, it is found that the static equilibrium of forces requires at this point is to satisfy the followings equations:

$$\left. \begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x &= 0 \\ \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} + F_y &= 0 \\ \frac{\partial \sigma_{zx}}{\partial z} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial x} + F_z &= 0 \end{aligned} \right\} \quad (2.1)$$

These equations are known as the equations of equilibrium, where F_x , F_y and F_z are the components of the body force in x, y and z directions respectively [2].

The six stress components satisfy the above mentioned three equations of equilibrium; but it is not practicable to obtain six stress components solving three equations. As such consideration of more relations is the option to have equity in number of variables as well as equations. In this regard, following six relations are defining the three strain components in terms of the three displacement components through partial differentiation [2].

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u_x}{\partial x}, \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \end{aligned} \quad (2.2)$$

In addition, the six stress-strain relations are also there. Thus one can have altogether 15 unknowns and 15 equations. This system of equations is generally sufficient for the solution of an elasticity problem.

By differentiation and simple manipulation of Eq. (2.2), the following set of differential equations can be obtained.

$$\begin{aligned}
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}; & 2 \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x} \left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}; & 2 \frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} &= \frac{\partial}{\partial y} \left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} &= \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}; & 2 \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z} \left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right)
\end{aligned} \tag{2.3}$$

These differential relations are called the conditions of compatibility. The solution of an elasticity problem must satisfy the equilibrium i.e. Eq (2.1) and the compatibility conditions i.e. Eq (2.3) along with the boundary conditions.

2.3 Hooke's Law

In the simplest approximation the relation between stress and strain is taken to be linear and called Hooke's law named after the 17th century British physicist Robert Hooke. The most general form of linear stress-strain relationship for anisotropic material is given by the following expression [38].

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \\ \sigma_{yx} \\ \sigma_{xz} \\ \sigma_{zy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} & c_{19} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} & c_{29} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} & c_{39} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} & c_{49} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & c_{57} & c_{58} & c_{59} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & c_{67} & c_{68} & c_{69} \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & c_{77} & c_{78} & c_{79} \\ c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & c_{88} & c_{89} \\ c_{91} & c_{92} & c_{93} & c_{94} & c_{95} & c_{96} & c_{97} & c_{98} & c_{99} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \\ \gamma_{yx} \\ \gamma_{xz} \\ \gamma_{zy} \end{Bmatrix} \tag{2.4}$$

Where the 81 coefficients $c_{11}, \dots, c_{99}$ are called elastic coefficients or stiffness. For the equilibrium condition it is found that $\sigma_{ij} = \sigma_{ji}$, $\gamma_{ij} = \gamma_{ji}$. As such $\sigma_{xy} = \sigma_{yx}$, $\sigma_{zx} = \sigma_{xz}$, $\sigma_{yz} = \sigma_{zy}$, $\gamma_{xy} = \gamma_{yx}$, $\gamma_{zx} = \gamma_{xz}$ and $\gamma_{yz} = \gamma_{zy}$.

Therefore, the stress–strain relation becomes as follows

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yx} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} \quad (2.5)$$

From the consideration of strain energy density, it can be shown that $c_{ij} = c_{ji}$

Therefore,

$$\begin{aligned} c_{12} &= c_{21}, \quad c_{13} = c_{31}, \quad c_{14} = c_{41}, \quad c_{15} = c_{51}, \quad c_{16} = c_{61} \\ c_{23} &= c_{32}, \quad c_{24} = c_{42}, \quad c_{25} = c_{52}, \quad c_{26} = c_{62}, \quad c_{34} = c_{43} \\ c_{35} &= c_{53}, \quad c_{36} = c_{63}, \quad c_{45} = c_{54}, \quad c_{46} = c_{64}, \quad c_{56} = c_{65} \end{aligned}$$

Thus, the 36 coefficients of the stiffness matrix in Eq. (2.5) come down to 21 and the stiffness matrix turns to a symmetric matrix as follows.

$$[C_{ij}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \quad (2.6)$$

Materials having symmetry with respect to one plane is referred to as monoclinic materials.

For such case of material, transformation of axis can be done and found that

$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{44} = c_{56} = 0$ and then the number of elastic coefficient becomes 13 only.

Thus, the stiffness matrix of Eq. (2.6) further reduces to

$$[C_{ij}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & 0 \end{bmatrix} \quad (2.7)$$

Again an orthotropic material has at least two orthogonal planes of symmetry, where material properties are independent of direction within each plane. Normally the reference system of coordinates is selected along the principal planes of material symmetry. Examples of an orthogonal material include a single lamina of continuous fibre composite arranged in a rectangular array, a wooden bar and rolled steel. For such case $c_{16} = c_{26} = c_{36} = c_{45} = 0$ and then this type of materials require 9 independent variables as elastic constants in their stiffness matrix as follows.

$$[C_{ij}](Orthotropic) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \quad (2.8)$$

Where

$$c_{11} = \frac{1 - \mu_{yz}\mu_{zy}}{E_y E_z \nabla}; \quad c_{12} = \frac{\mu_{xy} + \mu_{zy}\mu_{xz}}{E_x E_z \nabla}$$

$$c_{13} = \frac{\mu_{xz} + \mu_{zy}\mu_{yz}}{E_x E_y \nabla}; \quad c_{22} = \frac{1 - \mu_{xz}\mu_{zx}}{E_x E_z \nabla}$$

$$c_{23} = \frac{\mu_{yz} + \mu_{yx}\mu_{xz}}{E_x E_y \nabla}; \quad c_{33} = \frac{1 - \mu_{xy}\mu_{yx}}{E_x E_y \nabla}$$

$$c_{44} = G_{yz}; c_{55} = G_{zx}; c_{66} = G_{xy}$$

$$\nabla = \frac{1 - \mu_{xy}\mu_{yx} - \mu_{yz}\mu_{zy} - \mu_{zx}\mu_{xz} - 2\mu_{yx}\mu_{zy}\mu_{xz}}{E_x E_y E_z}$$

The reciprocal relations are given by.

$$\frac{\mu_{ij}}{E_i} = \frac{\mu_{ji}}{E_j}; i, j = x, y, z$$

Most metallic alloys and thermoset polymers are considered isotropic material, where by definition the mechanical properties are independent of direction. In this case there are

infinite planes of symmetry. Such materials have only two independent variables i.e. elastic constants in their stiffness matrix as

$$[C_{ij}]_{(Isotropic)} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c_{11} - c_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{c_{11} - c_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11} - c_{12}}{2} \end{bmatrix} \quad (2.9)$$

These constants are given by

$$c_{11} = \frac{E(1-\mu)}{(1-2\mu)(1+\mu)}; \quad c_{12} = \frac{\mu E}{(1-2\mu)(1+\mu)}; \quad \text{and} \quad \frac{c_{11} - c_{12}}{2} = \frac{E}{2(1+2\mu)(1-2\mu)}$$

The summarised form of independent elastic constants for general anisotropic, anisotropic with symmetric stress and strain components or with energy consideration, orthotropic and isotropic materials can be thus presented in table 2.1 as follows.

Table 2.1 Number of Elastic Constants

Serial	Material	Condition	No of constant
1	Anisotropic	General form	81
2	Anisotropic	Equilibrium condition	36
3	Anisotropic	Stain energy consideration	21
4	Monoclinic	Symmetric to a plane	13
5	Orthotropic	Having mutually perpendicular planes of symmetry	09
6	Isotropic	Same elastic properties in all directions (having infinite perpendicular planes of symmetry)	02

2.4 Two - Dimensionalization of the Problem

Although the elastic analysis in general form is of three dimensional, orthotropic and isotropic materials can be analyzed using two dimensions on the consideration of symmetry of planes. For such simplification there are two options i.e. (i) plane stress condition and (ii) plane strain condition.

Plane stress condition is considered to be a state of stress in which the normal stress σ_{zz} and the shear stresses σ_{xz} and σ_{yz} directed perpendicular to the plane are assumed to be zero (but not the strain). Generally, members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the x - y plane can be considered to be under plane stress. Thus, a state of plane stress exists in a thin object loaded in the plane of its largest dimensions. The non-zero stresses σ_{xx} , σ_{yy} , and σ_{xy} lie in the x - y plane and do not vary in the z direction. A thin beam loaded in its plane and a spur gear tooth are good examples of plane stress problems.

On the other hand plane strain is said to be a state of strain in which the strain normal to the x - y plane ϵ_{zz} and the shear strains γ_{xz} and γ_{yz} are assumed to be zero. The assumptions of the plane strain are realistic for long bodies (saying in the z direction) with constant cross-sectional area subjected to loads that act only in the x and/or y directions and do not vary in the z direction.

The option (i) i.e. the plane stress condition has been followed in the present study. Thus

$$\sigma_{zz} = 0; \sigma_{zx} = 0; \sigma_{yz} = 0 \quad (2.10)$$

At this condition, the equilibrium equation (2.1) having no body force reduces to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad (2.11a)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \quad (2.11b)$$

The stresses for a two dimensional element at plane stress condition are shown in fig. 2.2.

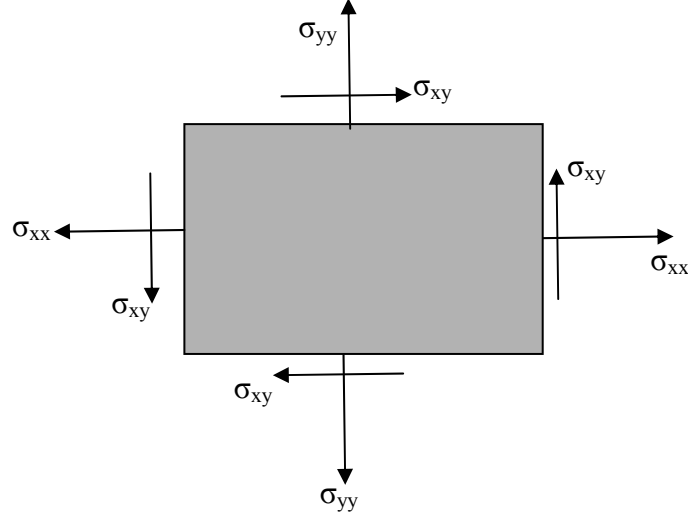


Fig. 2.2 Stress components on a plane

The two equilibrium equations (2.11) contain three unknown stress components. Thus, one more equation is required to obtain an exclusive solution of three unknowns. The third equation is the mathematical formulation of the condition for compatibility, which can be obtained from the strain displacement relations. For two dimensional cases, these relations are:

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \quad (2.12)$$

Differentiating the first equation of (2.12) twice with respect to y , the second twice with respect to x and the third once with respect to x and once with respect to y , the expression for condition of compatibility in term of strain then becomes as follows:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad (2.13)$$

But there would have been the necessity of one more equation in terms of stresses; which can be obtained using stiffness matrix in Eq. (2.13). For orthotropic material, while the stiffness matrix is given by Eq (2.8), the stress-strain relations in the case of plane stress can be reduced to:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{12} & K_{22} & 0 \\ 0 & 0 & K_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \quad (2.14)$$

It can be noted that the symbols of elastic constants (c) are replaced conveniently by the symbols K 's for the case of plane stress condition so that they can be identified easily, where

$$\left. \begin{aligned} K_{11} &= \frac{E_x}{1 - \mu_{xy}\mu_{yx}}; & K_{12} &= \frac{\mu_{xy}E_y}{1 - \mu_{xy}\mu_{yx}} = \frac{\mu_{yx}E_x}{1 - \mu_{xy}\mu_{yx}} \\ K_{22} &= \frac{E_y}{1 - \mu_{xy}\mu_{yx}}; & K_{66} &= G_{xy} \end{aligned} \right\} \quad (2.15)$$

From the elastic constant K_{12} of Eq. (2.15) the reciprocal relations can be reduced as:

$$\frac{\mu_{xy}}{E_x} = \frac{\mu_{yx}}{E_y} \quad (2.16)$$

Making use of Eqs. (2.13), (2.14), (2.15) and (2.16), the differential equation for compatibility condition in terms of stresses can be as follows:

$$\left(\frac{1}{E_x} \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{1}{E_y} \frac{\partial^2 \sigma_{yy}}{\partial x^2} \right) - \frac{\mu_{xy}}{E_x} \left(\frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} \right) = \frac{1}{G_{xy}} \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (2.17)$$

Now Eq.(2.11) and (2.17) are to be solved to obtain elastic fields satisfying the boundary conditions.

2.5 Usual Method for Solution

As per existing mathematical methods, the analytical solution of three simultaneous partial differential equations given by Eq (2.11) and (2.17) is fairly impossible. However, these equations may be solved numerically. The numerical solution procedure is even complicated and cumbersome for this type of equations. Moreover, it gives only approximate results. As such it continues to remain a challenging job for the researchers to obtain the solution of elastic fields analytically for a composite structural element under mixed mode boundary conditions using traditional formulation. In this study, attention is paid to the theoretical enhancement of suitable and reliable formulation for the solution for elastic fields of orthotropic as well as isotropic composite beams under mixed mode of boundary conditions. The principal viewpoint in this regard is summarized in subsequent paragraphs.

There are mathematical concepts on the reduction in number of unknowns by assuming intermediate functions, which in turn reduces the number of equations for solution. It is noticed that the number of partial differential equations (Eq. 2.11 and 2.17) and the unknown terms can be reduced to two when the stress components of these equations are replaced by displacement components. Using equations (2.12), (2.14) and (2.15) it is possible to get three expressions for three stresses in terms of two displacement components as follows:

$$\sigma_{xx} = \frac{E_x}{1 - \mu_{xy}\mu_{yx}} \left[\frac{\partial u_x}{\partial x} + \mu_{yx} \frac{\partial u_y}{\partial y} \right] \quad (2.18a)$$

$$\sigma_{yy} = \frac{E_y}{1 - \mu_{xy}\mu_{yx}} \left[\frac{\partial u_y}{\partial y} + \mu_{yx} \frac{\partial u_x}{\partial x} \right] \quad (2.18b)$$

$$\sigma_{xy} = G_{xy} \left[\frac{\partial u_x}{\partial y} + \mu_{yx} \frac{\partial u_y}{\partial x} \right] \quad (2.18c)$$

Substituting Eq. (2.16) and (2.18) in to equilibrium equations (2.11) the following two elliptical partial differential equations are obtained.

$$\left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \frac{\partial^2 u_y}{\partial x \partial y} + G_{xy} \frac{\partial^2 u_x}{\partial y^2} = 0 \quad (2.19a)$$

$$\left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \frac{\partial^2 u_y}{\partial y^2} + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \frac{\partial^2 u_x}{\partial x \partial y} + G_{xy} \frac{\partial^2 u_y}{\partial x^2} = 0 \quad (2.19b)$$

2.6 Stress Function Approach

It may be noted that in the analytical approach stress function is being used for long time, since it was introduced by George Biddell Airy, a British astrologer and mathematician, in 1862. Airy's stress function $\Phi(x,y)$ is defined in terms of stresses as a function of x and y for which following conditions are met [1-2]:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (2.20)$$

Stress function satisfies the equilibrium equations and compatibility conditions. After applying the above relations of stresses in terms of $\Phi(x,y)$ in Eq. (2.17) following expression is obtained.

$$\left\{ \frac{1}{E_x} \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 \phi}{\partial y^2} \right) + \frac{1}{E_y} \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \phi}{\partial x^2} \right) \right\} - \frac{\mu_{xy}}{E_x} \left\{ \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 \phi}{\partial x^2} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \phi}{\partial y^2} \right) \right\} = \frac{1}{G_{xy}} \frac{\partial^2}{\partial x \partial y} \left(- \frac{\partial^2 \phi}{\partial x \partial y} \right)$$

or, $\frac{1}{E_y} \frac{\partial^4 \phi}{\partial x^4} + \left(\frac{1}{G_{xy}} - \frac{2\mu_{xy}}{E_x} \right) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{1}{E_x} \frac{\partial^4 \phi}{\partial y^4} = 0$ (2.21)

This expression is the bi-harmonic partial differential equation for Airy's stress function. Now Eq. (2.21) is to be solved satisfying the boundary conditions to obtain stress components using Eq. (2.20) and then Hooke's law as well as strain displacement relations are used to obtain the displacement components.

For orthotropic case [Fig. 2.3] the young's moduli E_x and E_y may be replaced by E_1 and E_2 where they are used to denote the Young's modulus in the fibre direction and in perpendicular to the fibre direction respectively. Further G_{xy} is replaced by G_{12} to denote the shear modulus for on-axis orientation.

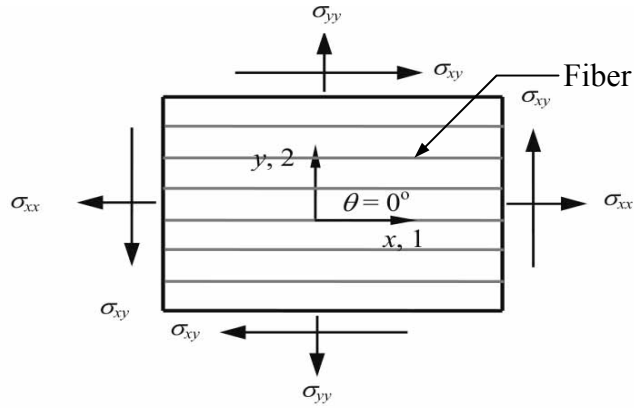


Fig. 2.3 Stress components on a composite plane

In such case the bi-harmonic governing differential equation orthotropic composites on the basis of stress function $\Phi(x,y)$ i.e. Eq. (2.21) becomes:

$$\frac{1}{E_2} \frac{\partial^4 \phi}{\partial x^4} + \left(\frac{1}{G_{12}} - \frac{2\mu_{12}}{E_1} \right) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{1}{E_1} \frac{\partial^4 \phi}{\partial y^4} = 0$$
 (2.22)

For an isotropic elastic materials under the condition of plane stress $E_x = E_y = E$, $\mu_{xy} = \mu_{yx} = \mu$ and $G_{xy} = G = \frac{E}{2(1 + \mu)}$. Substituting these relations in Eq. (2.21) the mathematical model for the isotropic condition is found as:

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (2.23)$$

Eqs. (2.22) and (2.23) developed basing on Airy's stress function can handle elastic problems of orthotropic and isotropic materials, whose boundary conditions are in terms of stress or load only. Thus attention is obviously necessary towards the solution of the two elliptical partial differential equations for those problems where the boundary restrains are to be satisfied. Again it is not even easy task to obtain the values of displacement components by the solution of equations (2.19). Consequently, further simplification of the solution method is the necessity.

2.7 Displacement Potential Formulation

Considering the difficulty of solving equations (2.19), a single function $\psi(x, y)$ is taken into consideration [30-33], which has to satisfy a single partial differential equation of equilibrium, somewhat similar in concept to that of Eq.(2.21). It is named as displacement potential function and defined as a function $\psi(x, y)$ of space variables x and y , where the displacement components are expressed as follows:

$$u_x = \alpha_1 \frac{\partial^2 \psi}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_3 \frac{\partial^2 \psi}{\partial y^2} \quad (2.24a)$$

$$u_y = \alpha_4 \frac{\partial^2 \psi}{\partial x^2} + \alpha_5 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_6 \frac{\partial^2 \psi}{\partial y^2} \quad (2.24b)$$

Here, α 's are unknown material constants.

Using the expressions of Eq. (2.24) in Eq (2.19a) following equation is obtained.

$$\left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \frac{\partial^2}{\partial x^2} \left(\alpha_1 \frac{\partial^2 \psi}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_3 \frac{\partial^2 \psi}{\partial y^2} \right) + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \frac{\partial^2}{\partial x \partial y} \left(\alpha_4 \frac{\partial^2 \psi}{\partial x^2} + \alpha_5 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_6 \frac{\partial^2 \psi}{\partial y^2} \right) + G_{xy} \frac{\partial^2}{\partial y^2} \left(\alpha_1 \frac{\partial^2 \psi}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_3 \frac{\partial^2 \psi}{\partial y^2} \right) = 0$$

or,

$$\left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \left(\alpha_1 \frac{\partial^4 \psi}{\partial x^4} + \alpha_2 \frac{\partial^4 \psi}{\partial x^3 \partial y} + \alpha_3 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} \right) + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \left(\alpha_4 \frac{\partial^4 \psi}{\partial x^3 \partial y} + \alpha_5 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \alpha_6 \frac{\partial^4 \psi}{\partial x \partial y^3} \right) + G_{xy} \left(\alpha_1 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \alpha_2 \frac{\partial^4 \psi}{\partial x \partial y^3} + \alpha_3 \frac{\partial^4 \psi}{\partial y^4} \right) = 0$$

or,

$$\begin{aligned} & \left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \alpha_1 \frac{\partial^4 \psi}{\partial x^4} + \left\{ \left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \alpha_2 + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_4 \right\} \frac{\partial^4 \psi}{\partial x^3 \partial y} + \\ & \left\{ \left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \alpha_3 + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_5 + G_{xy} \alpha_1 \right\} \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \\ & \left\{ \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_6 + G_{xy} \alpha_2 \right\} \frac{\partial^4 \psi}{\partial x \partial y^3} + G_{xy} \alpha_3 \frac{\partial^4 \psi}{\partial y^4} = 0 \end{aligned} \quad (2.25)$$

The material constants α 's are chosen in such a way that the equation (2.25) is automatically satisfied under all circumstances. This will happen when all of its coefficients are independently zero. In that situation,

$$\begin{aligned} & \left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \alpha_1 = 0 \\ & \left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \alpha_2 + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_4 = 0 \\ & \left(\frac{E_x^2}{E_x - \mu_{xy}^2 E_y} \right) \alpha_3 + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_5 + G_{xy} \alpha_1 = 0 \\ & \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_6 + G_{xy} \alpha_2 = 0 \\ & G_{xy} \alpha_3 = 0 \end{aligned}$$

Therefore,

$$\left. \begin{aligned} \alpha_1 &= \alpha_3 = \alpha_5 = 0 \\ \alpha_4 &= \frac{-E_x^2 \alpha_2}{\{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\}} \\ \alpha_6 &= \frac{-G_{xy} (E_x - \mu_{xy}^2 E_y) \alpha_2}{\{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\}} \end{aligned} \right\} \quad (2.26)$$

Again from the second equation of (2.19) and Eq (2.24) it is found that

$$\begin{aligned} & \left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \frac{\partial^2}{\partial y^2} \left(\alpha_4 \frac{\partial^2 \psi}{\partial x^2} + \alpha_5 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_6 \frac{\partial^2 \psi}{\partial y^2} \right) + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \\ & \frac{\partial^2}{\partial x \partial y} \left(\alpha_1 \frac{\partial^2 \psi}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_3 \frac{\partial^2 \psi}{\partial y^2} \right) + G_{xy} \frac{\partial^2}{\partial x^2} \left(\alpha_4 \frac{\partial^2 \psi}{\partial x^2} + \alpha_5 \frac{\partial^2 \psi}{\partial x \partial y} + \alpha_6 \frac{\partial^2 \psi}{\partial y^2} \right) = 0 \\ \text{or, } & \left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \left(\alpha_4 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \alpha_5 \frac{\partial^4 \psi}{\partial x \partial y^3} + \alpha_6 \frac{\partial^4 \psi}{\partial y^4} \right) + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \\ & \left(\alpha_1 \frac{\partial^4 \psi}{\partial x^3 \partial y} + \alpha_2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \alpha_3 \frac{\partial^4 \psi}{\partial x \partial y^3} \right) + G_{xy} \left(\alpha_4 \frac{\partial^4 \psi}{\partial x^4} + \alpha_5 \frac{\partial^4 \psi}{\partial x^3 \partial y} + \alpha_6 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} \right) = 0 \\ \text{or, } & G_{xy} \alpha_4 \frac{\partial^4 \psi}{\partial x^4} + \left\{ \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_1 + G_{xy} \alpha_5 \right\} \frac{\partial^4 \psi}{\partial x^3 \partial y} + \left\{ \left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \alpha_4 + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_2 + G_{xy} \alpha_6 \right\} \\ & \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \left\{ \left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \alpha_5 + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_3 \right\} \frac{\partial^4 \psi}{\partial x \partial y^3} + \left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \alpha_6 \frac{\partial^4 \psi}{\partial y^4} = 0 \end{aligned} \quad (2.27)$$

Now using Eqs (2.26) and (2.27) it is found that

$$\begin{aligned} & G_{xy} \alpha_4 \frac{\partial^4 \psi}{\partial x^4} + \left\{ \left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \alpha_4 + \left(\frac{\mu_{xy} E_x E_y}{E_x - \mu_{xy}^2 E_y} + G_{xy} \right) \alpha_2 + G_{xy} \alpha_6 \right\} \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \left(\frac{E_x E_y}{E_x - \mu_{xy}^2 E_y} \right) \alpha_6 \frac{\partial^4 \psi}{\partial y^4} = 0 \\ \text{or, } & G_{xy} \alpha_4 (E_x - \mu_{xy}^2 E_y) \frac{\partial^4 \psi}{\partial x^4} + [E_x E_y \alpha_4 + \{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\} \alpha_2 + G_{xy} (E_x - \mu_{xy}^2 E_y) \alpha_6] \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_x E_y \alpha_6 \frac{\partial^4 \psi}{\partial y^4} = 0 \\ \text{or, } & G_{xy} \left[\frac{-E_x^2 \alpha_2}{\{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\}} \right] (E_x - \mu_{xy}^2 E_y) \frac{\partial^4 \psi}{\partial x^4} + \\ & \left[E_x E_y \left\{ \frac{-E_x^2 \alpha_2}{\{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\}} \right\} + \{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\} \alpha_2 + G_{xy} (E_x - \mu_{xy}^2 E_y) \left\{ \frac{-G_{xy} (E_x - \mu_{xy}^2 E_y) \alpha_2}{\{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\}} \right\} \right] \\ & \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_x E_y \left\{ \frac{-G_{xy} (E_x - \mu_{xy}^2 E_y) \alpha_2}{\{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)\}} \right\} \frac{\partial^4 \psi}{\partial y^4} = 0 \end{aligned}$$

For non-zero values of α_2 ,

$$-E_x^2 G_{xy} (E_x - \mu_{xy}^2 E_y) \frac{\partial^4 \psi}{\partial x^4} + \left[-E_x^3 E_y + \left\{ \mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y) \right\}^2 - G_{xy}^2 (E_x - \mu_{xy}^2 E_y)^2 \right] \frac{\partial^4 \psi}{\partial x^2 \partial y^2} - E_x E_y G_{xy} (E_x - \mu_{xy}^2 E_y) \frac{\partial^4 \psi}{\partial y^4} = 0$$

or,

$$-E_x^2 G_{xy} (E_x - \mu_{xy}^2 E_y) \frac{\partial^4 \psi}{\partial x^4} + \left[-E_x^2 E_x E_y + \left\{ \mu_{xy} E_x E_y + 2G_{xy} (E_x - \mu_{xy}^2 E_y) \right\} \mu_{xy} E_x E_y \right] \frac{\partial^4 \psi}{\partial x^2 \partial y^2} - E_x E_y G_{xy} (E_x - \mu_{xy}^2 E_y) \frac{\partial^4 \psi}{\partial y^4} = 0$$

or,

$$E_x G_{xy} \frac{\partial^4 \psi}{\partial x^4} + E_y (E_x - 2\mu_{xy} G_{xy}) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_y G_{xy} \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (2.28)$$

The above fourth order differential equation (2.28) is only one governing equation for the solution of the displacement potential function ψ . Once the displacement potential function ψ is known, the components of displacement can be readily found from Eq. (2.24). Thereafter, using the stress displacement relations of Eq. (2.18) can be used for obtaining stress components.

Assuming the value of α_2 unity, and taking the values of α_1 , α_3 , α_4 , α_5 and α_6 from Eq.(2.26), one can obtain the components of displacement and stress using Eqs. (2.24) and (2.18) respectively as follows:

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y} \quad (2.29a)$$

$$u_y(x, y) = \left\{ \frac{-1}{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)} \right\} \left[E_x^2 \frac{\partial^2 \psi}{\partial x^2} + G_{xy} (E_x - \mu_{xy}^2 E_y) \frac{\partial^2 \psi}{\partial y^2} \right] \quad (2.29b)$$

$$\sigma_{xx}(x, y) = \left\{ \frac{-E_x G_{xy}}{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)} \right\} \left[E_x \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{xy} E_y \frac{\partial^2 \psi}{\partial y^2} \right] \quad (2.29c)$$

$$\sigma_{yy}(x, y) = \left\{ \frac{-E_x E_y}{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)} \right\} \left[(\mu_{xy} G_{xy} - E_x) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{xy} \frac{\partial^3 \psi}{\partial y^3} \right] \quad (2.29d)$$

$$\sigma_{xy}(x, y) = \left\{ \frac{-E_x G_{xy}}{\mu_{xy} E_x E_y + G_{xy} (E_x - \mu_{xy}^2 E_y)} \right\} \left[E_x \frac{\partial^3 \psi}{\partial x^3} - \mu_{xy} E_y \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \quad (2.29e)$$

2.7.1 Displacement Potential Formulation for Orthotropic Materials

For orthotropic materials the young's moduli E_x and E_y may be replaced by E_1 and E_2 where they are used to denote the Young's modulus in fibre direction and in perpendicular to fiber direction respectively. Further G_{xy} is replaced by G_{12} to denote the shear modulus for on-axis orientation. In such case the governing differential equation for the solution of two dimensional orthotropic composite structures becomes:

$$E_1 G_{12} \frac{\partial^4 \psi}{\partial x^4} + E_2 (E_1 - 2\mu_{12} G_{12}) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_2 G_{12} \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (2.30)$$

Then the components of displacement and stress are:

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y} \quad (2.31a)$$

$$u_y(x, y) = -\frac{1}{Z_{11}} \left[E_1^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12} (E_1 - \mu_{12}^2 E_2) \frac{\partial^2 \psi}{\partial y^2} \right] \quad (2.31b)$$

$$\sigma_{xx}(x, y) = -\frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{12} E_2 \frac{\partial^2 \psi}{\partial y^2} \right] \quad (2.31c)$$

$$\sigma_{yy}(x, y) = -\frac{E_1 E_2}{Z_{11}} \left[(\mu_{12} G_{12} - E_1) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12} \frac{\partial^3 \psi}{\partial y^3} \right] \quad (2.31d)$$

$$\sigma_{xy}(x, y) = -\frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^3} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \quad (2.31e)$$

where $Z_{11} = \mu_{12} E_1 E_2 + G_{12} (E_1 - \mu_{12}^2 E_2)$

2.7.2 Displacement Potential Formulation for Isotropic Materials

For an isotropic elastic solid under the condition of plane stress $E_x = E_y = E$, $\mu_{xy} = \mu_{yx} = \mu$

and $G_{xy} = G = \frac{E}{2(1+\mu)}$. Then the values of α 's of Eq. (2.24) are also obtained as follows:

$$\alpha_1 = \alpha_3 = \alpha_5 = 0; \quad \alpha_2 = 1; \quad \alpha_4 = \frac{-2}{1+\mu}; \quad \alpha_6 = \frac{1-\mu}{1+\mu} \quad (2.32)$$

Then the stress-displacement relations for the plane stress problems are obtained from the Hook's law as follows:

$$\sigma_{xx} = \frac{E}{1-\mu^2} \left[\frac{\partial u_x}{\partial x} + \mu \frac{\partial u_y}{\partial y} \right] \quad (2.33a)$$

$$\sigma_{yy} = \frac{E}{1-\mu^2} \left[\mu \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right] \quad (2.33b)$$

$$\sigma_{xy} = \frac{E}{2(1+\mu)} \left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right] \quad (2.33c)$$

When the displacement components in Eq. (2.19) are replaced by Eq. (2.24) having values of α as in Eq. (2.32), the single governing equation of equilibrium in partial differential form being satisfied by $\psi(x, y)$ is found for isotropic materials as follows:

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (2.34)$$

Combining Eqs. (2.18), (2.33), and (2.34), the expressions of displacement and stress components in terms of function $\psi(x, y)$ are obtained as follows:

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y} \quad (2.35a)$$

$$u_y(x, y) = -\frac{1}{1+\mu} \left[2 \frac{\partial^2 \psi}{\partial x^2} + (1-\mu) \frac{\partial^2 \psi}{\partial y^2} \right] \quad (2.35b)$$

$$\sigma_{xx}(x, y) = -\frac{E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu \frac{\partial^2 \psi}{\partial y^2} \right] \quad (2.35c)$$

$$\sigma_{yy}(x, y) = -\frac{E}{(1+\mu)^2} \left[(2+\mu) \frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3} \right] \quad (2.35d)$$

$$\sigma_{xy}(x, y) = -\frac{E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial x^3} - \mu \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \quad (2.35e)$$

2.8 Consideration of Boundary Conditions

The equilibrium equation (2.1) has to be satisfied at all points throughout the volume of the solid elastic body. The components of stress may vary over the volume of the body, and at the surface the stresses must be such as to be in equilibrium with external forces acting on the boundary of the body. As such the external forces would have a contribution over the internal stress distribution.

In practical situation, along the edge or boundary of a structure, there are two things to be known, i.e. (i) displacements and (ii) loading or stress. Both the displacements and stresses are identified by their respective components as follows:

- a. Normal displacement
- b. Tangential displacement
- c. Normal stress
- d. Tangential stress

The solution of the governing equation requires specific normal and tangential conditions. At any point on the boundary, any 2 components out of 4 are known at a time. Thus there are 6 types of following boundary conditions:

- i. Normal displacement + Tangential displacement
- ii. Normal displacement + Normal stress
- iii. Normal displacement + Tangential stress
- iv. Tangential displacement + Normal stress
- v. Tangential displacement + Tangential stress
- vi. Normal stress + Tangential stress

While both the components are purely normal or purely tangential, the boundary conditions do not practically exist. As such boundary conditions of (ii) and (v) are no longer required to be considered and the remaining four boundary conditions would be considered for solving the physical problems of elastic body. If the shape of the boundary surface is rectangular, the structure may be oriented so that its edges are parallel to the co-ordinate axes. In that case, the normal and the tangential components of displacement and stress at the boundary are the corresponding coordinate components inside the structure. Out of the above mentioned four possible boundaries, only the number (vi) is suitable for Airy's stress function; whereas all four boundary conditions can be dealt with using displacement potential function (ψ).

2.9 Solution Procedure Using Displacement Potential Approach

The concept of structural analysis consists of four essential matters like any engineering system, such as, proper understanding of physical phenomena, derivation of governing equation, proper application of boundary conditions, and development of routines for the solution and finally the interpretation of solutions. Thus the solution procedure is the uniting of physics and mathematics with a view to potential usefulness in practical problems.

The equilibrium problem is essentially one of describing the steady-state configuration of the physical system. This can usually be achieved by specifying the magnitudes of state variables like stresses, displacements, pressures, etc. at a finite number of points. In this thesis work, boundary-value problems are dealt with having equilibrium state of affairs. With this pretext a very powerful method of solving boundary-value problems is the so-called trial function or trial solution method.

Attempt is made here to solve the fourth order homogeneous partial differential equations, i.e., Eq.(2.30) for orthotropic and Eq. (2.34) for isotropic materials through utilization of different trial functions for $\psi(x, y)$. Since the Airy's stress function of similar pattern has been solved using polynomials for quite long time, similar type of functions are considered here at first as trial solutions. It is observed that pure polynomials do not actually help much in this regard. Rather it is seen that various combinations of trigonometric and hyperbolic functions offer suitable choices for analytical functions. If these functions can be expressed as an infinite series, then construction of solutions of differential equations becomes more accurate. In the light of the ubiquitous problems which display aspects of periodic and a discontinuous nature, those infinite series known as Fourier series attain a place of special importance.

The Fourier series is probably the most commonly used of all the series for the solution of physical problems. It is a trigonometric series which can be used for the expansion of an arbitrary function. The usefulness of the Fourier series is due to the fact that certain functions which can not be expanded in power series form can still be represented by Fourier series. The reason for this is that the coefficients of the power series contain derivatives of the function; hence these derivatives must exist uniquely in order to obtain the power series expansion. Many functions which are not differentiable, including certain

types of discontinues function, can be expanded in Fourier series. Thus a much greater degree of generality is attained by taking the function as Fourier series.

Taking all this in mind trial and error operations are done to reach to the possible best displacement potential function to be assumed. In this assumption process, boundary conditions of the two ends should be satisfied automatically. Then the solution can be progressed further to make the boundary conditions of remaining two ends of the beam satisfied.

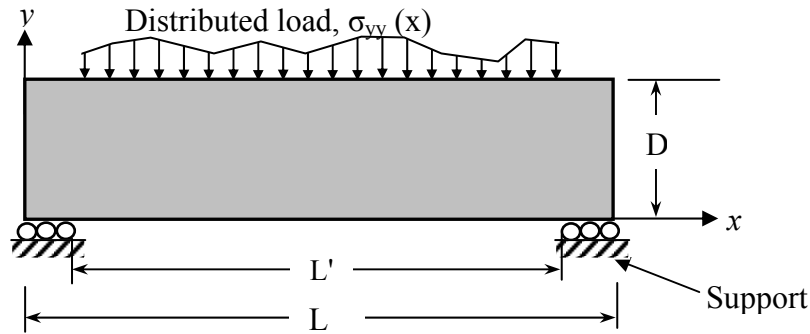
CHAPTER 3

SOLUTION TO ISOTROPIC DEEP BEAMS

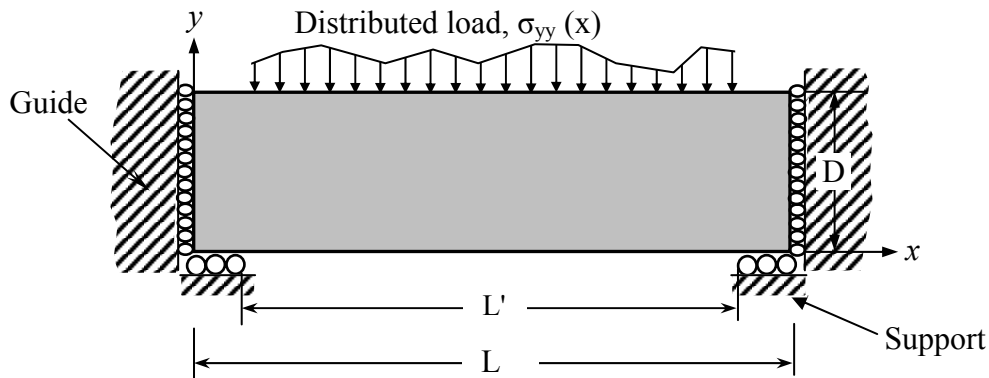
3.1 Analytical Solution of Guided Isotropic Deep Beam

3.1.1 Problem Articulation

A simply supported deep beam of rectangular cross section subjected to a distributed load is considered. The generalised form of such a beam is shown in Fig. 3.1a. While the two opposing lateral ends of this beam are guided by some means, it can be considered as guided simply supported beam. It is illustrated in Fig. 3.1b. The support at the bottom surface is inserted over a certain portion of beam length. Beam length, depth and width are denoted by L , D and W respectively. The loaded length can be taken as L' and the effective length for each support is $(L - L')/2$.



(a) Unguided simply supported beam



(b) Guided simply supported beam

Fig. 3.1 Geometry and loading of a simply supported deep beam: (a) Unguided simply supported beam, (b) Guided simply supported beam

As a particular enunciation for the development of analytical solution, the load acting over the top surface is considered as uniformly distributed with a magnitude of σ_0 acting over the length of $0.8L$. The support of the beam is also considered as uniformly distributed and the effective length for each support is assumed to be $0.1L$. The plane stress is assumed here taking unit thickness of the beam. Since the two opposing lateral ends of the beam are guided, there is no allowance for the axial displacements, but the lateral displacements are free to have any value [Fig. 3.2].

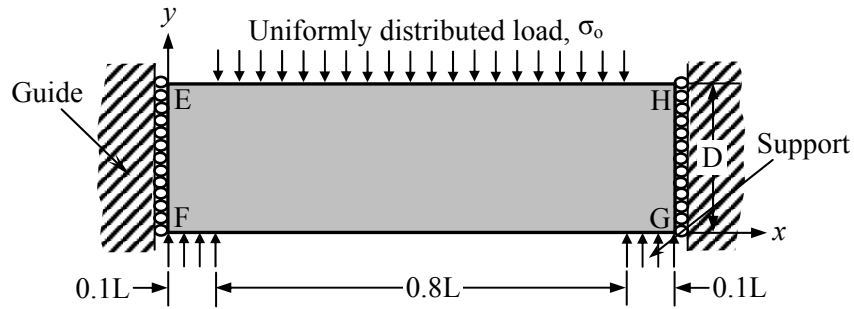


Fig. 3.2 Analytical model of a guided simply supported beam subjected to uniform loading

3.1.2 Boundary Conditions

For the present problem with reference to Fig. 3.2 the physical conditions is to be satisfied along the different boundaries of the beam. Due to the guides placed at both the ends, axial displacement is restricted to be zero over there. At the same time there can be no shear stress at those edges since the guide allows the beam to be free for lateral dislocation. So the conditions at the guided boundaries are of mixed mode. The top and bottom boundaries do not have conditions of displacements. On these surfaces the applied load and reactions are related to the boundary conditions within their striking region. Again there is no shearing effect over the top and bottom boundaries of the aforesaid beam. The boundary conditions of guided beam considered for the present problem can be expressed mathematically as follows:

- (a) Guided end, EF: There is no axial displacement and shearing stress. Thus,

$$u_x(0, y) = 0 \text{ and } \sigma_{xy}(0, y) = 0 \quad [0 \leq y \leq D].$$

- (b) Guided end, HG: There is no axial displacement and shearing stress. Thus,

$$u_x(L, y) = 0 \text{ and } \sigma_{xy}(L, y) = 0 \quad [0 \leq y \leq D]$$

- (c) Loaded boundary, EH: There is no shearing stress. But lateral stresses are there in the regions of applied loads, which is the function of load intensity. Thus, $\sigma_{xy}(x, D) = 0$ $[0 \leq x \leq L]$ and $\sigma_{yy}(x, D) = \sigma_0$ $[0.1L \leq x \leq 0.9L]$
- (d) Supporting end, FG: This is the supporting boundary whose total reaction force is to be the algebraic sum of reactions acting on both the supports and eventually same as to the applied force at the opposite site for equilibrium condition. The uniform load is distributed over 80% of beam length (from $x/L = 0.1$ to $x/L = 0.9$) and support reaction is distributed over 20% of beam length ($x/L = 0.0$ to 0.1 and $x/L = 0.9$ to 1.0). Therefore, $\sigma_{xy}(x, 0) = 0$ $[0 \leq x \leq L]$; $\sigma_{yy}(x, 0) = 4\sigma_0$ $[0 \leq x \leq 0.1L \text{ \& } 0.9L \leq x \leq L]$ and $\sigma_{yy}(x, 0) = 0$ $[0.1L < x < 0.9L]$

3.1.3 Analytical Solution

For isotropic material in case plane stress the governing equation based on displacement potential function $\psi(x, y)$ is obtained from Eq. (2.34) as follows:

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (3.1)$$

The expressions of displacement and stress components in terms of function $\psi(x, y)$ are also obtained from Eqs. (2.35) as follows:

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y} \quad (3.2a)$$

$$u_y(x, y) = -\frac{1}{1+\mu} \left[2 \frac{\partial^2 \psi}{\partial x^2} + (1-\mu) \frac{\partial^2 \psi}{\partial y^2} \right] \quad (3.2b)$$

$$\sigma_{xx}(x, y) = -\frac{E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu \frac{\partial^2 \psi}{\partial y^2} \right] \quad (3.2c)$$

$$\sigma_{yy}(x, y) = -\frac{E}{(1+\mu)^2} \left[(2+\mu) \frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3} \right] \quad (3.2d)$$

$$\sigma_{xy}(x, y) = -\frac{E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial x^3} - \mu \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \quad (3.2e)$$

The potential function $\psi(x, y)$ is first assumed in a way so that the physical conditions of the two opposing guided ends are automatically satisfied. At the same time solution has to

satisfy the 4th order bi-harmonic partial differential governing equation. After a long trial and error process, the solution of the governing equation (3.1) is thus approximated as follows, which gives the result:

$$\psi(x, y) = \sum_{m=1}^{\infty} Y_m(y) \cos \alpha x + K y^3 \quad (3.3)$$

where, $Y_m = f(y)$, $\alpha = (m\pi/L)$, K is an arbitrary constant and $m = 1, 2, 3, \dots, \infty$.

Derivatives of equation (3.3) with respect to x and y are

$$\frac{\partial \psi}{\partial x} = -\sum_{m=1}^{\infty} Y_m \alpha \sin \alpha x$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\sum_{m=1}^{\infty} Y_m \alpha^2 \cos \alpha x$$

$$\frac{\partial^3 \psi}{\partial x^3} = \sum_{m=1}^{\infty} Y_m \alpha^3 \sin \alpha x$$

$$\frac{\partial^4 \psi}{\partial x^4} = \sum_{m=1}^{\infty} Y_m \alpha^4 \cos \alpha x$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = -\sum_{m=1}^{\infty} Y'_m \alpha \sin \alpha x$$

$$\frac{\partial^3 \psi}{\partial x \partial y^2} = -\sum_{m=1}^{\infty} Y''_m \alpha \sin \alpha x$$

$$\frac{\partial^3 \psi}{\partial x^2 \partial y} = -\sum_{m=1}^{\infty} Y'_m \alpha^2 \cos \alpha x$$

$$\frac{\partial^4 \psi}{\partial x^2 \partial y^2} = -\sum_{m=1}^{\infty} Y''_m \alpha^2 \cos \alpha x$$

$$\frac{\partial \psi}{\partial y} = \sum_{m=1}^{\infty} Y'_m \cos \alpha x + 3Ky^2$$

$$\frac{\partial^2 \psi}{\partial y^2} = \sum_{m=1}^{\infty} Y''_m \cos \alpha x + 6Ky$$

$$\frac{\partial^3 \psi}{\partial y^3} = \sum_{m=1}^{\infty} Y_m''' \cos \alpha x + 6K$$

$$\frac{\partial^4 \psi}{\partial y^4} = \sum_{m=1}^{\infty} Y_m'''' \cos \alpha x$$

Substituting the expressions of above derivatives in Eq. (3.1) following equation is obtained.

$$\sum_{m=1}^{\infty} Y_m \alpha^4 \cos \alpha x - 2 \sum_{m=1}^{\infty} Y_m'' \alpha^2 \cos \alpha x + \sum_{m=1}^{\infty} Y_m'''' \cos \alpha x = 0$$

$$\text{or, } \sum_{m=1}^{\infty} [Y_m'''' - 2\alpha^2 Y_m'' + \alpha^4 Y_m] \cos \alpha x = 0$$

$$\text{or, } Y_m'''' - 2\alpha^2 Y_m'' + \alpha^4 Y_m = 0 \quad (3.4)$$

The solution of the above 4th order ordinary differential equation with constant coefficients [Eq. (3.4)] can normally be approximated as follows:

$$Y_m = A_m e^{r_1 y} + B_m y e^{r_2 y} + C_m e^{r_3 y} + D_m y e^{r_4 y} \quad (3.5)$$

But the ordinary differential equation (3.4) has the complementary function of repeated roots. Thus $r_1 = r_2 = \alpha$ and $r_3 = r_4 = -\alpha$ and the general solution of Eq. (3.4) can be written as

$$Y_m = (A_m + B_m y) e^{\alpha y} + (C_m + D_m y) e^{-\alpha y} \quad (3.6)$$

where A_m , B_m , C_m and D_m are arbitrary constants.

Differentiating equation (3.6) following expressions are found

$$Y_m' = (A_m \alpha + B_m \alpha y + B_m) e^{\alpha y} + (-C_m \alpha - D_m \alpha y + D_m) e^{-\alpha y}$$

$$Y_m'' = (A_m \alpha^2 + B_m \alpha^2 y + 2B_m \alpha) e^{\alpha y} + (C_m \alpha^2 + D_m \alpha^2 y - 2D_m \alpha) e^{-\alpha y}$$

$$Y_m''' = (A_m \alpha^3 + B_m \alpha^3 y + 3B_m \alpha^2) e^{\alpha y} + (-C_m \alpha^3 - D_m \alpha^3 y + 3D_m \alpha^2) e^{-\alpha y}$$

$$Y_m'''' = (A_m \alpha^4 + B_m \alpha^4 y + 4B_m \alpha^3) e^{\alpha y} + (C_m \alpha^4 + D_m \alpha^4 y - 4D_m \alpha^3) e^{-\alpha y}$$

Now substituting the derivatives of ψ and Y_m in the expressions for displacement (3.2) and stresses (3.3), following expressions are found.

$$\begin{aligned}
u_x(x, y) &= \frac{\partial^2 \psi}{\partial x \partial y} \\
&= -\sum_{m=1}^{\infty} Y'_m \alpha \sin \alpha x \\
&= -\sum_{m=1}^{\infty} \left[(A_m \alpha + B_m \alpha y + B_m) e^{\alpha y} + (-C_m \alpha - D_m \alpha y + D_m) e^{-\alpha y} \right] \alpha \sin \alpha x \\
&= -\sum_{m=1}^{\infty} \left[A_m \alpha e^{\alpha y} + B_m (\alpha y + 1) e^{\alpha y} - C_m \alpha e^{-\alpha y} - D_m (\alpha y - 1) e^{-\alpha y} \right] \alpha \sin \alpha x \quad (3.7a)
\end{aligned}$$

$$\begin{aligned}
u_y(x, y) &= -\frac{1}{(1+\mu)} \left[2 \frac{\partial^2 \psi}{\partial x^2} + (1-\mu) \frac{\partial^2 \psi}{\partial y^2} \right] \\
&= -\frac{1}{(1+\mu)} \left[2 \left\{ -\sum_{m=1}^{\infty} Y_m \alpha^2 \cos \alpha x \right\} + (1-\mu) \left\{ \sum_{m=1}^{\infty} Y''_m \cos \alpha x + 6Ky \right\} \right] \\
&= -\frac{1}{(1+\mu)} \left[-2 \sum_{m=1}^{\infty} \left\{ (A_m + B_m y) e^{\alpha y} + (C_m + D_m y) e^{-\alpha y} \right\} \alpha^2 \cos \alpha x + (1-\mu) \right. \\
&\quad \left. \sum_{m=1}^{\infty} \left\{ (A_m \alpha^2 + B_m \alpha^2 y + 2B_m \alpha) e^{\alpha y} + (C_m \alpha^2 + D_m \alpha^2 y - 2D_m \alpha) e^{-\alpha y} \right\} \cos \alpha x + 6K(1-\mu)y \right] \\
&= \frac{-1}{(1+\mu)} \left[\sum_{m=1}^{\infty} \left\{ \begin{aligned} &-A_m(1+\mu)\alpha^2 e^{\alpha y} + \\ &B_m(-\alpha y - \mu\alpha y - 2\mu + 2)\alpha e^{\alpha y} \\ &-C_m(1+\mu)\alpha^2 e^{-\alpha y} + \\ &D_m(-\alpha y - \mu\alpha y + 2\mu - 2)\alpha e^{-\alpha y} \end{aligned} \right\} \cos \alpha x + 6K(1-\mu)y \right] \quad (3.7b)
\end{aligned}$$

$$\begin{aligned}
\sigma_{xx}(x, y) &= \frac{E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu \frac{\partial^3 \psi}{\partial y^3} \right] \\
&= \frac{E}{(1+\mu)^2} \left[\left\{ -\sum_{m=1}^{\infty} Y'_m \alpha^2 \cos \alpha x \right\} - \mu \left\{ \sum_{m=1}^{\infty} Y'''_m \cos \alpha x + 6K \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} \left\{ (A_m \alpha + B_m \alpha y + B_m) e^{\alpha y} + (-C_m \alpha - D_m \alpha y + D_m) e^{-\alpha y} \right\} \alpha^2 \cos \alpha x \right. \\
&\quad \left. + \mu \sum_{m=1}^{\infty} \left\{ (A_m \alpha^3 + B_m \alpha^3 y + 3B_m \alpha^2) e^{\alpha y} + (-C_m \alpha^3 - D_m \alpha^3 y + 3D_m \alpha^2) e^{-\alpha y} \right\} \cos \alpha x + 6\mu K \right] \\
&= \frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} \left\{ A_m \alpha (1+\mu) e^{\alpha y} + B_m (\alpha y + \mu \alpha y + 3\mu + 1) e^{\alpha y} \right. \right. \\
&\quad \left. \left. - C_m \alpha (1+\mu) e^{-\alpha y} + D_m (-\alpha y - \mu \alpha y + 3\mu + 1) e^{-\alpha y} \right\} \alpha^2 \cos \alpha x + 6\mu K \right] \quad (3.7c)
\end{aligned}$$

$$\begin{aligned}
\sigma_{yy}(x, y) &= \frac{-E}{(1+\mu)^2} \left[(2+\mu) \frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3} \right] \\
&= \frac{-E}{(1+\mu)^2} \left[(2+\mu) \left\{ - \sum_{m=1}^{\infty} Y_m' \alpha^2 \cos \alpha x \right\} + \left\{ \sum_{m=1}^{\infty} Y_m''' \cos \alpha x + 6K \right\} \right] \\
&= \frac{-E}{(1+\mu)^2} \left[-(2+\mu) \sum_{m=1}^{\infty} \left\{ (A_m \alpha + B_m \alpha y + B_m) e^{\alpha y} + (-C_m \alpha - D_m \alpha y + D_m) e^{-\alpha y} \right\} \alpha^2 \cos \alpha x \right. \\
&\quad \left. + \sum_{m=1}^{\infty} \left\{ (A_m \alpha^3 + B_m \alpha^3 y + 3B_m \alpha^2) e^{\alpha y} + (-C_m \alpha^3 - D_m \alpha^3 y + 3D_m \alpha^2) e^{-\alpha y} \right\} \cos \alpha x + 6K \right] \\
&= \frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} \left\{ A_m \alpha (-1-\mu) e^{\alpha y} + B_m (-\alpha y - \mu \alpha y - \mu + 1) e^{\alpha y} + \right. \right. \\
&\quad \left. \left. C_m \alpha (1+\mu) e^{-\alpha y} + D_m (\alpha y + \mu \alpha y - \mu + 1) e^{-\alpha y} \right\} \alpha^2 \cos \alpha x + 6K \right] \quad (3.7d)
\end{aligned}$$

$$\begin{aligned}
\sigma_{xy}(x, y) &= \frac{-E}{(1+\mu)^2} \left[\frac{\partial^3 \psi}{\partial x^3} - \mu \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \\
&= \frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} Y_m \alpha^3 \sin \alpha x - \mu \left\{ - \sum_{m=1}^{\infty} Y_m'' \alpha \sin \alpha x \right\} \right] \\
&= \frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} \left\{ (A_m + B_m y) e^{\alpha y} + (C_m + D_m y) e^{-\alpha y} \right\} \alpha^3 \sin \alpha x + \right. \\
&\quad \left. \mu \sum_{m=1}^{\infty} \left\{ (A_m \alpha^2 + B_m \alpha^2 y + 2B_m \alpha) e^{\alpha y} + (C_m \alpha^2 + D_m \alpha^2 y - 2D_m \alpha) e^{-\alpha y} \right\} \alpha \sin \alpha x \right] \\
&= \frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} \left\{ A_m (1+\mu) \alpha e^{\alpha y} + B_m (\alpha y + \mu \alpha y + 2\mu) e^{\alpha y} \right. \right. \\
&\quad \left. \left. + C_m (1+\mu) \alpha e^{-\alpha y} + D_m (\alpha y + \mu \alpha y - 2\mu) e^{-\alpha y} \right\} \alpha^2 \sin \alpha x \right] \quad (3.7e)
\end{aligned}$$

Now, the reactions on the bottom boundary ($y = 0$) are acting over the two supports. It is considered that the supports are located at $x=0$ to $0.1L$ and $x=0.9L$ to L respectively. The total length for reaction is 20 percent of beam length, where the load is over the 80 percent. As a result the intensity of reaction is four times of the load intensity. Therefore, the reactions over the beam at the supports can be taken as Fourier series in the following manner:

$$\sigma_{yy}(x,0) = 4\sigma_0 = E_0 + \sum_{m=1}^{\infty} E_m \cos \alpha x \quad \text{for } x=0 \text{ to } 0.1L \text{ and } 0.9L \text{ to } L \quad (3.8)$$

Here

$$\begin{aligned} E_0 &= \frac{1}{L} \left[\int_0^{L/10} 4\sigma_0 dx + \int_{9L/10}^L 4\sigma_0 dx \right] \\ &= \frac{4\sigma_0}{L} \left[\frac{L}{10} + L - \frac{9L}{10} \right] \\ &= \frac{4\sigma_0}{5} \end{aligned} \quad (3.9a)$$

$$\begin{aligned} E_m &= \frac{2}{L} \left[\int_0^{L/10} 4\sigma_0 \cos \alpha x dx + \int_{9L/10}^L 4\sigma_0 \cos \alpha x dx \right] \\ &= \frac{8\sigma_0}{L} \left[\int_0^{L/10} \cos \alpha x dx \right] + \frac{8\sigma_0}{L} \left[\int_{9L/10}^L \cos \alpha x dx \right] \\ &= \frac{8\sigma_0}{L} \left[\frac{\sin \alpha x}{\alpha} \right]_0^{L/10} + \frac{8\sigma_0}{L} \left[\frac{\sin \alpha x}{\alpha} \right]_{9L/10}^L \\ &= \frac{8\sigma_0}{\alpha L} \left\{ \sin \left(\frac{\alpha L}{10} \right) + \sin(\alpha L) - \sin \left(\frac{9\alpha L}{10} \right) \right\} \\ &= \frac{8\sigma_0}{m\pi} \left\{ \sin \left(\frac{m\pi}{10} \right) + \sin(m\pi) - \sin \left(\frac{9m\pi}{10} \right) \right\}; \quad m = 1, 2, 3, \dots, \infty \end{aligned} \quad (3.9b)$$

The compressive load on the edge $y = D$ acting over $x=0.1L$ to $0.9L$ can also be given by a Fourier series as follows

$$\sigma_{yy}(x,0) = \sigma_0 = I_0 + \sum_{m=1}^{\infty} I_m \cos \alpha x \quad \text{for } x=0.1L \text{ to } 0.9L \quad (3.10)$$

Here

$$I_0 = \frac{1}{L} \left[\int_{L/10}^{9L/10} \sigma_0 dx \right]$$

$$\begin{aligned}
&= \frac{\sigma_0}{L} \left[\frac{9L}{10} - \frac{L}{10} \right] \\
&= \frac{4\sigma_0}{5}
\end{aligned} \tag{3.11a}$$

$$\begin{aligned}
I_m &= \frac{2}{L} \left[\int_{L/10}^{9L/10} \sigma_0 \cos \alpha x dx \right] \\
&= \frac{2\sigma_0}{L} \left[\frac{\sin \alpha x}{\alpha} \right]_{L/10}^{9L/10} \\
&= \frac{2\sigma_0}{\alpha L} \left\{ \sin \left(\frac{9\alpha L}{10} \right) - \sin \left(\frac{\alpha L}{10} \right) \right\} \\
&= \frac{2\sigma_0}{m\pi} \left\{ \sin \left(\frac{9m\pi}{10} \right) - \sin \left(\frac{m\pi}{10} \right) \right\}; \quad m = 1, 2, 3, \dots, \infty
\end{aligned} \tag{3.11b}$$

The loading considerations of equations (3.8a) and (3.9a) are to satisfy the boundary conditions at the bottom and top boundaries of the beam. Using boundary condition $\sigma_{xy}(x,0)=0$ at the edge of $y=0$, it is found that

$$\begin{aligned}
&\frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} (A_m + C_m) \alpha^3 \sin \alpha x + \mu \sum_{m=1}^{\infty} \left\{ \begin{aligned} &(A_m \alpha^2 + 2B_m \alpha) + \\ &(C_m \alpha^2 - 2D_m \alpha) \end{aligned} \right\} \alpha \sin \alpha x \right] = 0 \\
\text{or, } &\frac{-E}{(1+\mu)^2} \left[(A_m + C_m) \alpha^3 + \mu \{ (A_m \alpha^2 + 2B_m \alpha) + (C_m \alpha^2 - 2D_m \alpha) \} \alpha \right] = 0 \\
\text{or, } &\frac{-E}{(1+\mu)^2} \left[(1+\mu) \alpha^3 A_m + 2B_m \mu \alpha^2 + (1+\mu) \alpha^3 C_m - 2\mu \alpha^2 D_m \right] = 0 \\
\text{or, } &\frac{-E \alpha^2}{(1+\mu)^2} \left[(1+\mu) \alpha A_m + 2\mu B_m + (1+\mu) \alpha C_m - 2\mu D_m \right] = 0
\end{aligned} \tag{3.12a}$$

Using boundary condition $\sigma_{xy}(x,D)=0$ at the edge of $y=D$

$$\frac{-E}{(1+\mu)^2} \left[\sum_{m=1}^{\infty} \{ (A_m + B_m D) e^{\alpha D} + (C_m + D_m D) e^{-\alpha D} \} \alpha^3 \sin \alpha x + \right. \\
\left. \mu \sum_{m=1}^{\infty} \left\{ \begin{aligned} &(A_m \alpha^2 + B_m \alpha^2 D + 2B_m \alpha) e^{\alpha D} + \\ &(C_m \alpha^2 + D_m \alpha^2 D - 2D_m \alpha) e^{-\alpha D} \end{aligned} \right\} \alpha \sin \alpha x \right] = 0$$

$$\text{or, } \frac{-E}{(1+\mu)^2} \left[\left\{ (A_m + B_m D) e^{\alpha D} + (C_m + D_m D) e^{-\alpha D} \right\} \alpha^3 + \right. \\ \left. \mu \left\{ (A_m \alpha^2 + B_m \alpha^2 D + 2B_m \alpha) e^{\alpha D} + (C_m \alpha^2 + D_m \alpha^2 D - 2D_m \alpha) e^{-\alpha D} \right\} \alpha \right] = 0$$

$$\text{or, } \frac{-E \alpha^2}{(1+\mu)^2} \left[A_m (1+\mu) \alpha e^{\alpha D} + B_m (\alpha D + \mu \alpha D + 2\mu) e^{\alpha D} + \right. \\ \left. C_m (1+\mu) \alpha e^{-\alpha D} + D_m (\alpha D + \mu \alpha D - 2\mu) e^{-\alpha D} \right] = 0 \quad (3.12b)$$

Using boundary condition $\sigma_{yy}(x,0) = 4\sigma_0$ at the edge of $y = 0$

$$\frac{-E}{(1+\mu)^2} \left[- (2+\mu) \sum_{m=1}^{\infty} \left\{ (A_m \alpha + B_m) + (-C_m \alpha + D_m) \right\} \alpha^2 \cos \alpha x \right. \\ \left. + \sum_{m=1}^{\infty} \left\{ (A_m \alpha^3 + 3B_m \alpha^2) + (-C_m \alpha^3 + 3D_m \alpha^2) \right\} \cos \alpha x + 6K \right] = \sum_{m=1}^{\infty} E_m \cos \alpha x + E_o \quad (3.13a)$$

Therefore,

$$\frac{E \alpha^2}{(1+\mu)^2} \left[(2+\mu) \left\{ (A_m \alpha + B_m) + (-C_m \alpha + D_m) \right\} \right. \\ \left. - (A_m \alpha + 3B_m) - (-C_m \alpha + 3D_m) \right] = E_m$$

$$\text{or, } \frac{E \alpha^2}{(1+\mu)^2} [A_m (1+\mu) \alpha + B_m (-1+\mu) - C_m (1+\mu) \alpha + D_m (-1+\mu)] = E_m \quad (3.13b)$$

Using boundary condition $\sigma_{yy}(x,D) = \sigma_0$ at the edge of $y = D$

$$\frac{-E}{(1+\mu)^2} \left[- (2+\mu) \sum_{m=1}^{\infty} \left\{ (A_m \alpha + B_m \alpha D + B_m) e^{\alpha D} + (-C_m \alpha - D_m \alpha D + D_m) e^{-\alpha D} \right\} \alpha^2 \cos \alpha x \right. \\ \left. + \sum_{m=1}^{\infty} \left\{ (A_m \alpha^3 + B_m \alpha^3 D + 3B_m \alpha^2) e^{\alpha D} + (-C_m \alpha^3 - D_m \alpha^3 D + 3D_m \alpha^2) e^{-\alpha D} \right\} \cos \alpha x + 6K \right] = \sum_{m=1}^{\infty} I_m \cos \alpha x + I_o \quad (3.14a)$$

$$\frac{E}{(1+\mu)^2} \left[(2+\mu) \left\{ (A_m \alpha + B_m \alpha D + B_m) e^{\alpha D} + (-C_m \alpha - D_m \alpha D + D_m) e^{-\alpha D} \right\} \alpha^2 \right. \\ \left. - (A_m \alpha^3 + B_m \alpha^3 D + 3B_m \alpha^2) e^{\alpha D} + (-C_m \alpha^3 - D_m \alpha^3 D + 3D_m \alpha^2) e^{-\alpha D} \right] = I_m$$

$$\text{or, } \frac{E \alpha^2}{(1+\mu)^2} \left[A_m (1+\mu) \alpha e^{\alpha D} + B_m (\mu \alpha D + \alpha D + \mu - 1) e^{\alpha D} - \right. \\ \left. C_m (1+\mu) \alpha e^{-\alpha D} - D_m (\mu \alpha D + \alpha D - \mu + 1) e^{-\alpha D} \right] = I_m \quad (3.14b)$$

and using Eq. (3.9a) and Eq. (3.13a) the arbitrary constant K can be obtained as follows:

$$\begin{aligned} \frac{-E}{(1+\mu)^2} 6K &= E_0 = \frac{4\sigma_0}{5} \\ \text{or, } K &= \frac{-2\sigma_0(1+\mu)^2}{15E} \end{aligned} \quad (3.15)$$

The simultaneous equations (3.12a), (3.12b), (3.13b) and (3.14b) can be realized in a simplified matrix form for the solution of unknown terms like A_m , B_m , C_m and D_m as follows:

$$\begin{bmatrix} DD_1 & DD_2 & DD_3 & DD_4 \\ FF_1 & FF_2 & FF_3 & FF_4 \\ HH_1 & HH_2 & HH_3 & HH_4 \\ KK_1 & KK_2 & KK_3 & KK_4 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_m \\ I_m \end{bmatrix} \quad (3.16)$$

where

$$DD_1 = Z_{11}(1+\mu)\alpha$$

$$DD_2 = 2\mu Z_{11}$$

$$DD_3 = Z_{11}(1+\mu)\alpha$$

$$DD_4 = -2\mu Z_{11}$$

$$FF_1 = Z_{11}(1+\mu)\alpha e^{\alpha D}$$

$$FF_2 = Z_{11}\{(1+\mu)\alpha D + 2\mu\}e^{\alpha D}$$

$$FF_3 = Z_{11}(1+\mu)\alpha e^{-\alpha D}$$

$$FF_4 = Z_{11}\{(1+\mu)\alpha D - 2\mu\}e^{-\alpha D}$$

$$HH_1 = -Z_{11}(1+\mu)\alpha$$

$$HH_2 = -Z_{11}(-1+\mu)$$

$$HH_3 = Z_{11}(1+\mu)\alpha$$

$$HH_4 = -Z_{11}(-1+\mu)$$

$$KK_1 = -Z_{11}(1+\mu)\alpha e^{\alpha D}$$

$$KK_2 = -Z_{11}\{(1+\mu)\alpha D + \mu - 1\}e^{\alpha D}$$

$$KK_3 = Z_{11}(1+\mu)\alpha e^{-\alpha D}$$

$$KK_4 = Z_{11}\{(1+\mu)\alpha D - \mu + 1\}e^{-\alpha D}$$

$$Z_{11} = \frac{-E\alpha^2}{(1+\mu)^2}$$

Once the matrix Eq. (3.16) or four algebraic equations (3.12a), (3.12b), (3.13b) and (3.14b) are solved simultaneously and the values of four unknowns, namely, A_m , B_m , C_m and D_m , are obtained, equations of (3.7) are then used for subsequent finding of stress and displacement components of the beam at various points.

3.1.4 Results of Displacement Potential Solution

The analytical solutions of displacement and stress components are obtained for various aspect ratios (L/D) of the beam. At first the result of a guided steel beam ($\mu = 0.3$ and $E = 209$ GPa) having $L/D = 3$ and the uniform loading parameter = 40 N/mm is presented in sequence of axial displacement (u_x), lateral displacement (u_y), bending stress (σ_{xx}), normal stress (σ_{yy}) and shearing stress (σ_{xy}). Thereafter a comparative study is presented, which would be useful to examine the validity of the displacement potential solutions over the classical bending theory as well as finite element method (FEM). Finally the effects of beam aspect ratio on the elastic fields are analyzed.

i. Solution of Displacements

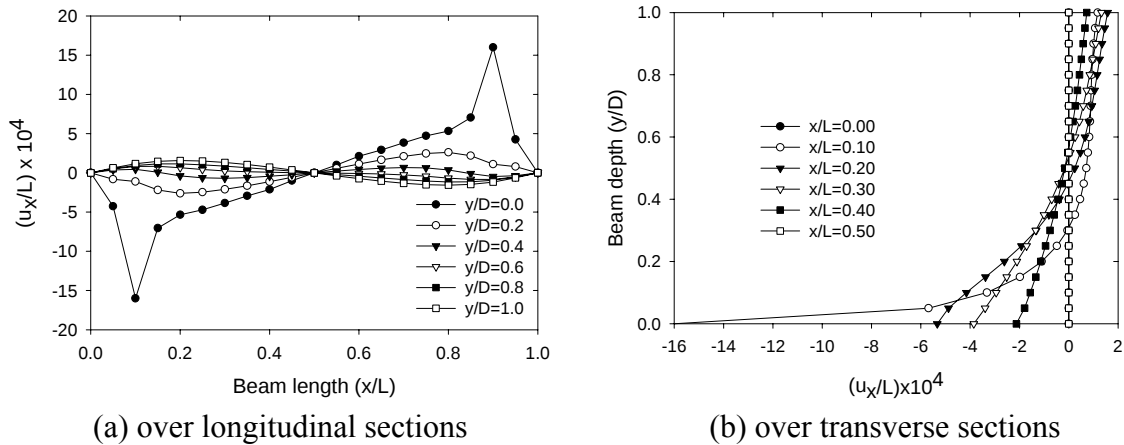


Fig. 3.3 Normalized axial displacement, $L/D = 3$: (a) distribution over longitudinal sections, (b) distribution over transverse sections from left guide to mid of span

Axial displacements (u_x) are found zero at the mid section of span, at the lateral guided boundaries and over the mid-horizontal plane [Fig. 3.3(a) and (b)]. Zero value of u_x at the guided boundaries confirms the satisfaction of boundary condition of those ends. Axial displacements are found to be symmetric about the mid-vertical plane. The magnitudes of axial displacement at the top half are quite less than those of bottom half

of the beam. The significant values of u_x are negative for sections $0 < x/L < 0.5$ and positive for $0.5 < x/L < 1.0$. The maximum magnitude of $u_x/L = 0.0016$ is observed on bottom fibre at the sections of $x/L = 0.1$ and 0.9 , where the loads terminate from both sides of the beam.

Lateral displacements (u_y) near the two lateral ends are found to take positive value because of the supports at the bottom boundary, and for the region $0.2 < x/L < 0.8$, displacements are negative [Fig. 3.4(a) and (b)]. In the present problem, there is no restriction on the lateral displacement other than the loading at the top edge and balanced at the bottom corners of the beam to bring the equilibrium condition. The result confirms this physical condition being pushed the corners up and mid-region down. The positive and negative maximum lateral displacements are $u_y/D = 0.000225$ and $u_y/D = -0.00017$ respectively. The positive maximum value is observed at the two ends on the lowest fibre and the negative maximum value is found on the top fibre at the mid section of the beam.

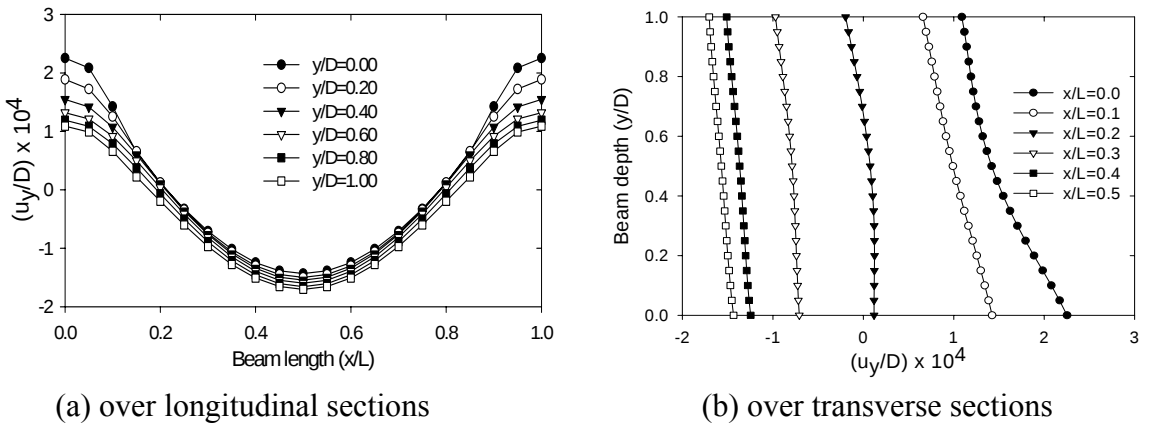


Fig. 3.4 Normalized lateral displacement distribution, $L/D = 3$: (a) over longitudinal sections (b) over transverse sections from left guide to mid of span

Fig. 3.5 presents the deformed shape of the beam together with the original shape with the magnification of 500 times of displacement. The guided ends have gone up and at the same time centre region of the beam have gone down. The deformation of the top edge is uniform through out the length of beam with the uniformly distributed loading. The bottom edge is also deformed uniformly except the support region, where there is very little non-uniformity of deformation. However, the overall vertical sliding type deformation is again in excellent agreement with the applied loading and support of the beam.

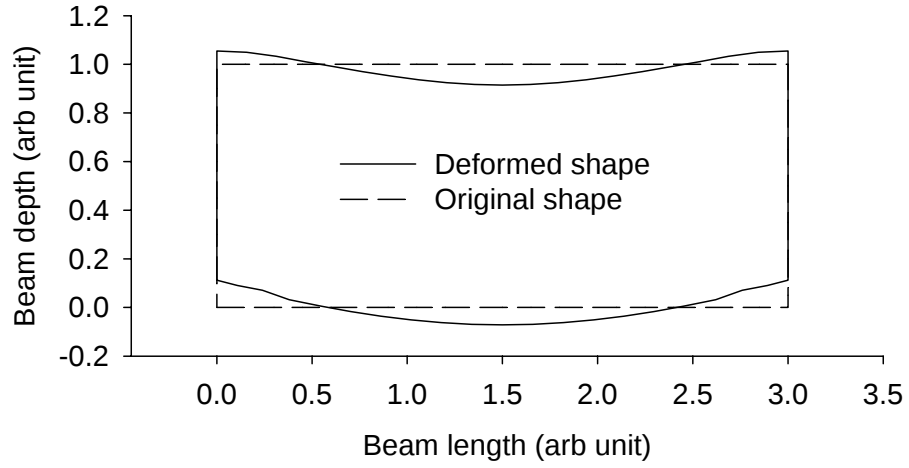


Fig. 3.5 Deformed shape of the guided isotropic deep beam, $L/D = 3$
(magnification factor $\times 500$)

ii. Solution of Stresses

Bending stress distribution is more or less non-linear over the whole span [Fig. 3.6(a) and (b)]. This non-linearity increases towards the guided ends. The stress (σ_{xx}) maximizes at the top and bottom edges of the beam but carries opposite sign. The maximum normalized values at the top and bottom fibre are 3.368 and -5.243 respectively. Near the guided ends, σ_{xx} is positive for the upper half and negative for the lower half of the beam, but the opposite is observed for sections away from the support.

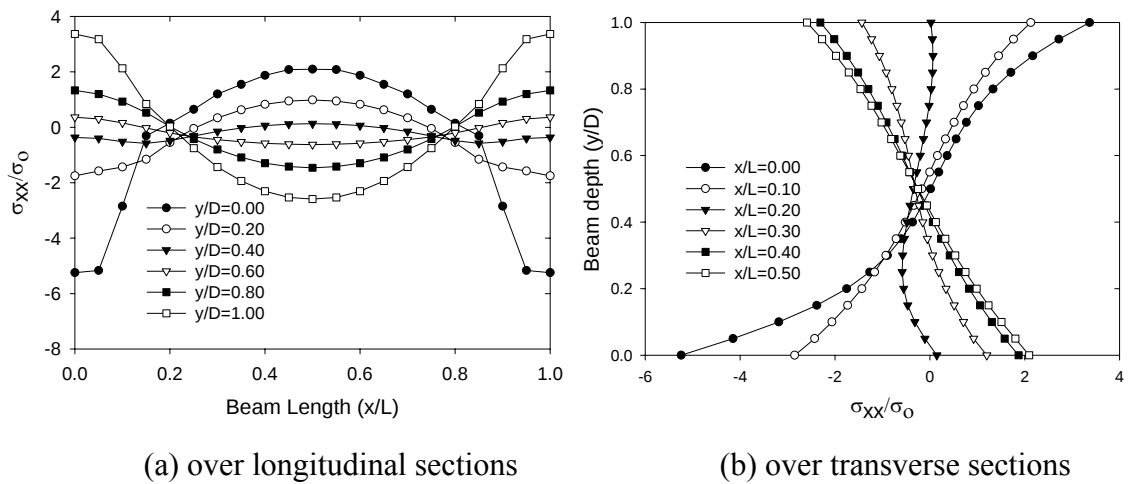


Fig. 3.6 Distribution of bending stress of the guided isotropic deep beam, $L/D=3$:
(a) over longitudinal sections, (b) over transverse sections

Fig. 3.7 reveals that the lateral stress (σ_{yy}) gets its highest value around the position of supports on the bottom edge. As per the loading configuration, the uniform load is distributed over 80% of beam length (from $x/L = 0.1$ to $x/L=0.9$) and support reaction is distributed over 20% of beam length ($x/L = 0.0$ to 0.1 and $x/L=0.9$ to 1.0). Therefore, the reaction at each support is four times of the load density. From the solution of displacement potential approach it is comprehensible that the normalized value of the lateral stress is zero where load is absent at the top layer and unity in the loaded region. The magnitude of the normalized value is four at the bottom layer of the support region and zero in the free region. This is a full agreement with the applied loading as well as boundary condition.

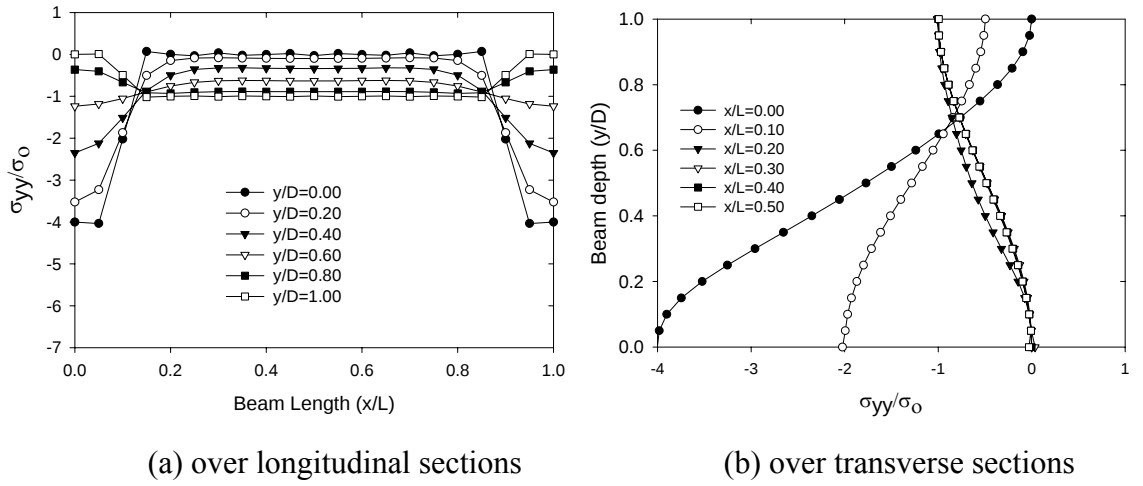


Fig. 3.7 Normalized lateral stress distribution of the guided isotropic deep beam, $L/D=3$:
(a) over longitudinal sections, (b) over transverse sections

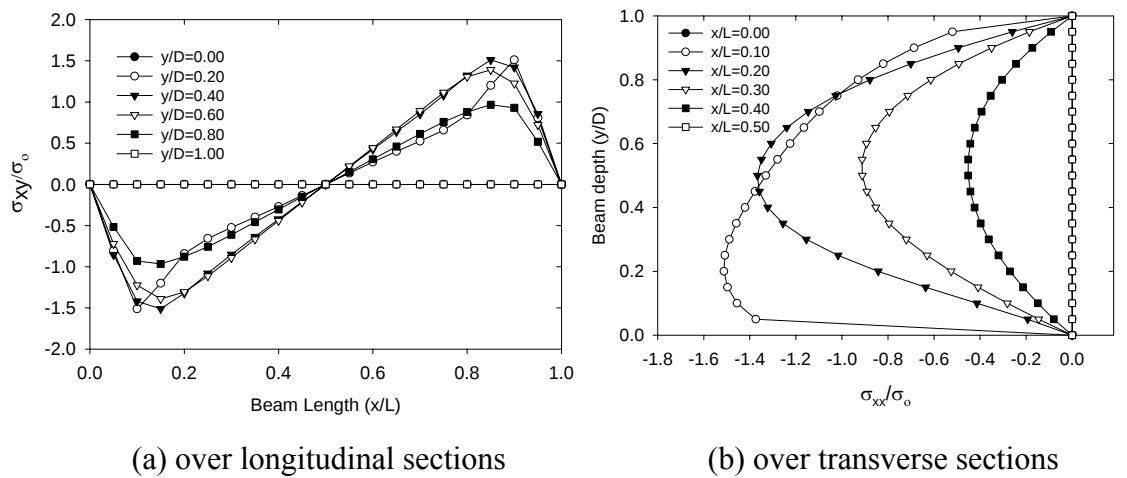


Fig. 3.8 Shear stress distribution of the guided isotropic deep beam for $L/D=3$:
(a) over longitudinal sections, (b) over transverse sections

Four edges and mid-span section of the guided deep beam are found free from shearing stress [Fig. 3.8(a) and (b)]. The distributions of shearing stress (σ_{xy}) conform to the standard parabolic profile except that at sections $x/L= 0.1$ and 0.9 , *i.e.*, the termination point of loading [Fig. 3.1]. The maximum shear stress is observed at $y/D=0.2$ of sections $x/L=0.1$ (normalized value = - 1.51233) and $x/L=0.9$ (normalized value =1.51233).

Distribution of normalized maximum principal stress obtained using the results of displacement potential approach is presented as contour in Fig. 3.9. The relation used in

this regard is $\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2}$, where σ_1 is the maximum principal

stress. No reference of the principal stress distribution for a guided deep beam could be found. As such the verification of present results remains to be investigated with the availability of any other results. However, the contour pattern of maximum principal stress of the current solution seems to be satisfactory in a general sense on visual basis.

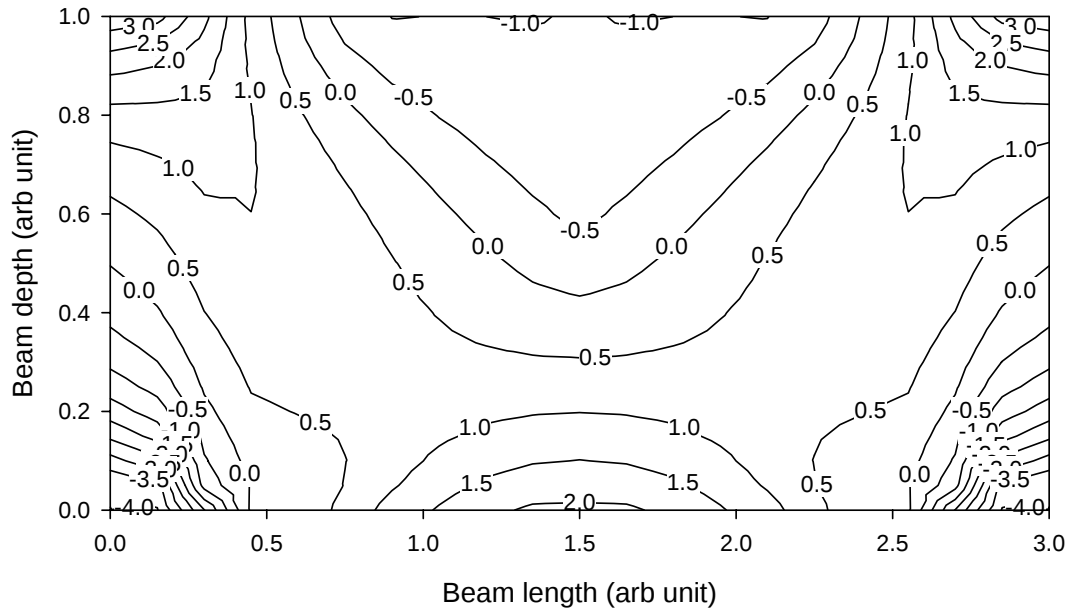


Fig. 3.9 Normalized maximum principal stress contour of the guided isotropic beam, $L/D=3$

3.1.5 Comparison of Results

The present beam problem with its guided ends cannot be solved appropriately by the classical beam theory. However, for the sake of comparison, the elementary solutions are obtained for the deep beam, which is basically a one dimensional solution to actual beam without guides. Since the distribution of reaction at the bottom surface cannot be addressed appropriately using the elementary theory, the beam is considered here to be simply supported taking the resultant of the reaction forces at $x=0.05L$ and $0.95L$. At this condition the available text book formula are modified to meet the requirement of the present problem and the solutions for the lateral deflection, bending stress and shear stress are obtained as follows:

$$u_y(x) = \frac{\sigma_0}{2E} \left(\frac{L}{D}\right)^3 \left[2 \left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3 - 1 \right] x \quad (3.17)$$

$$\begin{aligned} \sigma_{xx}(x) &= \frac{24\sigma_0}{5} \left(\frac{L}{D}\right)^2 \left(\frac{x}{L} - \frac{1}{20}\right) \left(\frac{1}{2} - \frac{y}{D}\right) && \text{for } x/L < 0.1 \text{ and } > 0.9 \\ &= 6\sigma_0 \left(\frac{L}{D}\right)^2 \left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{1}{25} \right] \left(\frac{1}{2} - \frac{y}{D}\right) && \text{for } x/L = 0.1 \text{ to } 0.9 \end{aligned} \quad (3.18)$$

$$\sigma_{xy}(x) = 6\sigma_0 \left(\frac{L}{D}\right) \left[\frac{1}{4} - \left(\frac{y}{D} - \frac{1}{2}\right)^2 \right] \left(\frac{1}{2} - \frac{x}{L}\right) \quad (3.19)$$

FEM solutions are obtained using the standard facilities of the commercial software ANSYS. For that the relevant boundary conditions used are the same as those used in the analytical solution. Four noded rectangular plane elements are used to construct the corresponding mesh network of the beam. All the elements are of same size and their distribution is kept uniform all over the domain. The total number of finite elements used to construct the element mesh network for the present problem is 400 (20 x 20) as shown in Figure 3.10.

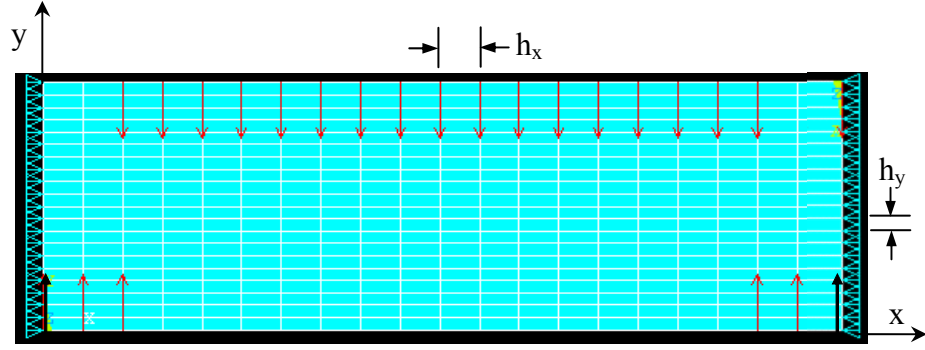
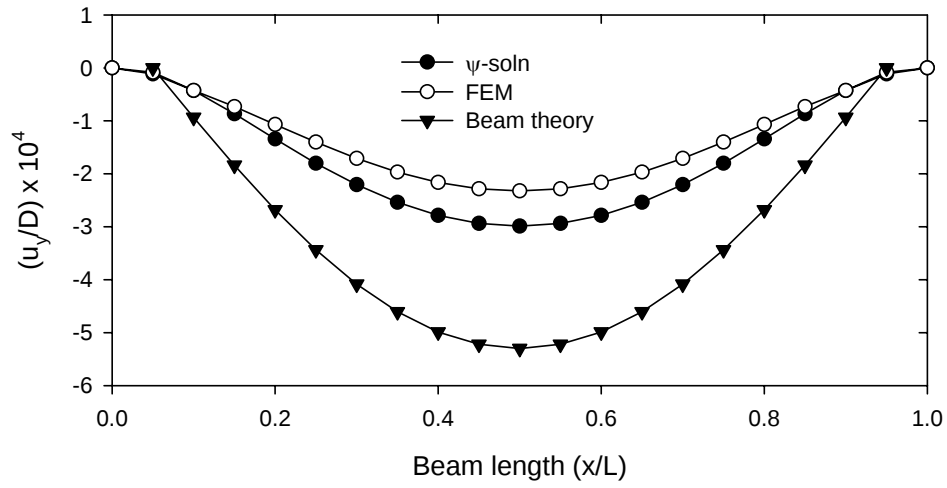


Fig. 3.10 FEM meshing (20 x 20) and loading arrangement of steel beam, $L/D=3$

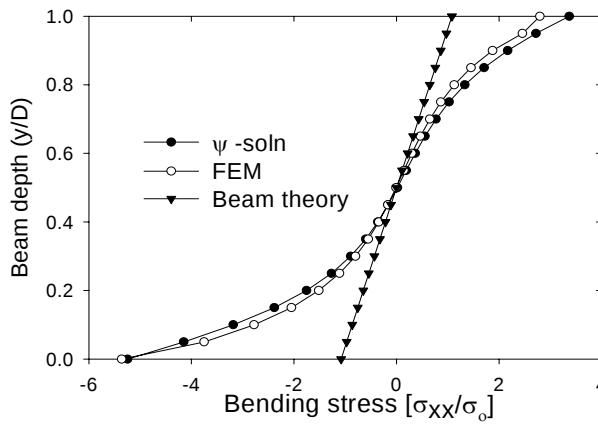
The comparison of the present ψ -solution with those obtained by the elementary theory and finite element method (FEM) is presented in Fig. 3.11.1 and 3.11.2. As appears from Fig. 3.11.1(a), the deflection is over predicted by the elementary beam theory, whereas FEM result is very close to the present ψ -solution result. Basically the ψ -solution and FEM indicate that the displacement at the guided edges is positive i.e. beam moves up and the middle of span displacement is negative i.e. the beam goes down. So, the results of ψ -solution and FEM has been translated to have the zero value at the guided ends for the comparison with beam theory results.

Fig. 3.11.1(b), (c) and (d) illustrate that the bending stress distribution obtained by the elementary theory varies linearly over the beam depth. The bending stress is under predicted at the end/support [Fig. 3.11.1(b) and (d)] but over predicted at the mid span [Fig. 3.11.1(c)] by the elementary theory from those of the corresponding ψ -solution as well as FEM solution. Moreover, beam theory predicts that the bending stress at the bottom and top edge is of same magnitude with opposite sign. But it is not the case obtained by the ψ -solution and the FEM solution; here it is of dissimilar magnitude.

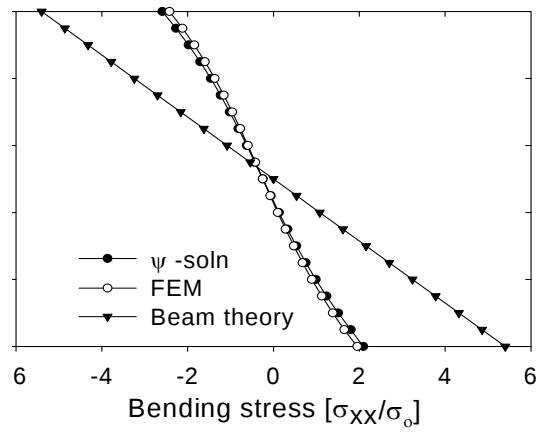
The lateral stress distribution at the support position as well as the mid section of the beam obtained using ψ -solution is very close to FEM solution [Fig. 3.11.2 (a) and (b)]. Here the result of beam theory is absent due to obvious reason. The shear stress distribution at section of $x/L = 0.25$ obtained by the three approaches is found to be very close to each other. But the solutions differ in the approaches at sections $x/L = 0.1$ and 0.9 , where the loadings on the two boundaries terminate [Fig. 3.11.2 (c) and (d)]. The elementary theory cannot address the local effect of load termination on shear stress, whereas the present ψ -solution is free from such limitations and provides reliable and accurate results at any section of the guided beam.



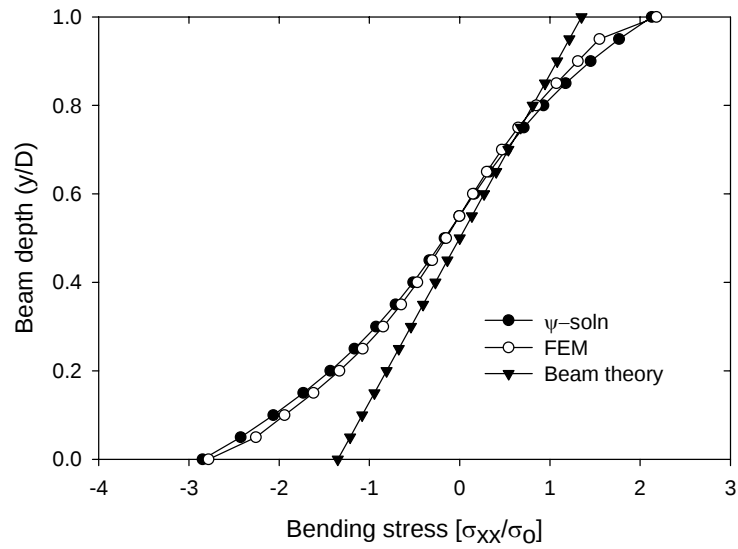
(a) Lateral displacement [$y/D=0.5$]



(b) Bending stress [$x/L=0.0$]



(c) Bending stress [$x/L=0.5$]



(d) Bending stress [$x/L=0.1$]

Fig. 3.11.1 Comparison of ψ -solution with those of beam theory and FEM: (a) lateral displacement [$y/D=0.5$], (b) bending stress [$x/L=0.0$], (c) bending stress [$x/L=0.5$], (d) bending stress [$x/L=0.1$]

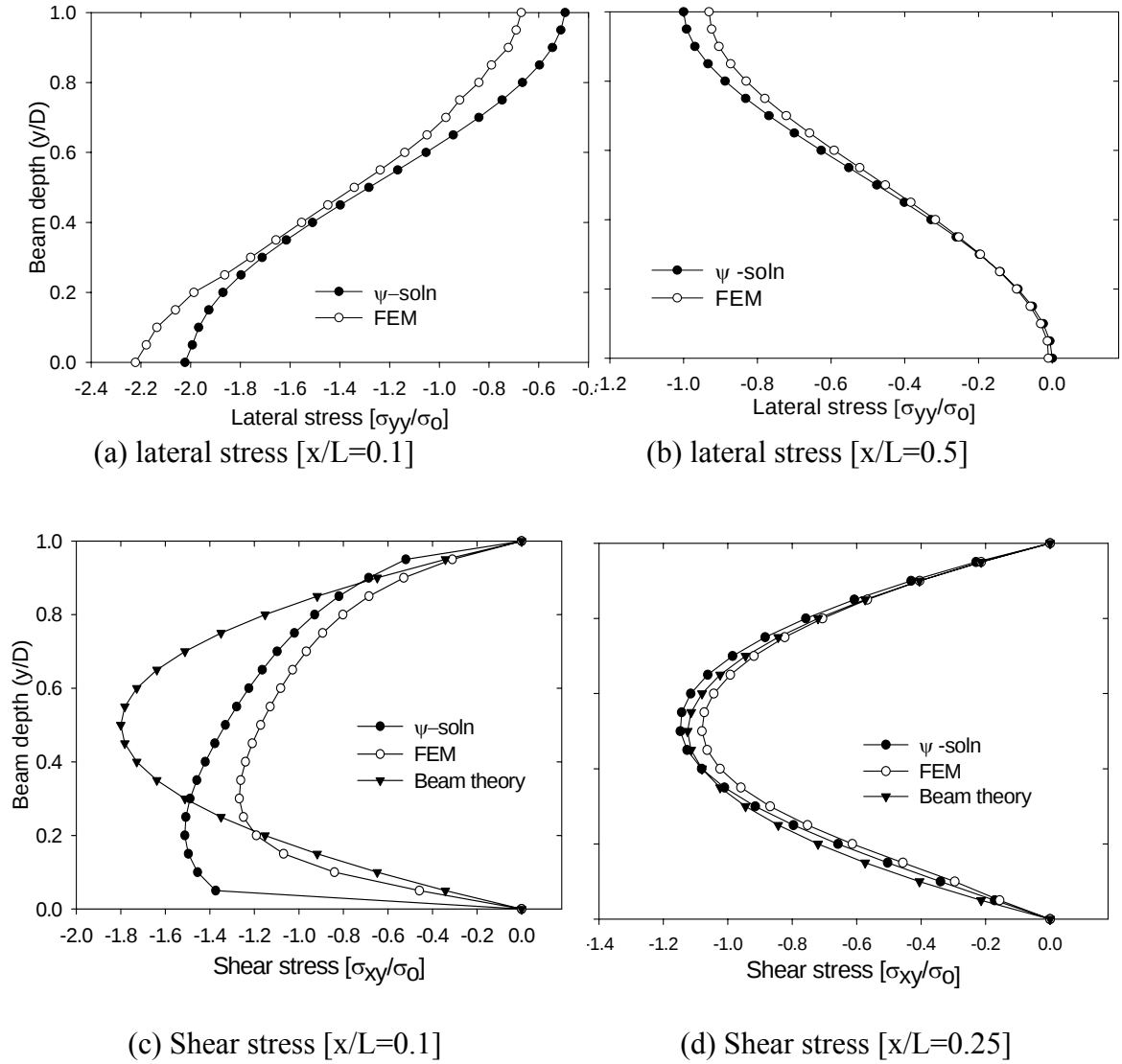


Fig. 3.11.2 Comparison of ψ -solution with those of classical beam theory and FEM:
 (a) lateral stress [$x/L=0.1$], (b) lateral stress [$x/L=0.5$], (c) shear stress [$x/L=0.1$],
 (d) shear stress [$x/L=0.25$]

3.1.6 Effect of Aspect Ratio on Stresses

The bending stresses for various aspect ratios ($L/D=1$ to 4) at different sections of the beam are observed as shown in Fig. 3.12 (a) to (f). It is seen that the nonlinearity of bending stress is the highest at the guide while aspect ratio is one and it gets reduced with the increase of aspect ratio. For higher L/D ratio at the mid region ($x/L=0.4$ to 0.6) of the beam, the bending stress distribution is quite linear [Fig. 3.12(e) and (f)], whereas it is still nonlinear

for aspect ratio one or two. However, the maximum magnitude of bending stress is increasing with the increase of L/D ratio.

The effect of aspect ratio variation is also observed using FEM for the same physical condition. The maximum bending stress observed in this regard is shown in the following tables 3.1(a) and (b).

Table 3.1 Maximum Normalized Bending Stress at $x/L = 0.1$ and 0.5

(a) at $x/L = 0.1$

L/D ratio	Simple theory	ψ -solution		FEM	
	y/D=0 and 1	y/D=0	y/D=1	y/D=0	y/D=1
1	± 0.15	-1.49	0.13	-1.45	0.14
2	± 0.60	-1.84	0.84	-1.76	0.53
3	± 1.35	-2.85	2.13	-2.78	2.18
4	± 2.40	-4.44	3.86	-4.14	3.31

(b) at $x/L = 0.5$

L/D ratio	Simple theory	ψ -solution		FEM	
	y/D=0 and 1	y/D=0	y/D=1	y/D=0	y/D=1
1	± 0.63	0.53	-0.47	0.57	-0.45
2	± 2.52	0.97	-1.30	0.93	-1.23
3	± 5.67	2.07	-2.57	1.95	-2.42
4	± 10.08	3.75	-4.27	3.51	-3.98

It is evident from the above tables that the ψ -solution and FEM results are closer at different aspect ratios. However, with the rise of aspect ratio FEM results are found to be little more than that of ψ -solution. In general, the bending stress magnitude rises with the rise of aspect ratio in a gradual nature. Both ψ -solution and FEM indicate that the bending stress magnitudes are not the same at top and bottom fibres of the beam for all aspect ratio. On the other hand the simple beam theory indicates that the rise bending stress magnitude is much higher than that of other two approaches.

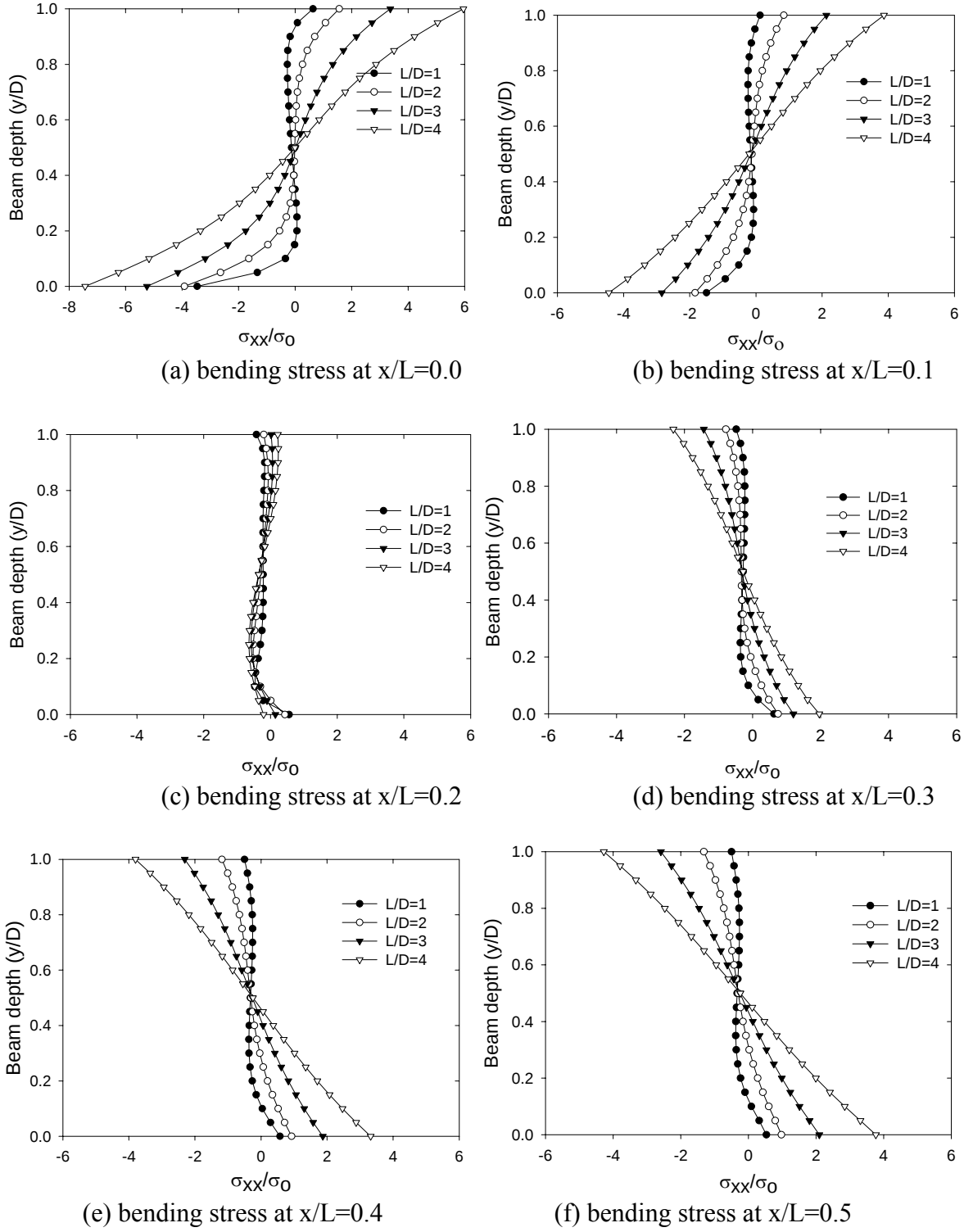


Fig. 3.12 Effect of beam aspect ratio on normalized bending stress at (a) $x/L=0.0$, (b) $x/L=0.1$, (c) $x/L=0.2$, (d) $x/L=0.3$, (e) $x/L=0.4$, (f) $x/L=0.5$

The lateral stress distributions for $L/D=1$ to 4 at sections from guide end to middle of the span ($x/L=0.0$ to 0.5) are shown in Fig. 3.13 (a) to (f). As appeared from the figures that lateral stresses in all L/D ratios follow the loading pattern. With the increase of L/D ratio the

nonlinearity reduces, but some degree of nonlinearity remains. Lateral stress at the guided ends as shown in Fig. 3.13 (a) indicates that the distribution pattern changes while the length depth ratio is changed. This pattern changing phenomenon is less with respect to sections moving away from the guide. At the mid sections it is almost insignificant.

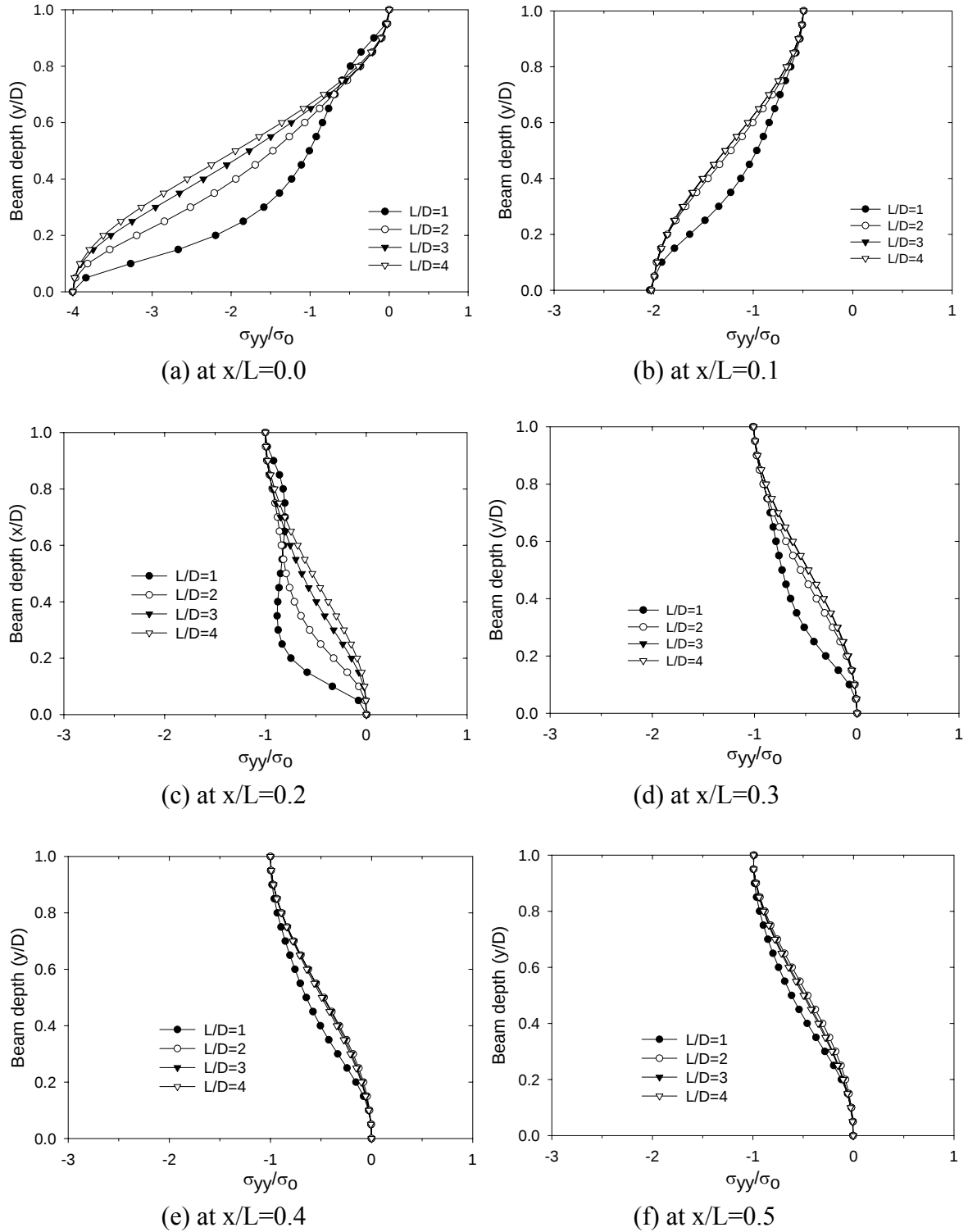


Fig. 3.13 Effect of beam aspect ratio on normalized lateral stress at: (a) $x/L=0.0$, (b) $x/L=0.1$, (c) $x/L=0.2$, (d) $x/L=0.3$, (e) $x/L=0.4$, (f) $x/L=0.5$

Table 3.2 represents the lateral stresses obtained using different approaches. The ψ -solution and FEM are very close to each other in finding the lateral stress distribution. Both the results indicate that the change of aspect ratio does not have significant effect on the lateral stress.

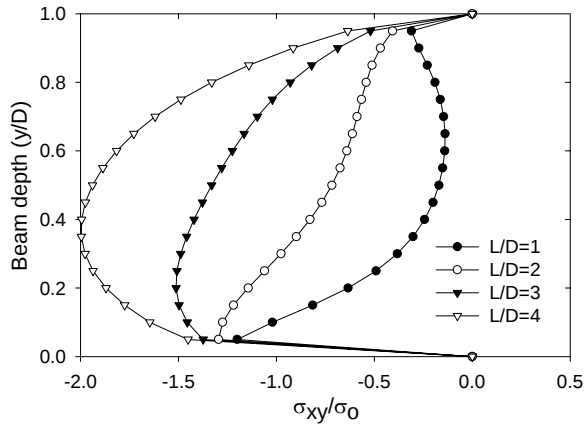
Table 3.2 Maximum Normalized Lateral Stress at $x/L=0.0, 0.1$ and 0.25

L/D	at $x/L=0.0$ ($y/D=0$)			at $x/L=0.1$ ($y/D=0$)			at $x/L=0.25$ ($y/D=1$)		
ratio	Simple theory	ψ -soln	FEM	Simple theory	ψ -soln	FEM	Simple theory	ψ -soln	FEM
1	-	-4.00	-3.99	-	-2.00	-1.99	-	-1.00	-0.99
2	-	-4.00	-3.99	-	-2.00	-2.00	-	-1.00	-0.99
3	-	-4.00	-4.00	-	-2.00	-2.00	-	-1.00	-0.99
4	-	-4.00	-4.00	-	-2.00	-2.00	-	-1.00	-1.00

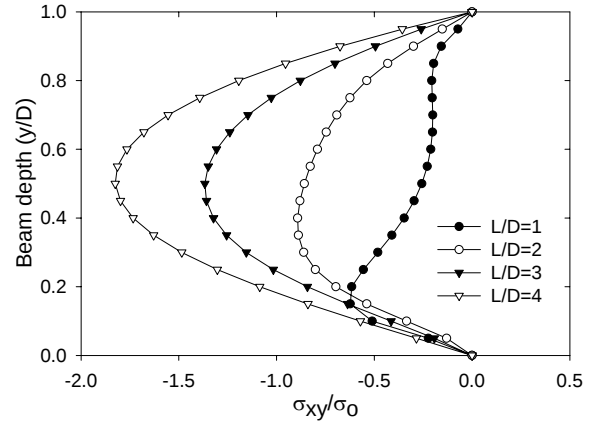
Shear stress is zero at the guided ends and at the mid span for all L/D ratios. Therefore, the results at $x/L=0.1$ to 0.4 (4 sections) are shown in Figs. 3.14 (a) to (d), where the shearing stress distribution patterns for $L/D=1$ are found quite different from others. With the rise of length depth ratio parabolic pattern of shear stress distribution is observed with higher value of maximum shear stress magnitude. The accuracy of parabolic pattern becomes finer at the sections away from the guided ends. The effect of L/D ratio variation is observed in simple beam theory, ψ -solution and FEM is shown in table 3.3.

Table 3.3 Maximum Normalized Shearing Stress at $x/L=0.1$ and $x/L=0.25$

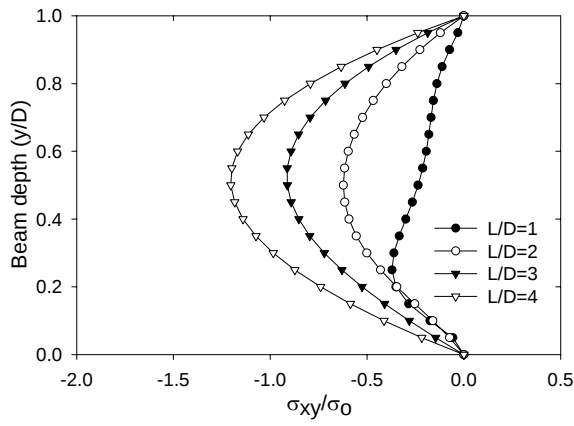
L/D ratio	at $x/L=0.1$			at $x/L=0.25$		
	Simple theory	ψ -soln	FEM	Simple theory	ψ -soln	FEM
1	0.60	1.20	0.85	0.38	0.48	0.48
2	1.20	1.29	0.97	0.75	0.76	0.72
3	1.80	1.51	1.27	1.12	1.14	1.08
4	2.40	1.99	1.72	1.50	1.51	1.43



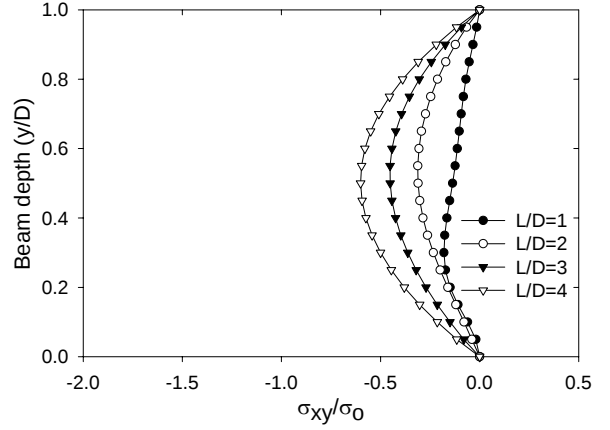
(a) at $x/L=0.1$



(b) at $x/L=0.2$



(c) at $x/L=0.3$



(d) at $x/L=0.4$

Fig. 3.14 Effect of beam aspect ratio on normalized shear stress at: (a) $x/L=0.1$, (b) $x/L=0.2$, (c) $x/L=0.3$, (d) $x/L=0.4$

3.1.7 Effect of Distribution of Support Reaction

3.1.7.1 Solution with Parabolic Reaction

The problem articulated in section 3.1.1 is here reframed with changed distribution of support reaction. Earlier the reactions at the supports were considered as uniformly distributed; here the effect of parabolic reaction pattern is investigated keeping all other parameters intact. The modification is shown in Fig. 3.15.

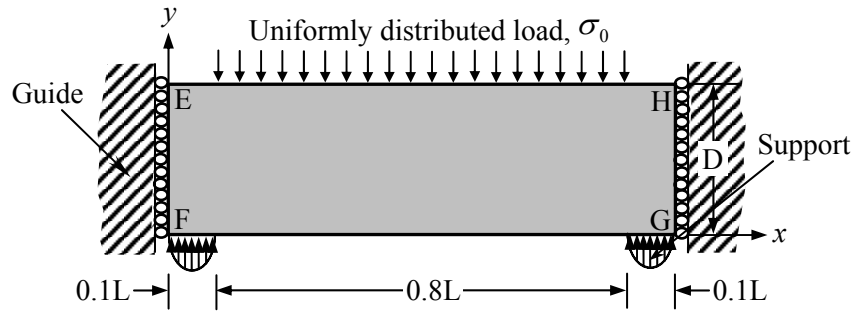


Fig. 3.15 Geometry and loading of guided isotropic beam with parabolic reaction

The physical conditions at different boundaries of the beam are also similar to that of section 3.1.2 except the conditions supporting end, FG. Since the lateral stress at this edge is in relation to the reaction and the shear stress is zero, the boundary conditions would be here as follows:

$$\sigma_{xy}(x,0)=0 \quad [0 \leq x \leq L]$$

$$\sigma_{yy}(x,0)=\frac{-24\sigma_0}{(L/10)^2}(x^2-0.1Lx) \quad [0 \leq x \leq 0.1L]$$

$$\sigma_{yy}(x,0)=\frac{-24\sigma_0}{(L/10)^2}(x-L)(x-0.9L) \quad [0.9L \leq x \leq L]$$

$$\text{and } \sigma_{yy}(x,0)=0 \quad [0.1L < x < 0.9L]$$

The governing equation and expressions elastic fields are obtained from Eq. (2.32), (2.33) and (2.34) as done for isotropic guided beam with distributed load and distributed support (section 3.1.3). Since all conditions in the boundaries except the bottom one are same, the trial solution of ψ -function is also same. Then the 4th order

differential equation has been solved in the same way as done in section 3.1.3 taking the difference in conditions of the bottom boundary.

The reactions on the bottom boundary ($y = 0$) at the supports can be taken as Fourier series in the following manner:

$$\sigma_{yy}(x,0) = E_0 + \sum_{m=1}^{\infty} E_m \cos \alpha x \quad \text{for } x=0 \text{ to } 0.1L \text{ and } 0.9L \text{ to } L \quad (3.20)$$

Here

$$\begin{aligned} E_0 &= \frac{1}{L} \left\{ \frac{-24\sigma_0}{(L/10)^2} \right\} \left[\int_0^{L/10} (x^2 - Lx/10) dx + \int_{9L/10}^L (x-L)(x-9L/10) dx \right] \\ &= \frac{-24\sigma_0}{L^3/100} \left[\left\{ \frac{-L^3}{3000} - \frac{-L^3}{2000} \right\} + \left\{ \frac{-L^3}{3000} - \frac{-L^3}{2000} \right\} \right] \\ &= \frac{-24\sigma_0}{L^3/100} \left\{ \frac{-L^3}{3000} \right\} \\ &= \frac{4\sigma_0}{5} \end{aligned} \quad (3.21a)$$

$$\begin{aligned} E_m &= \frac{2}{L} \left\{ \frac{-24\sigma_0}{(L/10)^2} \right\} \left[\int_0^{L/10} (x^2 - Lx/10) \cos \alpha x dx + \int_{9L/10}^L (x-L)(x-9L/10) \cos \alpha x dx \right] \\ &= \frac{2}{L} \left\{ \frac{-24\sigma_0}{(L/10)^2} \right\} \left[\left\{ \frac{L}{10\alpha^2} \cos\left(\frac{\alpha L}{10}\right) - \frac{2}{\alpha^3} \sin\left(\frac{\alpha L}{10}\right) + \frac{L}{10\alpha^2} \right\} \left\{ 1 + \cos\left(\frac{9\alpha L}{10}\right) \right\} \right. \\ &\quad \left. + \left\{ \frac{L}{10\alpha^2} \sin\left(\frac{\alpha L}{10}\right) - \frac{2}{\alpha^3} \cos\left(\frac{\alpha L}{10}\right) - \frac{2}{\alpha^3} \right\} \sin\left(\frac{9\alpha L}{10}\right) \right] \\ &= -4800\sigma_0 \left[\left\{ \frac{1}{10\alpha^2 L^2} \cos\left(\frac{\alpha L}{10}\right) - \frac{2}{\alpha^3 L^3} \sin\left(\frac{\alpha L}{10}\right) + \frac{1}{10\alpha^2 L^2} \right\} \left\{ 1 + \cos\left(\frac{9\alpha L}{10}\right) \right\} \right. \\ &\quad \left. + \left\{ \frac{1}{10\alpha^2 L^2} \sin\left(\frac{\alpha L}{10}\right) - \frac{2}{\alpha^3 L^3} \cos\left(\frac{\alpha L}{10}\right) - \frac{2}{\alpha^3 L^3} \right\} \sin\left(\frac{9\alpha L}{10}\right) \right] \\ &= -4800\sigma_0 \left[\left\{ \frac{1}{10m^2 \pi^2} \cos\left(\frac{\alpha L}{10}\right) - \frac{2}{m^3 \pi^3} \sin\left(\frac{\alpha L}{10}\right) + \frac{1}{10m^2 \pi^2} \right\} \left\{ 1 + \cos\left(\frac{9\alpha L}{10}\right) \right\} \right. \\ &\quad \left. + \left\{ \frac{1}{10m^2 \pi^2} \sin\left(\frac{\alpha L}{10}\right) - \frac{2}{m^3 \pi^3} \cos\left(\frac{\alpha L}{10}\right) - \frac{2}{m^3 \pi^3} \right\} \sin\left(\frac{9\alpha L}{10}\right) \right] \end{aligned} \quad (3.21b)$$

where $m = 1, 2, 3, \dots, \infty$

The Fourier series for compressive load on the edge $y = D$ can also be as similar to that of Eq. (3.9a) and then I_o and I_m are as to Eq. (3.9b) and (3.9c) respectively.

The loading considerations of equations (3.15) and (3.9a) are investigated for the bottom ($y=0$) and top ($y=D$) boundaries respectively to be satisfied using boundary conditions (c) and (d), for which following algebraic equations are true.

$$\frac{-E\alpha^2}{(1+\mu)^2} [(1+\mu)\alpha A_m + 2\mu B_m + (1+\mu)\alpha C_m - 2\mu D_m] = 0 \quad (3.22a)$$

$$\frac{-E\alpha^2}{(1+\mu)^2} \left[A_m (1+\mu)\alpha e^{\alpha D} + B_m (\alpha D + \mu\alpha D + 2\mu)e^{\alpha D} + C_m (1+\mu)\alpha e^{-\alpha D} + D_m (\alpha D + \mu\alpha D - 2\mu)e^{-\alpha D} \right] = 0 \quad (3.22b)$$

$$\frac{E\alpha^2}{(1+\mu)^2} [A_m (1+\mu)\alpha + B_m (-1+\mu) - C_m (1+\mu)\alpha + D_m (-1+\mu)] = E_m \quad (3.22c)$$

$$\frac{E\alpha^2}{(1+\mu)^2} \left[A_m (1+\mu)\alpha e^{\alpha D} + B_m (\mu\alpha D + \alpha D + \mu - 1)e^{\alpha D} - C_m (1+\mu)\alpha e^{-\alpha D} - D_m (\mu\alpha D + \alpha D - \mu + 1)e^{-\alpha D} \right] = I_m \quad (3.22d)$$

The four algebraic equations (3.22a), (3.22b), (3.22c) and (3.22d) are solved simultaneously taking the expression for E_m from Eq. (3.16a) and I_m from (3.9c) to find out the values of arbitrary constants, such as A_m , B_m , C_m and D_m . Then stress and displacement components of the beam at various points are obtained using equations of (3.7).

3.1.7.2 Results and Discussion for Parabolic Reaction

The analytical solution results of displacement and stress components are obtained for parabolic reaction various aspect ratios (L/D) of the beam. The effects of various aspect ratios for the guided steel beam in the same physical and loading conditions along with parabolic supports are observed. The graphs for variations of stress components with respect to the variation of aspect ratio are presented in appendix A. All other results are presented in the subsequent paragraphs.

i. Displacement Field

Axial displacements (u_x) are found negative for sections $0 < x/L < 0.5$ and positive for $0.5 < x/L < 1.0$ with the maximum magnitudes at $x/L = 0.1$ and 0.9 , where the loads terminate from both sides of the beam. Notable point is that the maximum magnitude of axial displacement is found quite less for parabolic support than that of uniform reaction (support) with same top loading [Fig. 3.16]. Normalised axial displacement at the bottom end section of $x/L=0.1$ and 0.9 has the maximum value 16×10^{-4} uniform support and 12.9×10^{-4} .

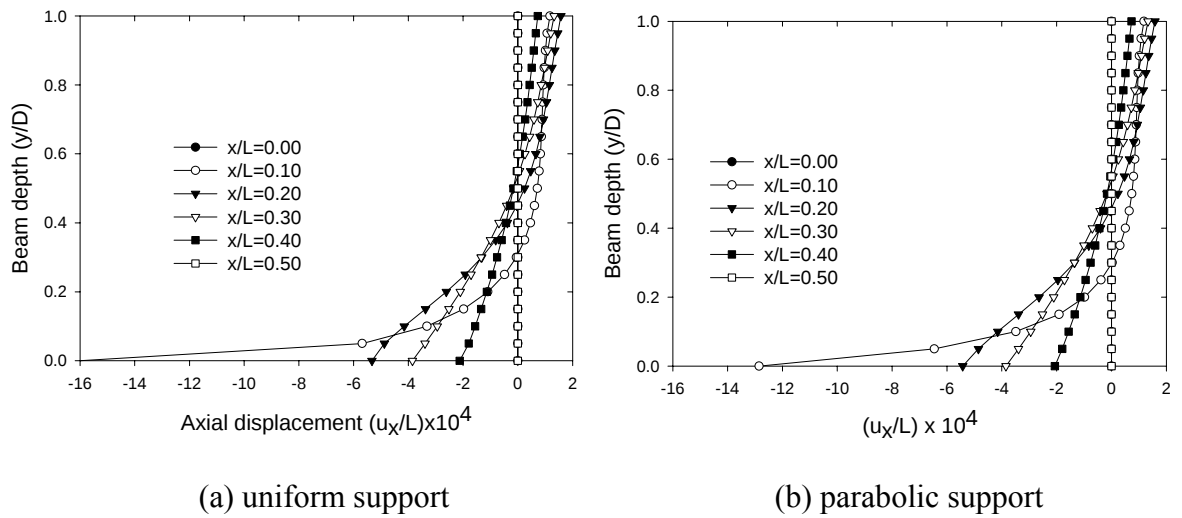


Fig. 3.16 Axial displacements of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

Lateral displacements (u_y) near the two lateral ends are found to take positive value because of the supports at the bottom boundary. In case of uniform support loading the lateral displacement is gradually reducing and becomes negative for the region $0.2 < x/L < 0.8$. For parabolic support loading positive and negative region of the lateral displacement is also same, but the bottom fibre of beam ($y/D=0.0$) takes a lift at the section of $x/L=0.05$, where the pick reaction loading is being acted, and then falls gradually up to the mid span of the beam [Fig. 3.17]. All other fibres demonstrate the similar lateral displacements in both the supporting conditions. The maximum magnitude of lateral displacement for parabolic support is also found to similar to that of uniform support.

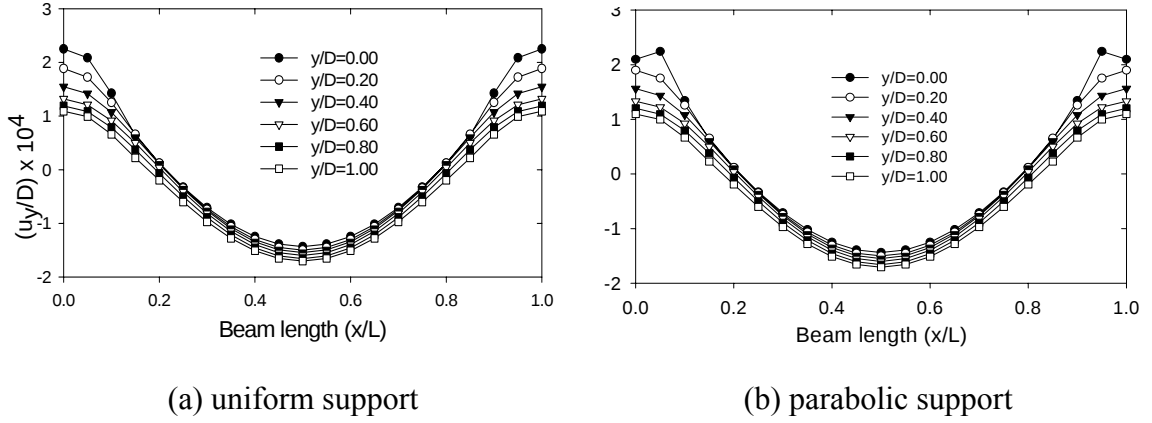


Fig. 3.17 Lateral displacements of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

ii. Stress Field

The distribution of bending stress (σ_{xx}) is not linear as the classical theory expresses for simply supported condition of a beam, rather it is more or less non-linear over the whole span. This non-linearity increases towards the guided ends. For the uniform support the stress (σ_{xx}) maximizes at the top and bottom edges of the beam. But for parabolic supports σ_{xx} the maximum magnitude of σ_{xx} does not appear at the bottom edge of the beam. Rather it is observed little inside (near $y/D=0.08$) at the sections $x/L=0.0$ and $x/L=1.0$ of the beam. Moreover, the normalised maximum value of bending stress reduces from 5.24 to 4.33 for the change of support pattern and results the avoidance of higher stress concentration. Near the guided ends, σ_{xx} is positive for the upper half and negative for the lower half of the beam, but the opposite situation reverses for sections away from the support. The distribution of bending stress for the beam with $L/D=3$ having uniform as well as parabolic support is shown in figure 3.18 section wise from guide to mid span for the half length of beam due to symmetry.

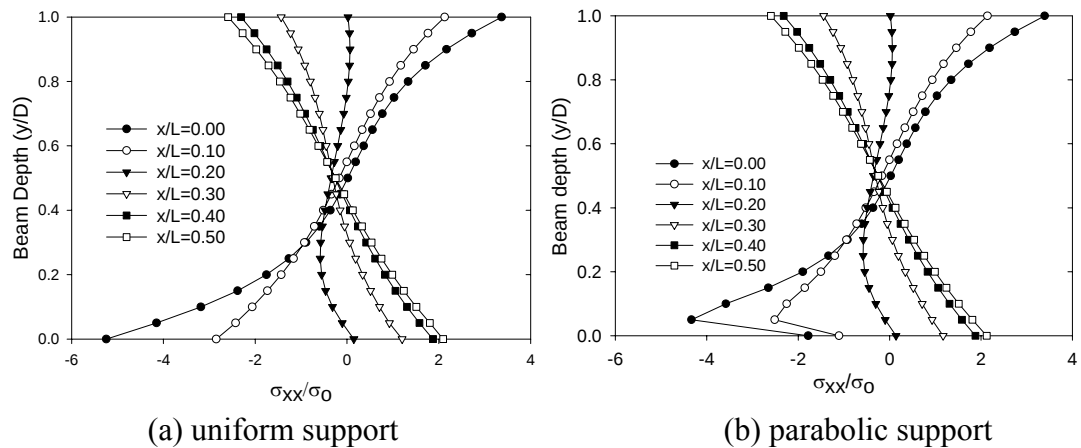


Fig. 3.18 Bending stress of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

It is clear from Fig. 3.19 that the distribution of lateral stress (σ_{yy}) is non-linear over the depth at all sections of the beam. For parabolic reactions, lateral stress (σ_{yy}) concentration occurs at the lateral ends little up (near $y/D=0.2$) from the bottom edge of the beam, which is just on the bottom edge for uniformly distributed supports. The normalized maximum value of σ_{yy} is less in case of parabolic reaction than that of uniform support. For the top loaded region, normalized value of the lateral stress varies from zero to unity, which is in full agreement with the applied loading. The distribution of normal stress for the beam with $L/D=3$ having uniform as well as parabolic support section wise from the guided to mid span is shown in figure 3.19 (a) and (b) respectively.

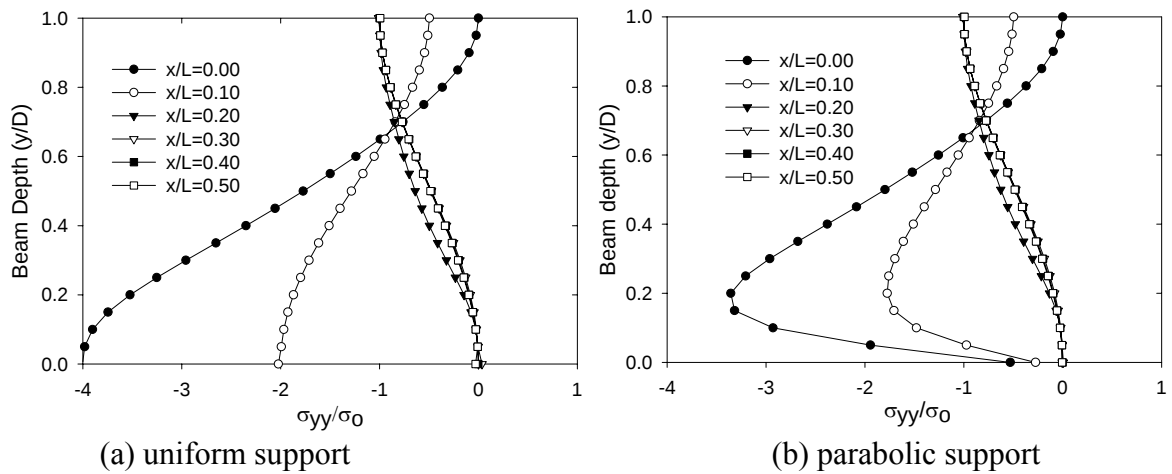
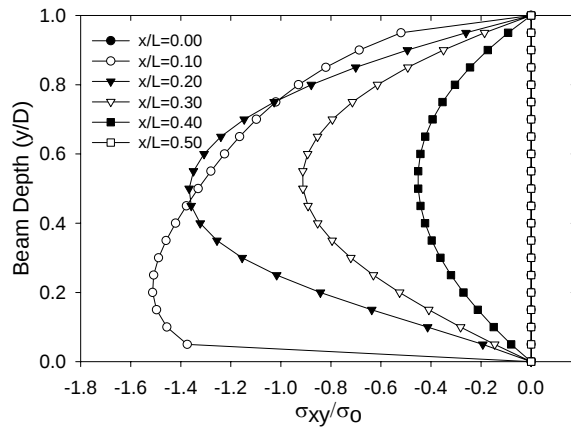
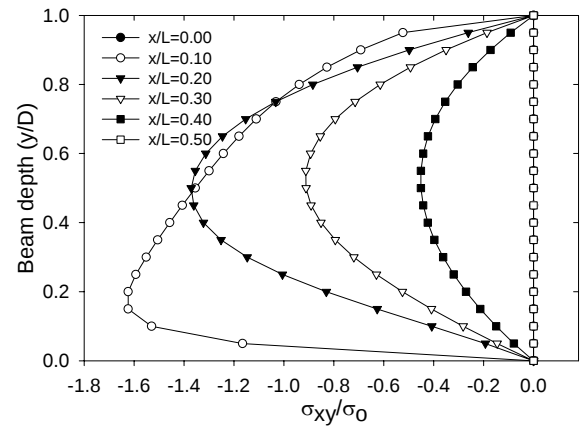


Fig. 3.19 Lateral stress of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

Four edges and mid-span section are found free from shearing stress as same as to the uniform support reaction [Fig. 3.20]. The distribution of shearing stress (σ_{xy}) is obeying the rules to the standard profile except that at sections $x/L = 0.1$ and 0.9 , *i.e.*, the termination point of loading. At these sections, shearing stress takes more value in the bottom half than that of top half of the beam. Here the point may be noted that the shearing stress concentration is found more for parabolic support than that of uniform support reaction. The maximum normalized shear stress is 1.62 for parabolic and it is 1.51 for uniform support.



(a) uniform support



(b) parabolic support

Fig. 3.20 Shear stress of the guided steel beam, $L/D = 3$: (a) uniform support reaction (b) parabolic support reaction

3.2 Solution of Unguided Simply Supported Beams

A rectangular section simply supported steel beam of length L , depth D and width W (assumed to be unity for plane stress consideration) is subjected to a uniformly distributed load on its top boundary. The generalized consideration of its geometry and distribution of load is shown in figure 3.21.

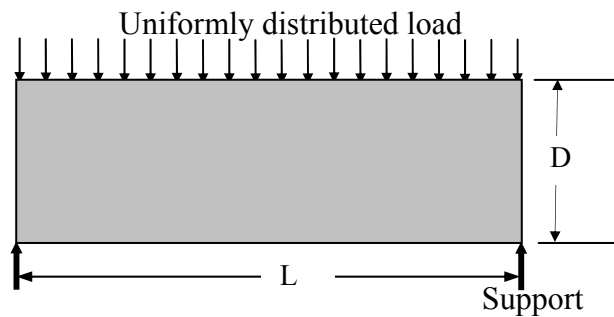


Fig. 3.21 Geometry and loading of an unguided simply supported beam

3.2.1 Analytical Solution Using Displacement Potential

The problem of simply supported guided beam can be solved using displacement potential approach very conveniently. However, the analytical solution of simply supported beam without any guide at the lateral ends cannot be obtained directly. In this regard following assumptions are considered pertinent and helpful for the solution purpose:

- i. The unguided deep beam is considered to be a portion of a long beam. The long beam is taken as guided one with the aspect ratio ten. In the present study the unguided beam is taken as the 20 percent of guided beam from the mid portion of the long span. Thus the aspect ratio of the unguided beam is then two, where the beam depth is as same the long beam.
- ii. The lateral ends of the long beam are guided, where the axial displacements are restrained, but the lateral displacements are free to have any value.
- iii. The effect of guided boundary conditions of the long beam is insignificant for the section taken as the short beam on account of Saint Venant's principle.

- iv. The condition of the support (reaction) is not simulated by a point loading at the two bottom corners of the beam. Rather they are assumed to be distributed reaction forces for certain lengths of the beam.
- v. Since the reactions at the supports are distributed, the reaction forces at the supports are located in such a way that their resultants act just at the ends of the beam. The details of the geometry and loading of the unguided deep beam along with long guided beam are illustrated in Fig. 3.22.

3.2.1.1 Particular Enunciation

The support reactions at both the ends are distributed keeping half portion over and half beyond the considered length (L) for the unguided beam as shown in figure 3.22. In this consideration the resultant line of reaction force remains just at the termination section of the distributed loading at the top edge as well as at the end of the beam. The uniformly distributed load is considered to be located over the whole length of unguided beam and each support reaction is assumed to be over 5 percent of that length (L). Therefore, support reactions are acting at $x'/L'=0.4$ (distributed over $x'/L'=0.395 - 0.405$) and $x'/L'=0.6$ (distributed over $x'/L'=0.595 - 0.605$).

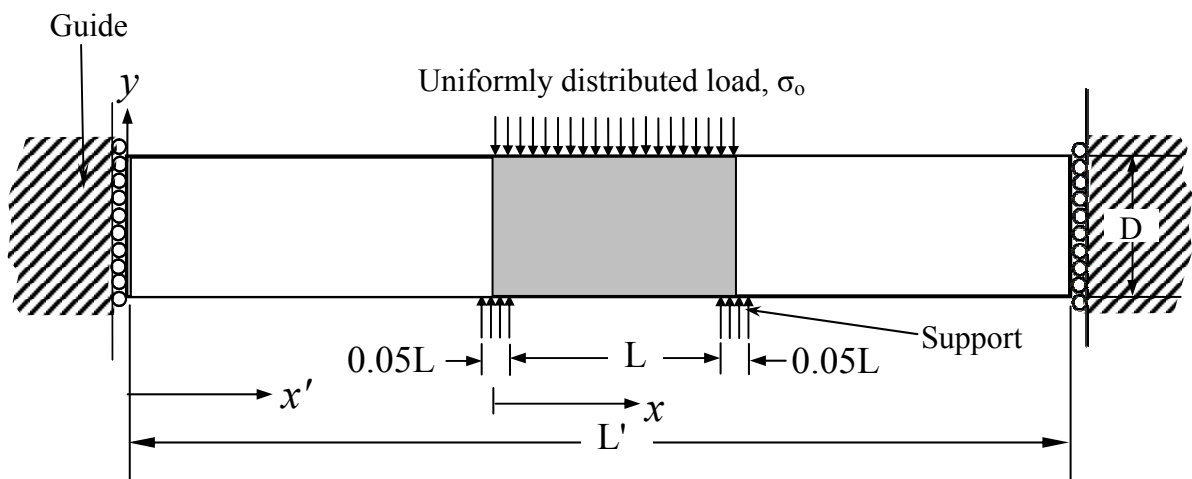


Fig. 3.22 Geometry and loading of model for analytical solution of unguided beam

3.2.1.2 Analytical solution

The solution is done here as it was obtained for isotropic guided beam in section 3.1.3 with the exception for expression for loading and reaction distribution only. The compressive load exerted on the edge $y = D$ is acting over the length of considered unguided portion of the beam, i.e., $x'/L' = 0.4$ to 0.6 and the reactions at the supports on the edge $y = 0$ are acting over $x'/L' = 0.395$ to 0.405 and 0.595 to 0.605 . Therefore, the load and reactions may be taken as Fourier function in the following manner.

First the compressive load on the edge $y = D$:

$$\sigma_{yy}(x,0) = \sigma_{yy}^0 = I_0 + \sum_{m=1}^{\infty} I_m \cos \alpha x \quad \text{for } x'/L' = 0.4 \text{ to } 0.6 \quad (3.23)$$

Here

$$I_0 = \frac{1}{L'} \left[\int_{0.4L'}^{0.6L'} \sigma_{yy}^0 dx \right] = \frac{\sigma_{yy}^0}{5} \quad (3.24a)$$

and

$$\begin{aligned} I_m &= \frac{2}{L'} \left[\int_{0.4L'}^{0.6L'} \sigma_{yy}^0 \cos \alpha x dx \right] \\ &= \frac{2\sigma_{yy}^0}{L'} \left[\frac{\sin \alpha x}{\alpha} \right]_{0.4L'}^{0.6L'} \\ &= \frac{2\sigma_{yy}^0}{\alpha L'} \{ \sin 0.6\alpha L' - \sin 0.4\alpha L' \} \\ &= \frac{2\sigma_{yy}^0}{m\pi} \{ \sin 0.6m\pi - \sin 0.4m\pi \} \end{aligned} \quad (3.24b)$$

where $m = 1, 2, 3, \dots, \infty$.

Now compressive reactions at the edge $y = 0$:

$$\sigma_{yy}(x,0) = 10\sigma_{yy}^0 = E_0 + \sum_{m=1}^{\infty} E_m \cos \alpha x \quad [x'/L' = 0.395 \text{ to } 0.405 \text{ and } 0.595 \text{ to } 0.605] \quad (3.25)$$

Here

$$E_0 = \frac{1}{L'} \left[\int_{0.395L'}^{0.405L'} 10\sigma_{yy}^0 dx + \int_{0.595L'}^{0.605L'} 10\sigma_{yy}^0 dx \right] = \frac{\sigma_{yy}^0}{5} \quad (3.26a)$$

and

$$\begin{aligned} E_m &= \frac{2}{L'} \left[\int_{0.395L'}^{0.405L'} 10\sigma_{yy}^0 \cos \alpha x dx + \int_{0.595L'}^{0.605L'} 10\sigma_{yy}^0 \cos \alpha x dx \right] \\ &= \frac{20\sigma_{yy}^0}{L'} \left[\frac{\sin \alpha x}{\alpha} \right]_{0.395L'}^{0.405L'} + \frac{20\sigma_{yy}^0}{L'} \left[\frac{\sin \alpha x}{\alpha} \right]_{0.595L'}^{0.605L'} \\ &= \frac{20\sigma_{yy}^0}{\alpha L'} \{ \sin(0.405\alpha L') - \sin(0.395\alpha L') + \sin(0.605\alpha L') - \sin(0.595\alpha L') \} \\ &= \frac{20\sigma_{yy}^0}{m\pi} \{ \sin(0.405m\pi) - \sin(0.395m\pi) + \sin(0.605m\pi) - \sin(0.595m\pi) \} \end{aligned} \quad (3.26b)$$

where $m = 1, 2, 3, \dots, \infty$.

The arbitrary constant K of assumed ψ -function [Eq. (3.3)] can be obtained from Eq. (3.7d) and Eq. (3.19) as follows

$$\begin{aligned} \frac{-E}{(1+\mu)^2} 6K &= E_0 = \frac{\sigma_0}{5} \\ \text{or, } K &= \frac{-\sigma_0(1+\mu)^2}{30E} \end{aligned} \quad (3.27)$$

Then Eqs. (3.24b), (3.26b) and (3.27) are used to solve the simultaneous equations of Eq.(3.9) to obtain the values of A_m , B_m , C_m and D_m . Then stress and displacement components of the beam at various points are obtained using equations of (3.7).

3.2.2 Results of Unguided Deep Beam

The analytical solutions of displacement and stress components are obtained for the unguided beam using displacement potential approach taking the assumptions mentioned in section 3.2.1. the result presented in this section is of a steel beam ($\mu = 0.3$ and $E = 209$ GPa) having $L/D = 2$ and the uniform loading parameter = 40 N/mm of length.

i. Displacement Field

Axial displacements (u_x) are found symmetric but in two opposite directions with respect to the mid section of the span. It is negative for sections $0 < x/L < 0.5$ and positive for $0.5 < x/L < 1.0$. The maximum magnitudes are located at the bottom fibre of the end sections $x/L = 0.1$ and 0.9 . The normalized value of maximum axial displacement is found to be 6.64×10^{-4} . Axial displacements are found zero at the mid-span of the beam as shown in figure 3.23.

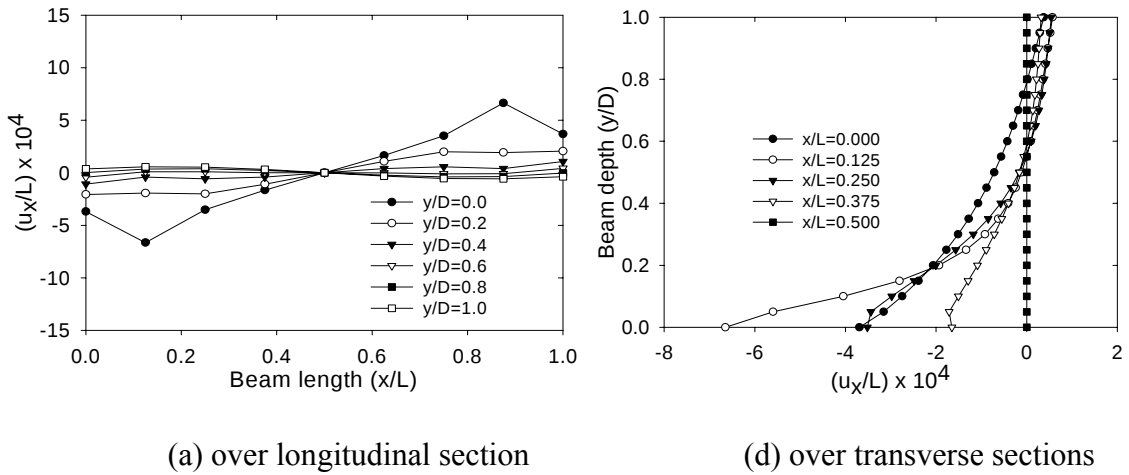


Fig. 3.23 Axial displacement of unguided beam, $L/D=2$: (a) at longitudinal sections, (b) over transverse sections

Lateral displacements (u_y) are found negative through out the span with the maximum value at the mid section and minimum at the end. The lateral displacements for both the support conditions are presented in figure 3.24.

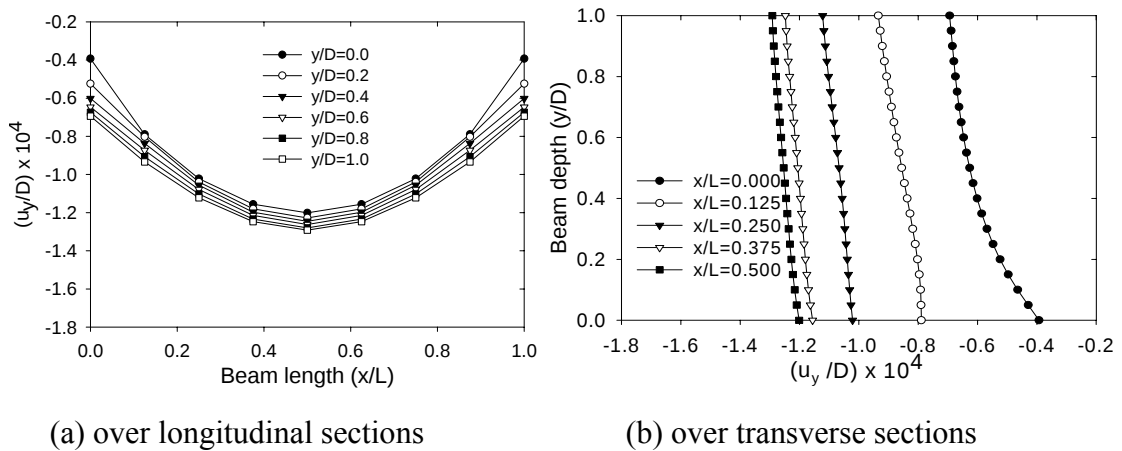


Fig. 3.24 Lateral displacement of unguided beam, $L/D=2$: (a) over longitudinal sections, (b) over transverse sections

Since the constraint of axial displacement at the lateral end of the long beam is far away from the location of the lateral boundary of short beam, its effect would be negligible on account of Saint Venant's principle. As such, both the longitudinal and transverse displacements observed at the lateral ends are all right. The lower part of the beam is thus elongated and the upper part is contracted freely at the same time.

ii. Stress Field

Bending stress distribution is not linear as normally considered on the basis of mechanics of material. However, this non-linearity is quite less than that of guided beam and it may be neglected at the mid sections of the span. But for both the cases of support location, the condition of non-linearity at the ends of the beam has to be considered for proper assessment of beam stresses. The stress (σ_{xx}) maximizes at the top and bottom edges at every section of the beam. At the top edge it follows the gradual augmentation of compressive stress towards the load centre. At the bottom edge bending stress is more non-linear and maximises twice: once at the load centre and again at the location of supports. Bending stress distribution is shown in figure 3.25 for both cases of support condition.

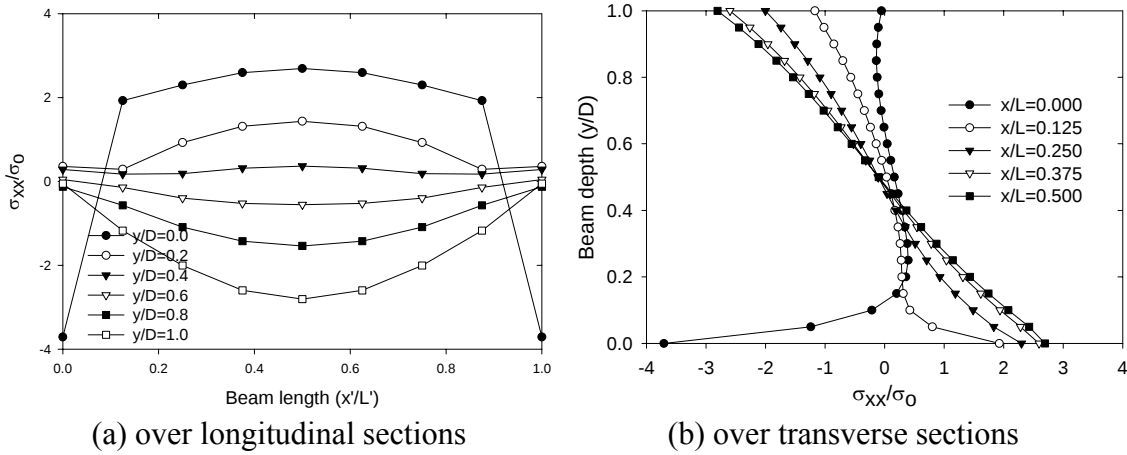


Fig. 3.25 Distribution of bending stress for unguided consideration, $L/D = 2$:
(a) over longitudinal sections, (b) over transverse sections

The lateral stress (σ_{yy}) distribution follows the loading of the beam for both the case of support location. For the loaded region, normalized value of the lateral stress varies from zero to unity [Fig. 3.26 (a)]. The section wise transverse distribution of lateral stress is non-linear over the depth at all sections of the beam [Fig. 3.26 (b)].

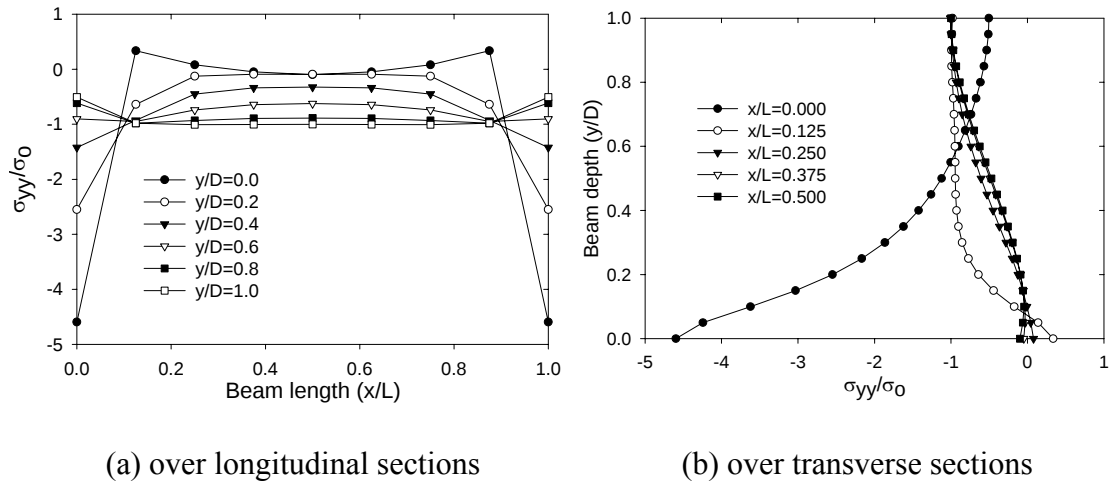


Fig. 3.26 Distribution of lateral stress for unguided consideration, $L/D = 2$:
(a) over longitudinal sections, (b) over transverse sections

Top and bottom edges and mid-span section are found free from shearing stress, but the support end are not free from shearing stress here as it is observed for guided ends. So the significant difference between guided and unguided consideration is observed on the shearing stress issue. It is not such remarkable as for bending and lateral stresses. The vertical distribution of shearing stress (σ_{xy}) is obeying the rules to the standard profile except that at sections where the termination of loading takes place [Fig. 3.27 (a) and (b)]. At these sections, shearing stress is not having the same magnitude at the same depth from top and bottom; it takes more value in the bottom half than that of top half of the beam.

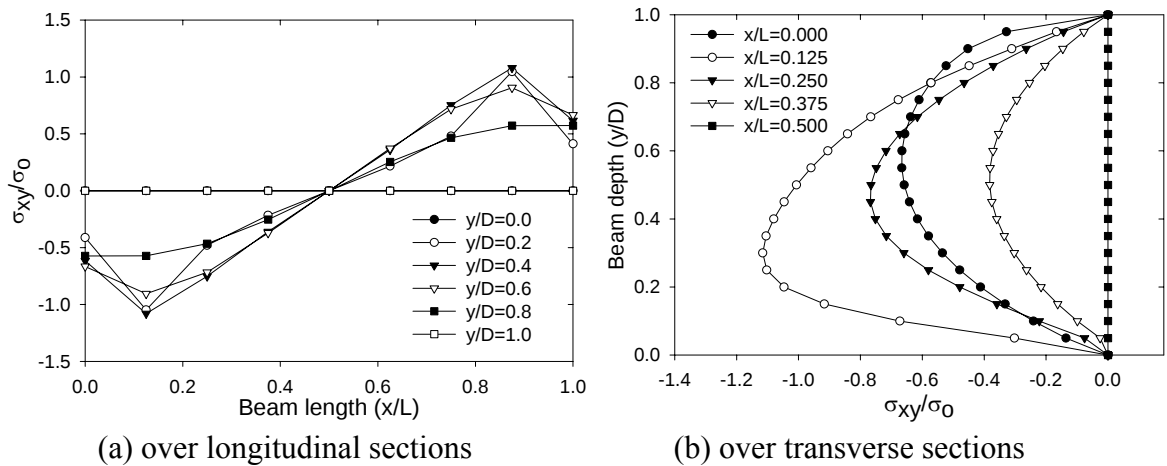


Fig. 3.27 Shear stress distribution for unguided consideration, $L/D = 2$:
(a) over longitudinal sections, (b) over transverse sections

3.3 Stress Function Solution of a Simply Supported Beam

3.3.1 Particular Assumptions

To solve the problem of a simply supported deep beam with uniformly distributed load (σ_o per unit area) of length L , depth D and thickness W (taken as unity) following assumptions are necessary:

- The supports act just at the end points of the beam symmetrically, thereby reactions are equal ($\sigma_o L/2$) on each end.
- No axial force is acting at the ends of the beam where the supports are located causing the bending moment to be zero at points of the support.
- Supports do exert only shear stresses ($\sigma_o L/2$) at end edges of the beam (over the lateral boundary).

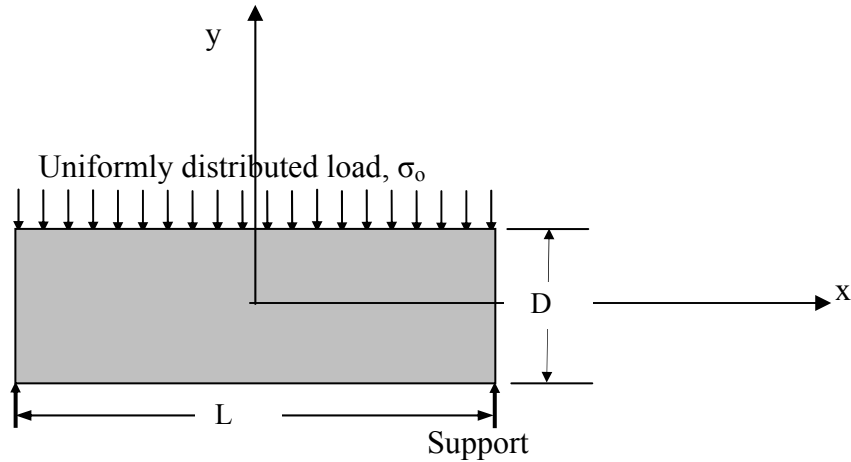


Fig. 3.28 Configuration of unguided beam for stress function solution

3.3.2 Boundary conditions

$$\text{At the bottom boundary } (y = -\frac{D}{2}): \quad \sigma_{yy} = 0 \quad \text{and} \quad \sigma_{xy} = 0 \quad (3.29a)$$

$$\text{At the top boundary } (y = +\frac{D}{2}): \quad \sigma_{yy} = -\sigma_o \quad \text{and} \quad \sigma_{xy} = 0 \quad (3.29b)$$

At lateral boundaries ($x = \pm L/2$):

$$F_x(L/2) = \int_{-D/2}^{D/2} \sigma_{xx}(a, y) dy = 0 \quad (3.29c)$$

$$M_x(L/2) = \int_{-D/2}^{D/2} \sigma_{xx}(a, y) y dy = 0 \quad (3.29d)$$

$$F_y(L/2) = \int_{-D/2}^{D/2} \sigma_{xy}(a, y) dy = \frac{\sigma_o L}{2} \quad (3.29e)$$

3.3.3 Analytical solution

Since the traction load on the top surface is uniformly distributed i.e varies linearly with x^0 , the polynomial of stress function may be considered as:

$$\begin{aligned} \phi = & C_1 x^2 + C_2 xy + C_3 y^2 + C_4 x^3 + C_5 x^2 y + C_6 xy^2 + C_7 y^3 + C_8 x^4 + C_9 x^3 y + C_{10} x^2 y^2 \\ & + C_{11} xy^3 + C_{12} y^4 + C_{13} x^5 + C_{14} x^4 y + C_{15} x^3 y^2 + C_{16} x^2 y^3 + C_{17} xy^4 + C_{18} y^5 \end{aligned} \quad (3.30)$$

Differentiating the equation (3.30) with respect to x and y up to fourth order following expressions are obtained.

$$\frac{\partial^2 \phi}{\partial x^2} = 2C_1 + 6C_4 x + 2C_5 y + 12C_8 x^2 + 6C_9 xy + 2C_{10} y^2 + 20C_{13} x^3 + 12C_{14} x^2 y + 6C_{15} xy^2 + 2C_{16} y^3$$

$$\frac{\partial^4 \phi}{\partial x^4} = 24C_8 + 120C_{13} x + 24C_{14} y$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = C_2 + 2C_5 x + 2C_6 y + 3C_9 x^2 + 4C_{10} xy + 3C_{11} y^2 + 4C_{14} x^3 + 6C_{15} x^2 y + 6C_{16} xy^2 + 4C_{17} y^3$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 4C_{10} + 12C_{15} x + 12C_{16} y$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2C_3 + 2C_6 x + 6C_7 y + 2C_{10} x^2 + 6C_{11} xy + 12C_{12} y^2 + 2C_{15} x^3 + 6C_{16} x^2 y + 12C_{17} xy^2 + 20C_{18} y^3$$

$$\frac{\partial^4 \phi}{\partial y^4} = 24C_{12} + 24C_{17} x + 120C_{18} y$$

Substituting the above derivatives of Φ into the relations of stress components as defined by Airy (2.20), the following expressions are obtained

$$\sigma_{xx} = 2C_3 + 2C_6 x + 6C_7 y + 2C_{10} x^2 + 6C_{11} xy + 12C_{12} y^2 + 2C_{15} x^3 + 6C_{16} x^2 y + 12C_{17} xy^2 + 20C_{18} y^3 \quad (3.31a)$$

$$\sigma_{yy} = 2C_1 + 6C_4x + 2C_5y + 12C_8x^2 + 6C_9xy + 2C_{10}y^2 + 20C_{13}x^3 + 12C_{14}x^2y + 6C_{15}xy^2 + 2C_{16}y^3 \quad (3.31b)$$

$$\sigma_{xy} = -(C_2 + 2C_5x + 2C_6y + 3C_9x^2 + 4C_{10}xy + 3C_{11}y^2 + 4C_{14}x^3 + 6C_{15}x^2y + 6C_{16}xy^2 + 4C_{17}y^3) \quad (3.31c)$$

Again using the bi-harmonic partial differential equation for stress function (2.21), following equation is obtained:

$$24C_8 + 120C_{13}x + 24C_{14}y + 2(4C_{10} + 12C_{15}x + 12C_{16}y) + 24C_{12} + 24C_{17}x + 120C_{18}y = 0$$

or, $(120C_{13} + 24C_{15} + 24C_{17})x + (24C_{14} + 24C_{16} + 120C_{18})y + (24C_8 + 8C_{10} + 24C_{12}) = 0 \quad (3.32)$

To have the Eq. (3.32) valid for all the values of x and y following three constraint equations are to be satisfied.

$$\left. \begin{aligned} 120C_{13} + 24C_{15} + 24C_{17} &= 0 \\ 24C_{14} + 24C_{16} + 120C_{18} &= 0 \\ 24C_8 + 8C_{10} + 24C_{12} &= 0 \end{aligned} \right\} \quad (3.33)$$

For the boundary conditions at top edge of the beam ($y=D/2$)

$$2C_1 + 6C_4x + 2C_5D + 12C_8x^2 + 3C_9xD + (1/2)C_{10}D^2 + 20C_{13}x^3 + 6C_{14}x^2D + (3/2)C_{15}xD^2 + (1/4)C_{16}D^3 = -\sigma_o \quad (3.34a)$$

$$C_2 + 2C_5x + C_6D + 3C_9x^2 + 2C_{10}xD + (3/4)C_{11}D^2 + 4C_{14}x^3 + 3C_{15}x^2D + (3/2)C_{16}xD^2 + (1/2)C_{17}D^3 = 0 \quad (3.34b)$$

The equations of (3.34) are to be satisfied with all values of x, which can provide the following two sets of equations, which can be solved simultaneously to find out the values of coefficients.

$$\left. \begin{aligned} 20C_{13} &= 0 \\ 12C_8 + 6C_{14}D &= 0 \\ 6C_4 + 3C_9D + (3/2)C_{15}D^2 &= 0 \\ 2C_1 + 2C_5D + (1/2)C_{10}D^2 + (1/4)C_{16}D^3 &= -\sigma_o \end{aligned} \right\} \quad (3.35a)$$

$$\left. \begin{aligned} 4C_{14} &= 0 \\ 3C_9 + 3C_{15}D &= 0 \\ 2C_5 + 2C_{10}D + (3/2)C_{16}D^2 &= 0 \\ C_2 + C_6D + (3/4)C_{11}D^2 + (1/2)C_{17}D^3 &= 0 \end{aligned} \right\} \quad (3.35b)$$

Similarly for the boundary conditions at the bottom edge of the beam ($y = -D/2$)

$$\left. \begin{aligned} 20C_{13} &= 0 \\ 12C_8 - 6C_{14}D &= 0 \\ 6C_4 - 3C_9D + (3/2)C_{15}D^2 &= 0 \\ 2C_1 - 2C_5D + (1/2)C_{10}D^2 - (1/4)C_{16}D^3 &= 0 \end{aligned} \right\} \quad (3.36a)$$

$$\left. \begin{aligned} 4C_{14} &= 0 \\ 3C_9 - 3C_{15}D &= 0 \\ 2C_5 - 2C_{10}D + (3/2)C_{16}D^2 &= 0 \\ C_2 - C_6D + (3/4)C_{11}D^2 - (1/2)C_{17}D^3 &= 0 \end{aligned} \right\} \quad (3.36b)$$

Substituting $x=L/2$ in equations (3.31a) and (3.31c), the expressions for bending and shearing stresses at the lateral boundaries are found respectively as follows.

$$\sigma_{xx}(L/2) = 2C_3 + C_6L + 6C_7y + (1/2)C_{10}L^2 + 3C_{11}Ly + 12C_{12}y^2 + (1/4)C_{15}L^3 + (3/2)C_{16}L^2y + 6C_{17}Ly^2 + 20C_{18}y^3 \quad (3.37a)$$

$$\sigma_{xy} = - \left(C_2 + C_5L + 2C_6y + (3/4)C_9L^2 + 2C_{10}Ly + 3C_{11}y^2 + (1/2)C_{14}L^3 + (3/2)C_{15}L^2y + 3C_{16}Ly^2 + 4C_{17}y^3 \right) \quad (3.37b)$$

Substituting equation (3.37a) in the expression for bending force and moment equations

mentioned as the lateral boundary conditions i.e. $F_x(L/2) = \int_{-D/2}^{D/2} \sigma_{xx}(a, y) dy = 0$ and

$M_x(L/2) = \int_{-D/2}^{D/2} \sigma_{xx}(a, y) y dy = 0$ respectively and evaluating the integrals, following

equations are obtained.

$$2C_3D + C_6LD + (1/2)C_{10}L^2D + C_{12}D^3 + (1/4)C_{15}L^3D + (1/2)C_{17}LD^3 = 0 \quad (3.38)$$

$$(1/2)C_7D^3 + (1/4)C_{11}LD^3 + (1/8)C_{16}L^2D^3 + (1/4)C_{18}D^5 = 0 \quad (3.39)$$

Similarly substituting equation (3.37b) in the expression for shearing force equation mentioned as the lateral boundary condition, i.e., $F_y(L/2) = \int_{-D/2}^{D/2} \sigma_{xy}(a, y) dy = \frac{\sigma_o L}{2}$ and evaluating the integrals, following equations are obtained.

$$-\{C_2 D + (1/2)C_5 L D + (3/4)C_9 L^2 D + (1/4)C_{11} D^3 + (1/2)C_{14} L^3 D + (1/4)C_{16} L D^3\} = \frac{\sigma_o L}{2} \quad (3.40)$$

Solving the equations of (3.33), (3.35), (3.36), (3.38), (3.39) and (3.40) for the unknown arbitrary constants, we can find

$$C_2 = C_3 = C_4 = C_6 = C_8 = C_9 = C_{10} = C_{11} = C_{12} = C_{13} = C_{14} = C_{15} = C_{17} = 0$$

$$C_1 = -\frac{\sigma_o}{4}, \quad C_5 = \frac{3\sigma_o}{4D}, \quad C_7 = \frac{\sigma_o}{D^3} \left(\frac{L^2}{4} - \frac{D^2}{10} \right), \quad C_{16} = -\frac{\sigma_o}{D^3}, \quad C_{18} = \frac{\sigma_o}{5D^3}$$

Substituting the above values of constants in Eqs. (3.30) and (3.31) following expressions are found for stress function and stress components:

$$\phi(x, y) = \left\{ -\frac{\sigma_o L^2}{4} + \left(\frac{3\sigma_o L^2}{4} \right) \left(\frac{y}{D} \right) - \sigma_o L^2 \left(\frac{y}{D} \right)^3 \right\} \left(\frac{x}{L} \right)^2 + \sigma_o \left(\frac{L^2}{4} - \frac{D^2}{10} \right) \left(\frac{y}{D} \right)^3 + \left(\frac{\sigma_o D^2}{5} \right) \left(\frac{y}{D} \right)^5 \quad (3.41)$$

$$\sigma_{xx}(x, y) = 6\sigma_o \left(\frac{L}{D} \right)^2 \left(1 - \left(\frac{x}{L} \right)^2 \right) \left(\frac{y}{D} \right) + 4\sigma_o \left(\frac{y}{D} \right)^3 - \frac{3\sigma_o}{5} \left(\frac{y}{D} \right) \quad (3.42a)$$

$$\sigma_{yy}(x, y) = -\sigma_o \left[2 \left(\frac{y}{D} \right)^3 - \frac{3}{2} \left(\frac{y}{D} \right) + \frac{1}{2} \right] \quad (3.42b)$$

$$\sigma_{xy}(x, y) = -6\sigma_o \left(\frac{L}{D} \right) \left[\frac{1}{4} - \left(\frac{y}{D} \right)^2 \right] \left(\frac{x}{L} \right) \quad (3.42c)$$

Now the stress-strain and strain-displacement relation for the element available in the text book may be applied to obtain the displacement of the beam. In the case of plane stress distribution, following equations are considered [1].

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = \frac{1}{E} [\sigma_{xx} - \mu \sigma_{yy}] \quad (3.43a)$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = \frac{1}{E} [\sigma_{yy} - \mu \sigma_{xx}] \quad (3.43b)$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{2(1+\mu)}{E} \sigma_{xy} \quad (3.43c)$$

Substituting the expression for bending and normal stress of equations (3.42a) and (3.42b) in the relation of axial displacement and stresses [Eq. (3.43a)], and integrating

$$u_x(x, y) = \frac{6\sigma_o}{ED^3} \left[\frac{L^2}{4} - \frac{x^2}{3} + \frac{2y^2}{3} - \frac{D^2}{10} \right] xy + \frac{6\sigma_o\mu}{ED^3} \left[\frac{y^3}{3} - \frac{D^2y}{4} + \frac{D^3}{12} \right] x + f(y) \quad (3.44)$$

where $f(y)$ is the integrating solitary function of y . Assuming that the u_x is zero at the mid section ($x=0, y=0$), $f(y)$ is found to be zero. Thus the expression of the horizontal displacement component becomes

$$u_x(x, y) = \frac{6\sigma_o}{ED^3} \left[\frac{L^2}{4} - \frac{x^2}{3} + \frac{2y^2}{3} - \frac{D^2}{10} \right] xy + \frac{6\sigma_o\mu}{ED^3} \left[\frac{y^3}{3} - \frac{D^2y}{4} + \frac{D^3}{12} \right] x$$

In normalised form it can be written as

$$u_x(x, y) = 6 \left(\frac{\sigma_o}{E} \right) \left[\left(\frac{L}{D} \right)^3 \left\{ \frac{1}{4} \left(\frac{x}{L} \right) - \frac{1}{3} \left(\frac{x}{L} \right)^3 \right\} y + \left\{ \left(\frac{2}{3} + \frac{\mu}{3} \right) \left(\frac{y}{D} \right)^3 - \left(\frac{1}{10} + \frac{\mu}{4} \right) \left(\frac{y}{D} \right) + \frac{\mu}{12} \right\} x \right] \quad (3.45)$$

Again expression for bending and normal stress of equations (3.42a) and (3.42b) in the relation of lateral displacement and stresses [Eq. (3.43b)], and integrating

$$u_y(x, y) = -\frac{6q}{ED^3} \left[\frac{y^3}{12} - \frac{D^2y}{8} + \frac{D^3}{12} \right] y - \frac{3q\mu}{ED^3} \left[\left(\frac{L^2}{4} - x^2 \right) - \frac{y^2}{3} + \frac{D^2}{10} \right] y^2 + g(x) \quad (3.46)$$

where $g(x)$ is the integrating solitary function of x .

Applying equations (3.42c), (45) and (46) in the relation of shearing stress and strain [Eq.(3.43c)], following expression is found.

$$\begin{aligned} & \frac{6\sigma_o}{ED^3} \left[\frac{L^2}{4} - \frac{x^2}{3} + \frac{2y^2}{1} - \frac{D^2}{10} \right] x + \frac{6\sigma_o\mu}{ED^3} \left[\frac{y^2}{1} - \frac{D^2}{4} \right] x \\ & - \frac{6\sigma_o\mu}{ED^3} xy^2 + \frac{\partial g(x)}{\partial x} = - \frac{12(1+\mu)\sigma_o}{ED^3} \left[\frac{D^2}{4} - y^2 \right] x \end{aligned} \quad (3.47)$$

Integrating equation (3.47) with respect to x,

$$g(x) = - \frac{6\sigma_o}{ED^3} \left[\frac{L^2 x^2}{8} - \frac{x^4}{12} - \frac{D^2 x^2}{20} + (2+\mu) \frac{D^2 x^2}{8} \right] + C \quad (3.48)$$

where C is the constant for integration.

Using equations (3.46) and (3.48)

$$\begin{aligned} u_y(x, y) = & - \frac{6\sigma_o}{ED^3} \left[\frac{y^3}{12} - \frac{D^2 y}{8} + \frac{D^3}{12} \right] y - \frac{3\sigma_o\mu}{ED^3} \left[\left(\frac{L^2}{4} - x^2 \right) - \frac{y^2}{3} + \frac{D^2}{10} \right] y^2 \\ & - \frac{6\sigma_o}{ED^3} \left[\frac{L^2 x^2}{8} - \frac{x^4}{12} - \frac{D^2 x^2}{20} + (2+\mu) \frac{D^2 x^2}{8} \right] + C \end{aligned} \quad (3.49)$$

Since the lateral displacement on the centre line (y=0) is considered as the deflection of the beam, equation (3.49) for deflection can be written as

$$(u_y)_{y=0} = - \frac{6\sigma_o}{ED^3} \left[\frac{L^2 x^2}{8} - \frac{x^4}{12} - \frac{D^2 x^2}{20} + (2+\mu) \frac{D^2 x^2}{8} \right] + C \quad (3.50)$$

For the simply supported condition, the deflection at the end of the beam (at x=±L/2) is assumed to be zero. Then

$$C = \frac{6\sigma_o}{ED^3} \left[\frac{5L^4}{192} + \frac{D^2 L^2}{20} + \frac{\mu D^2 x^2}{32} \right] \quad (3.51)$$

Therefore, the lateral displacement component can be given by

$$\begin{aligned} u_y(x, y) = & - \frac{6\sigma_o}{ED^3} \left[\frac{y^3}{12} - \frac{D^2 y}{8} + \frac{D^3}{12} \right] y - \frac{3\sigma_o\mu}{ED^3} \left[\left(\frac{L^2}{4} - x^2 \right) - \frac{y^2}{3} + \frac{D^2}{10} \right] y^2 \\ & - \frac{6\sigma_o}{ED^3} \left[\frac{L^2 x^2}{8} - \frac{x^4}{12} - \frac{D^2 x^2}{20} + (2+\mu) \frac{D^2 x^2}{8} \right] + \frac{6\sigma_o}{ED^3} \left[\frac{5L^4}{192} + \frac{D^2 L^2}{20} + \frac{\mu D^2 x^2}{32} \right] \end{aligned} \quad (3.52a)$$

The normalised form of expression for the lateral displacement component is then

$$u_y(x,y) = -6\left(\frac{\sigma_o}{E}\right)\left\{\frac{1}{12}\left(\frac{y}{D}\right)^3 - \frac{1}{8}\left(\frac{y}{D}\right) + \frac{1}{12}\right\}y - \frac{\mu}{2}\left\{\frac{1}{4} - \left(\frac{x}{L}\right)\right\}\left\{\left(\frac{y}{D}\right)\left(\frac{L}{D}\right)^2 y + \frac{\mu}{6}\left(\frac{y}{D}\right)^3 y - \frac{\mu}{20}\left(\frac{y}{D}\right)y\right. \\ \left. - \left(\frac{L}{D}\right)^3\left\{\frac{1}{8}\left(\frac{x}{L}\right) - \frac{1}{12}\left(\frac{x}{L}\right)^3 + \left(\frac{1}{5} + \frac{\mu}{8}\right)\left(\frac{D}{L}\right)^2\left(\frac{x}{L}\right)\right\}x + L\left(\frac{L}{D}\right)^3\left\{\frac{5}{192} + \frac{1}{20}\left(\frac{D}{L}\right)^2 + \frac{\mu}{32}\left(\frac{x}{L}\right)^2\left(\frac{D}{L}\right)^2\right\}\right\} \quad (3.52b)$$

The deflection of the beam at the centre line of the beam becomes

$$(u_y)_{y=0} = 6L\left(\frac{\sigma_o}{E}\right)\left(\frac{L}{D}\right)^3\left[\frac{1}{12}\left\{\frac{5}{16} - \frac{3}{2}\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^4\right\} + \frac{1}{4}\left(\frac{D}{L}\right)^2\left\{\frac{1}{4} - \left(\frac{x}{L}\right)\right\}\left\{\frac{4}{5} + \frac{\mu}{2}\right\}\right] \quad (3.53)$$

3.4 Comparison of ψ -Solution with Stress Function and Numerical Solutions

The present problem of unguided beam is solved using potential approach taking a segment as guideless from a long beam of guided ends. From the discussions of the above paragraphs it is understandable that the result of such an especial consideration for both the cases of support location is not giving any unrealistic solution. The comparison study on the results of the present ψ -solution with those obtained by the stress function (Φ) and finite element method (FEM) would give the basis of the aforesaid statement. Here the FEM solutions are obtained using the standard facilities of the commercial software ANSYS taking equal size elements of 20 x 20 grids for the considered portion of the beam span.

As appears from figure 3.29 and table 3.4, the displacement potential approach would provide the results of bending stress distribution (at the sections of $x/L=0.25$ and 0.5) reasonably close to that of Φ -function and FEM. However, ψ -solution seems to be deviated little from other two solutions in the lower portion of the beam; but ψ -solution presents a very identical result to that of FEM on the upper portion of the beam. Here, Φ -solution is rather deviated away from the other two.

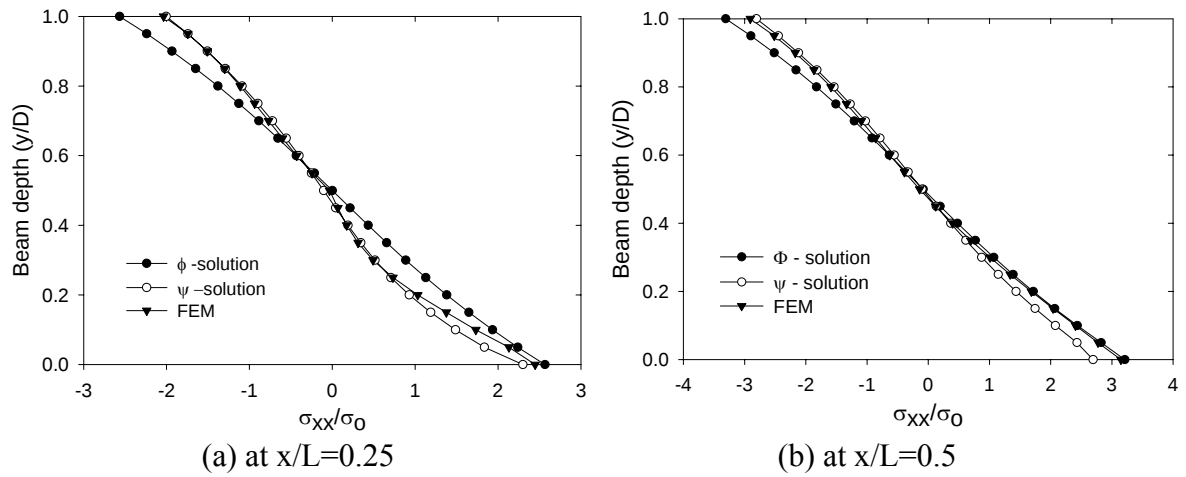


Fig. 3.29 Comparison of bending stress amongst ψ -solutions with those of stress function and FEM, $L/D=2$: (a) at $x/L=0.25$, (b) at $x/L=0.5$

Table 3.4 Bending Stress Comparison at $x/L=0.25$ and $x/L=0.5$

Approach	at $x/L=0.25$		at $x/L=0.5$	
	$y/D=0$	$y/D=1$	$y/D=0$	$y/D=1$
Φ -soln	2.57	-2.57	3.21	-3.30
ψ -soln	2.30	-2.00	2.69	-2.80
FEM	2.45	-2.04	3.15	-2.91

Fig. 3.30 and table 3.5 present the lateral stress distribution along with the solutions of stress function and FEM (at the sections of $x/L=0.25$ and 0.5). The average deviation in lateral stress magnitude between ψ -solution and FEM is 0.11. The average deviation in lateral stress magnitude between Φ -solution and FEM is 0.12. On the basis of this average value someone may consider that stress function solution is similar to the solution of ψ -function. But the lateral stress distribution pattern for ψ -solution and Φ -solution are not the same. The ψ -solution gives closer pattern to that of FEM, but Φ -solution displays more curvilinear pattern. The deviation on FEM and ψ -solution remains almost steady, whereas deviation on FEM and Φ -solution changes from a negative to a positive magnitude. As a result there is an apparent steadiness in the average deviation value on Φ -solution and ψ -solution.

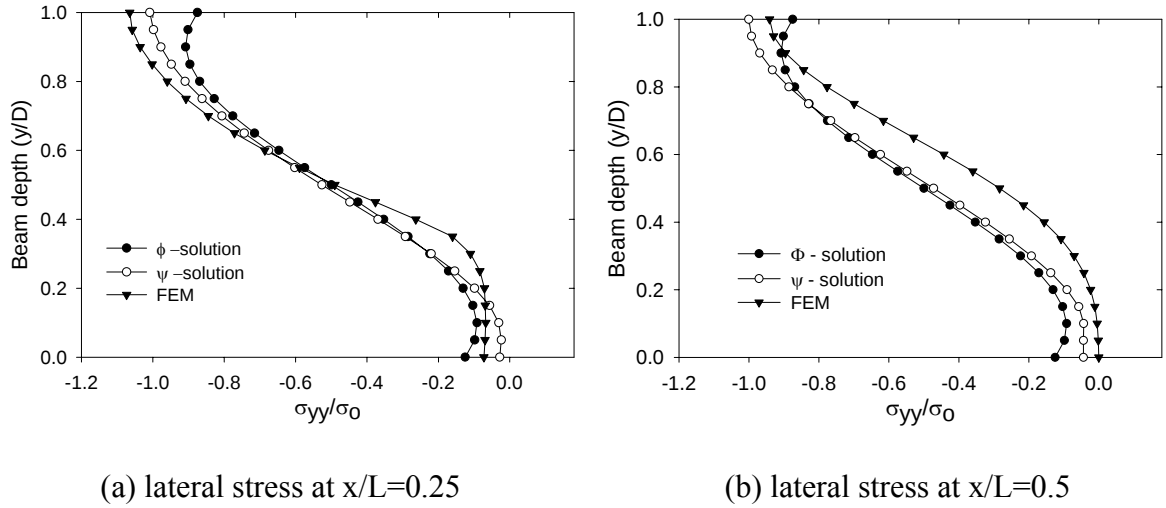


Fig. 3.30 Comparison of lateral stress amongst ψ -solutions with those of stress function and FEM, $L/D=2$: (a) at $x/L=0.25$, (b) at $x/L=0.5$

Table 3.5 Lateral Stress Comparison at $x/L=0.25$ and $x/L=0.5$

Approach	at $x/L=0.25$		at $x/L=0.5$	
	$y/D=0$	$y/D=1$	$y/D=0$	$y/D=1$
Φ -soln	-0.12	-0.87	-0.12	-0.87
ψ -soln	-0.02	-1.00	-0.02	-1.00
FEM	-0.04	-1.03	-0.00	-0.94

All three approaches provide almost zero shearing stress at the top and bottom edges, but the support ends are not free from shearing stress in this case of unguided beam as it is observed for guided ends. Fig. 3.31 (a) and (b) and table 3.6 represent the shear stresses at $x/L=0.0$ and $x/L=0.25$. The transverse distribution of shearing stress (σ_{xy}) is parabolic as per the solution of stress function at all the sections of the beam. But it is not fully obeying the parabolic profile pattern at the section of $x/L=0.0$ as per displacement potential solution. The FEM solution is more different from stress function pattern. However, while moving away with respect to support the displacement potential solution is becoming very close to stress function solution whereas FEM solution remains still of different pattern. At the bottom half of the beam FEM solution takes a little positive notion at first then moves towards the negative direction and forms a parabola for the rest portion of the beam. In no where FEM gives the pure parabolic pattern of shearing stress distribution in the solution of the present problem.

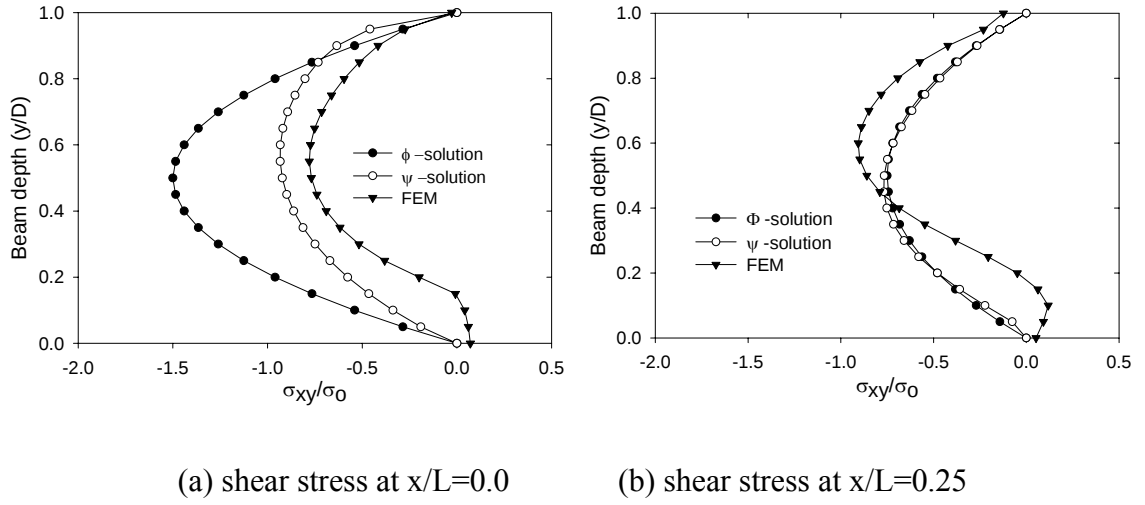


Fig. 3.31 Shear stress comparison amongst ψ -solutions with stress function and FEM, $L/D=2$: (a) at $x/L=0.0$, (b) at $x/L=0.25$

Table 3.6 Shear Stress Comparison at $x/L=0.0$ and $x/L=0.25$

Approach	at $x/L=0.0$			at $x/L=0.25$		
	$y/D=0$	$y/D=1$	σ_{xx} max	$y/D=0$	$y/D=1$	σ_{xx} max
Φ -soln	0.00	0.00	1.50	0.00	0.00	0.75
ψ -soln	0.00	0.00	0.90	0.00	0.00	0.76
FEM	0.08	0.00	0.75	0.05	-0.10	0.90

Actually the deflection in displacement potential approach and FEM is positive at the ends and negative at the mid-span, whereas it is negative in the case of stress function solution over the whole span. So for comparison the results of displacement potential approach and FEM for both the ends of the span are shifted to zero by linear transformation and the graphs are drawn accordingly. Now as appears from Fig. 3.32, the deflection of the unguided beam is almost similar in all three approaches. However, the maximum deflection ($u_y \times 10^4$) occurred at the mid-span for stress function solution, displacement potential approach and FEM are 0.75, 0.62 and 0.67 respectively. Therefore, any of the three approaches can be used for estimating the deflection of an unguided beam.

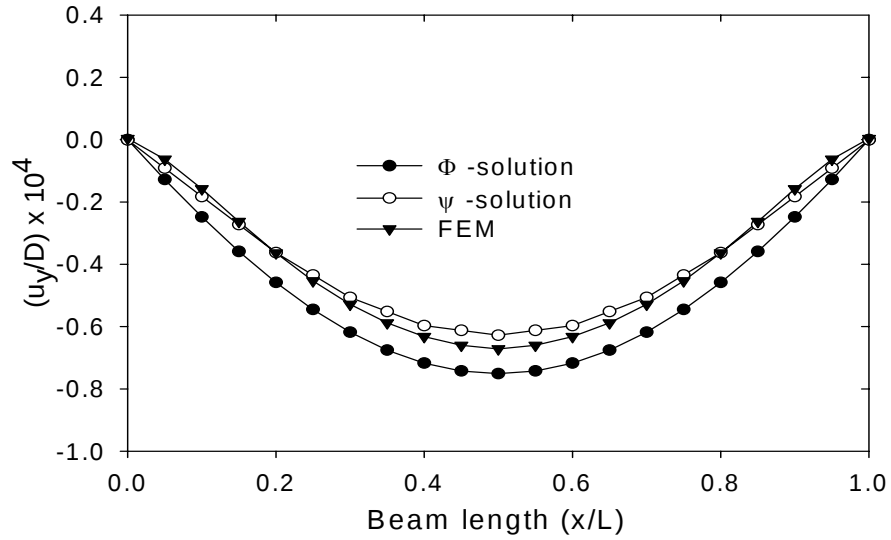


Fig. 3.32 Comparison of lateral displacement amongst ψ -solutions with the solution of stress function and FEM [$y/D=0.5$]

3.5 Summary

Analytical solution of the elastic field of a guided deep beam is investigated using displacement potential approach. The solution of the present mixed boundary-value elastic problem is obtained by satisfying all the physical conditions of the beam appropriately. The guided ends of the beam are identified to be the most critical sections in terms of stresses. The stress distribution at the load transition region is also found to be of critical in nature. The comparative study reveals that the classical theory of bending is simply inadequate to predict the stress and displacements fields of the guided beam. However, the solutions obtained by FEM are found to approach towards the present analytical solution. While the aspect ratio is varied from 1 to 4, the non-linearity of both bending and shearing stress distributions are also changing remarkably. This variation is also observed in FEM solution but not in classical Bernoulli- Euler beam theory. With lower aspect ratio the maximum magnitude of shear stress is not at the centreline of the beam as observed for higher aspect ratio. In beam theory bending stress distribution is always linear and shear stress parabolic, which do not reflect the reality of the problem for short beam.

While the analytical solutions of a guided deep beam subjected to uniformly distributed load on the top edge and parabolic reaction at the bottom two corners are done using displacement potential approach satisfying all the physical conditions of the beam appropriately, the guided ends and the load transition regions of the beam are again identified to be the most critical sections in terms of stresses. However, bending and lateral stress concentrations are found to be less in the beam of parabolic support than that of distributed support. But the concentration of shear stress is increased a little bit in comparison to uniform support.

Analytical solution of an unguided deep beam is also investigated using a special technique of displacement potential approach taking the St. Venant's principle into account. Analytical solution for the unguided deep beam under distributed loading condition is also obtained using stress function approach. The comparative study reveals that both the approaches may work satisfactorily in this regard. Therefore, the displacement potential solution can easily acts as a good replacement of stress function solution for an unguided simply supported beam under distribution loading.

CHAPTER 4

SOLUTION TO COMPOSITE DEEP BEAMS

4.1 Analytical Solution of Guided Composite Deep Beam

4.1.1 Problem Articulation

A rectangular section deep orthotropic composite beam of length L , wide D and thickness W (considered to be unity for plane stress condition) is subjected to a uniformly distributed load at its top boundary and uniformly distributed simply supporting at its bottom two corner. The beam is in equilibrium with loading σ_0 per unit length from $0.1L$ to $0.9L$ at the top side ($y=D$) and balanced by loading (reaction) distributed from $x=0$ to $0.1L$ and $x=0.9L$ to L as shown in figure 4.1. Its guided lateral edges at $x=0$ and $x=L$ are simulated here by the roller guided boundaries so that the axial displacements are restrained, but the lateral displacements are free to envisage any value. The fibre of the composite material is assumed to be parallel to the x -axis.

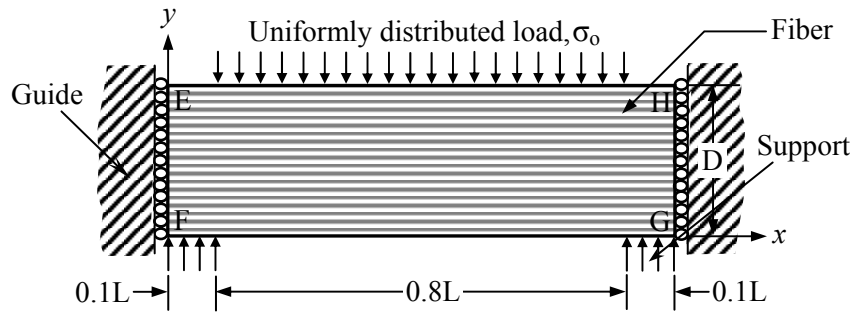


Fig. 4.1 Analytical model of an orthotropic guided beam

4.1.2 Boundary Conditions

The physical conditions of the present problem with reference to Fig. 4.1 are to be satisfied along the all four boundaries of the beam. Due to the guides located at both the ends the beam can not have elongation over lateral edges. At the same time there can be no shear stress on lateral edges since the guide allows the beam to be free for lateral dislocation. So the conditions at the guided boundaries are of mixed mode. The displacement conditions of

top and bottom boundaries are not known and to be solved out. On these surfaces the applied load and reactions are related to the boundary conditions within their affecting region. Again there is no shearing effect over the top and bottom boundaries of the aforesaid beam. The boundary conditions of guided composite beam considered for the present problem can be expressed mathematically as follows:

- (a) Guided end, EF: There is no axial displacement and shearing stress. Thus,
 $u_x(0, y) = 0$ and $\sigma_{xy}(0, y) = 0$ $[0 \leq y \leq D]$.
- (b) Guided end, HG: There is no axial displacement and shearing stress. Thus,
 $u_x(L, y) = 0$ and $\sigma_{xy}(L, y) = 0$ $[0 \leq y \leq D]$
- (c) Loaded boundary, EH: There is no shearing stress. But lateral stresses are there in the regions of applied loads, which is the function of load intensity. Thus,
 $\sigma_{xy}(x, D) = 0$ $[0 \leq x \leq L]$ and $\sigma_{yy}(x, D) = \sigma_0$ $[0.1L \leq x \leq 0.9L]$
- (d) Supporting end, FG: This is the supporting boundary whose total reaction force is to be the algebraic sum of reactions acting on both the supports and eventually same as to the applied force at the opposite site for equilibrium condition. The uniform load is distributed over 80% of beam length (from $x/L = 0.1$ to $x/L = 0.9$) and support reaction is distributed over 20% of beam length ($x/L = 0.0$ to 0.1 and $x/L = 0.9$ to 1.0). Therefore, $\sigma_{xy}(x, 0) = 0$ $[0 \leq x \leq L]$; $\sigma_{yy}(x, 0) = 4\sigma_0$ $[0 \leq x \leq 0.1L \text{ \& } 0.9L \leq x \leq L]$ and $\sigma_{yy}(x, 0) = 0$ $[0.1L < x < 0.9L]$

4.1.3 Analytical Solution

Mathematical model here is the partial differential equation derived from the equations of equilibrium and equations of compatibility based on Displacement Potential Function $\psi(x, y)$ obtained from Eq. (2.30) as follows.

$$E_1 G_{12} \frac{\partial^4 \psi}{\partial x^4} + E_2 (E_1 - 2\mu_{12} G_{12}) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_2 G_{12} \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (4.1)$$

In this case the displacement and stress components are also obtained from Eq. (2.29) as follows:

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y} \quad (4.2a)$$

$$u_y(x,y) = -\frac{1}{Z_{11}} \left[E_1^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12} (E_1 - \mu_{12}^2 E_2) \frac{\partial^2 \psi}{\partial y^2} \right] \quad (4.2b)$$

$$\sigma_{xx}(x,y) = \frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial y^3} \right] \quad (4.2c)$$

$$\sigma_{yy}(x,y) = \frac{E_1 E_2}{Z_{11}} \left[(\mu_{12} G_{12} - E_1) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12} \frac{\partial^3 \psi}{\partial y^3} \right] \quad (4.2d)$$

$$\sigma_{xy}(x,y) = -\frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^3} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \quad (4.2e)$$

$$\text{Where } Z_{11} = \mu_{12} E_1 E_2 + G_{12} (E_1 - \mu_{12}^2 E_2)$$

The displacement potential trial function is assumed in such a way that the boundary conditions at the guided edges are satisfied instantly. Actually, the trial function should be in terms of cosine function so that its first derivative and third derivative with respect to x can be found in terms of sine function. By this way the requirement of physical conditions of the two opposing guided ends are automatically satisfied, i.e., automatic satisfaction of boundary conditions of (a) and (b). At the same time the expression for ψ should also be compatible to the distribution of load on the top boundary and reactions over the supports. Considering all these factors the expression for ψ may be approximated as follows:

$$\psi = \sum_{m=1}^{\infty} Y_m \cos \alpha x + Ky^3 \quad (4.3)$$

where $Y_m = f(y)$, $\alpha = \frac{m\pi}{L}$ and $K = \text{Arbitrary constant}$ and $m = 1, 2, 3, \dots, \infty$.

Derivatives of equation (4.3) with respect to x and y are

$$\frac{\partial \psi}{\partial x} = -\sum_{m=1}^{\infty} Y_m \alpha \sin \alpha x$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\sum_{m=1}^{\infty} Y_m \alpha^2 \cos \alpha x$$

$$\frac{\partial^3 \psi}{\partial x^3} = \sum_{m=1}^{\infty} Y_m \alpha^3 \sin \alpha x$$

$$\frac{\partial^4 \psi}{\partial x^4} = \sum_{m=1}^{\infty} Y_m \alpha^4 \cos \alpha x$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = -\sum_{m=1}^{\infty} Y'_m \alpha \sin \alpha x$$

$$\frac{\partial^3 \psi}{\partial x \partial y^2} = -\sum_{m=1}^{\infty} Y''_m \alpha \sin \alpha x$$

$$\frac{\partial^3 \psi}{\partial x^2 \partial y} = -\sum_{m=1}^{\infty} Y'_m \alpha^2 \cos \alpha x$$

$$\frac{\partial^4 \psi}{\partial x^2 \partial y^2} = -\sum_{m=1}^{\infty} Y''_m \alpha^2 \cos \alpha x$$

$$\frac{\partial \psi}{\partial y} = \sum_{m=1}^{\infty} Y'_m \cos \alpha x + 3Ky^2$$

$$\frac{\partial^2 \psi}{\partial y^2} = \sum_{m=1}^{\infty} Y''_m \cos \alpha x + 6Ky$$

$$\frac{\partial^3 \psi}{\partial y^3} = \sum_{m=1}^{\infty} Y'''_m \cos \alpha x + 6K$$

$$\frac{\partial^4 \psi}{\partial y^4} = \sum_{m=1}^{\infty} Y''''_m \cos \alpha x$$

Using the derivatives of equation (4.3) equation (4.1) yields

$$E_1 G_{12} \sum_{m=1}^{\infty} Y_m \alpha^4 \cos \alpha x - E_2 (E_1 - 2\mu_{12} G_{12}) \sum_{m=1}^{\infty} Y''_m \alpha^2 \cos \alpha x + E_2 G_{12} \sum_{m=1}^{\infty} Y''''_m \cos \alpha x = 0$$

$$E_2 G_{12} \sum_{m=1}^{\infty} \left[Y'''_m - \frac{E_2 (E_1 - 2\mu_{12} G_{12})}{E_2 G_{12}} Y''_m \alpha^2 + \frac{E_1 G_{12}}{E_2 G_{12}} Y_m \alpha^4 \right] \cos \alpha x = 0$$

$$Y'''_m - \frac{E_2 (E_1 - 2\mu_{12} G_{12})}{E_2 G_{12}} Y''_m \alpha^2 + \frac{E_1 G_{12}}{E_2 G_{12}} Y_m \alpha^4 = 0$$

$$Y'''_m - \left(\frac{E_1}{G_{12}} - 2\mu_{12} \right) Y''_m \alpha^2 + \frac{E_1}{E_2} Y_m \alpha^4 = 0 \quad (4.4)$$

The general solution of ordinary differential equation be

$$Y_m = A_m e^{r_1 y} + B_m e^{r_2 y} + C_m e^{r_3 y} + D_m e^{r_4 y} \quad (4.5)$$

where A_m , B_m , C_m and D_m are arbitrary constants

and r_1 , r_2 , r_3 and r_4 are the roots of following equation

$$r^4 - \left(\frac{E_1}{G_{12}} - 2\mu_{12} \right) \alpha^2 r^2 + \frac{E_1}{E_2} \alpha^4 = 0$$

Thus

$$r_1, r_2 = \frac{\alpha}{\sqrt{2}} \left[\left(\frac{E_1}{G_2} - 2\mu_{12} \right) \pm \sqrt{\left(\frac{E_1}{G_2} - 2\mu_{12} \right)^2 - 4 \frac{E_1}{E_2}} \right]^{\frac{1}{2}} \quad (4.6a)$$

$$r_3, r_4 = -\frac{\alpha}{\sqrt{2}} \left[\left(\frac{E_1}{G_2} - 2\mu_{12} \right) \pm \sqrt{\left(\frac{E_1}{G_2} - 2\mu_{12} \right)^2 - 4 \frac{E_1}{E_2}} \right]^{\frac{1}{2}} \quad (4.6b)$$

Now substituting the derivatives of ψ and Y_m using equation (4.3) and (4.5) respectively in the expressions for displacement and stresses (4.2a, 4.2b, 4.2c, 4.2d and 4.2e).

$$\begin{aligned} u_x(x, y) &= \frac{\partial^2 \psi}{\partial x \partial y} \\ &= -\sum_{m=1}^{\infty} Y'_m \alpha \sin \alpha x \\ &= -\sum_{m=1}^{\infty} (A_m r_1 e^{r_1 y} + B_m r_2 e^{r_2 y} + C_m r_3 e^{r_3 y} + D_m r_4 e^{r_4 y}) \alpha \sin \alpha x \end{aligned} \quad (4.7a)$$

$$\begin{aligned} u_y(x, y) &= -\frac{1}{Z_{11}} \left[E_1^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12} (E_1 - \mu_{12}^2 E_2) \frac{\partial^2 \psi}{\partial y^2} \right] \\ &= -\frac{1}{Z_{11}} \left[E_1^2 \left\{ -\sum_{m=1}^{\infty} Y_m \alpha^2 \cos \alpha x \right\} + G_{12} (E_1 - \mu_{12}^2 E_2) \left\{ \sum_{m=1}^{\infty} Y_m'' \cos \alpha x + 6Ky \right\} \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{Z_{11}} \left[E_1^2 \left\{ -\sum_{m=1}^{\infty} (A_m e^{r_1 y} + B_m e^{r_2 y} + C_m e^{r_3 y} + D_m e^{r_4 y}) \alpha^2 \cos \alpha x \right\} + G_{12} (E_1 - \mu_{12}^2 E_2) \right] \\
&\quad \left[\sum_{m=1}^{\infty} (A_m r_1^2 e^{r_1 y} + B_m r_2^2 e^{r_2 y} + C_m r_3^2 e^{r_3 y} + D_m r_4^2 e^{r_4 y}) \cos \alpha x + 6Ky \right] \\
&= \frac{1}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ (E_1^2 \alpha^2 - E_1 G_{12} r_1^2 + \mu_{12}^2 E_2 G_{12} r_1^2) e^{r_1 y} A_m + (E_1^2 \alpha^2 - E_1 G_{12} r_2^2 + \mu_{12}^2 E_2 G_{12} r_2^2) e^{r_2 y} B_m + \right. \right. \\
&\quad \left. \left. (E_1^2 \alpha^2 - E_1 G_{12} r_3^2 + \mu_{12}^2 E_2 G_{12} r_3^2) e^{r_3 y} C_m + (E_1^2 \alpha^2 - E_1 G_{12} r_4^2 + \mu_{12}^2 E_2 G_{12} r_4^2) e^{r_4 y} D_m \right\} \right. \\
&\quad \left. \cos \alpha x - (E_1 G_{12} - \mu_{12}^2 E_2 G_{12}) 6Ky \right] \quad (4.7b)
\end{aligned}$$

$$\begin{aligned}
\sigma_{xx}(x, y) &= \frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial y^3} \right] \\
&= \frac{E_1 G_{12}}{Z_{11}} \left[E_1 \left\{ -\sum_{m=1}^{\infty} Y_m' \alpha^2 \cos \alpha x \right\} - \mu_{12} E_2 \left\{ \sum_{m=1}^{\infty} Y_m''' \cos \alpha x + 6K \right\} \right] \\
&= -\frac{E_1 G_{12}}{Z_{11}} [E_1 \left\{ \sum_{m=1}^{\infty} (A_m r_1 e^{r_1 y} + B_m r_2 e^{r_2 y} + C_m r_3 e^{r_3 y} + D_m r_4 e^{r_4 y}) \alpha^2 \cos \alpha x \right\} + \\
&\quad \mu_{12} E_2 \left\{ \sum_{m=1}^{\infty} (A_m r_1^3 e^{r_1 y} + B_m r_2^3 e^{r_2 y} + C_m r_3^3 e^{r_3 y} + D_m r_4^3 e^{r_4 y}) \cos \alpha x + 6K \right\}] \\
&= -\frac{E_1 G_{12}}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ (E_1 r_1 e^{r_1 y} \alpha^2 + \mu_{12} E_2 r_1^3 e^{r_1 y}) A_m + (E_1 r_2 e^{r_2 y} \alpha^2 + \mu_{12} E_2 r_2^3 e^{r_2 y}) B_m + \right. \right. \\
&\quad \left. \left. (E_1 r_3 e^{r_3 y} \alpha^2 + \mu_{12} E_2 r_3^3 e^{r_3 y}) C_m + (E_1 r_4 e^{r_4 y} \alpha^2 + \mu_{12} E_2 r_4^3 e^{r_4 y}) D_m \right\} \cos \alpha x + 6K \mu_{12} E_2 \right] \quad (4.7c)
\end{aligned}$$

$$\begin{aligned}
\sigma_{yy}(x, y) &= \frac{E_1 E_2}{Z_{11}} \left[(\mu_{12} G_{12} - E_1) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12} \frac{\partial^3 \psi}{\partial y^3} \right] \\
&= \frac{E_1 E_2}{Z_{11}} \left[(\mu_{12} G_{12} - E_1) \left\{ -\sum_{m=1}^{\infty} Y_m' \alpha^2 \cos \alpha x \right\} - G_{12} \left\{ \sum_{m=1}^{\infty} Y_m''' \cos \alpha x + 6K \right\} \right] \\
&= -\frac{E_1 E_2}{Z_{11}} [(\mu_{12} G_{12} - E_1) \left\{ \sum_{m=1}^{\infty} (A_m r_1 e^{r_1 y} + B_m r_2 e^{r_2 y} + C_m r_3 e^{r_3 y} + D_m r_4 e^{r_4 y}) \alpha^2 \cos \alpha x \right\} + \\
&\quad G_{12} \left\{ \sum_{m=1}^{\infty} (A_m r_1^3 e^{r_1 y} + B_m r_2^3 e^{r_2 y} + C_m r_3^3 e^{r_3 y} + D_m r_4^3 e^{r_4 y}) \cos \alpha x + 6K \right\}]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{E_1 E_2}{Z_{11}} \sum_{m=1}^{\infty} \left\{ (\mu_{12} G_{12} r_1 e^{r_1 y} \alpha^2 - E_1 r_1 e^{r_1 y} \alpha^2 + G_{12} r_1^3 e^{r_1 y}) A_m + (\mu_{12} G_{12} r_2 e^{r_2 y} \alpha^2 - E_1 r_2 e^{r_2 y} \alpha^2 + G_{12} r_2^3 e^{r_2 y}) B_m \right. \\
&\quad + (\mu_{12} G_{12} r_3 e^{r_3 y} \alpha^2 - E_1 r_3 e^{r_3 y} \alpha^2 + G_{12} r_3^3 e^{r_3 y}) C_m + \\
&\quad \left. (\mu_{12} G_{12} r_4 e^{r_4 y} \alpha^2 - E_1 r_4 e^{r_4 y} \alpha^2 + G_{12} r_4^3 e^{r_4 y}) D_m \right\} \cos \alpha x + 6K G_{12}
\end{aligned} \tag{4.7d}$$

$$\begin{aligned}
\sigma_{xy}(x, y) &= -\frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^3} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \\
&= -\frac{E_1 G_{12}}{Z_{11}} \left\{ E_1 \sum_{m=1}^{\infty} (A_m e^{r_1 y} + B_m e^{r_2 y} + C_m e^{r_3 y} + D_m e^{r_4 y}) \alpha^3 \sin \alpha x + \mu_{12} E_2 \right. \\
&\quad \left. \sum_{m=1}^{\infty} (A_m r_1^2 e^{r_1 y} + B_m r_2^2 e^{r_2 y} + C_m r_3^2 e^{r_3 y} + D_m r_4^2 e^{r_4 y}) \alpha \sin \alpha x \right\} \\
&= -\frac{E_1 G_{12}}{Z_{11}} \sum_{m=1}^{\infty} \left\{ (E_1 e^{r_1 y} \alpha^3 + \mu_{12} E_2 r_1^2 e^{r_1 y} \alpha) A_m + (E_1 e^{r_2 y} \alpha^3 + \mu_{12} E_2 r_2^2 e^{r_2 y} \alpha) B_m + \right. \\
&\quad \left. (E_1 e^{r_3 y} \alpha^3 + \mu_{12} E_2 r_3^2 e^{r_3 y} \alpha) C_m + (E_1 e^{r_4 y} \alpha^3 + \mu_{12} E_2 r_4^2 e^{r_4 y} \alpha) D_m \right\} \sin \alpha x
\end{aligned} \tag{4.7e}$$

From the expressions of (4.7a) and (4.7e) it is evident that the boundary conditions (i), (ii), (iii) and (iv) are automatically satisfied. Thus the next requirement is to satisfy the remaining boundary conditions i.e. (v), (vi), (vii) and (viii).

Now the compressive load exerted at the two corners on the edge $y = 0$ may be taken as Fourier function in the following manner:

$$\sigma_{yy}(x, 0) = 4\sigma_0 = E_0 + \sum_{m=1}^{\infty} E_m \cos \alpha x \quad \text{for } x=0 \text{ to } 0.1L \text{ and } 0.9L \text{ to } L \tag{4.8a}$$

Here

$$\begin{aligned}
E_0 &= \frac{1}{L} \left[\int_0^{L/10} 4\sigma_0 dx + \int_{9L/10}^L 4\sigma_0 dx \right] \\
&= \frac{4\sigma_0}{L} \left[\frac{L}{10} - 0 + L - \frac{9L}{10} \right] \\
&= \frac{4\sigma_0}{5}
\end{aligned} \tag{4.8b}$$

$$\begin{aligned}
E_m &= \frac{2}{L} \left[\int_0^{L/10} 4\sigma_0 \cos \alpha x dx + \int_{9L/10}^L 4\sigma_0 \cos \alpha x dx \right] \\
&= \frac{8\sigma_0}{L} \left[\int_0^{L/10} \cos \alpha x dx \right] + \frac{8\sigma_0}{L} \left[\int_{9L/10}^L \cos \alpha x dx \right] \\
&= \frac{8\sigma_{yy}^0}{L} \left[\frac{\sin \alpha x}{\alpha} \right]_0^{L/10} + \frac{8\sigma_{yy}^0}{L} \left[\frac{\sin \alpha x}{\alpha} \right]_{9L/10}^L \\
&= \frac{8\sigma_0}{\alpha L} \left\{ \sin \left(\frac{\alpha L}{10} \right) - 0 + \sin(\alpha L) - \sin \left(\frac{9\alpha L}{10} \right) \right\} \\
&= \frac{8\sigma_0}{m\pi} \left\{ \sin \left(\frac{m\pi}{10} \right) + \sin(m\pi) - \sin \left(\frac{9m\pi}{10} \right) \right\} \tag{4.8c}
\end{aligned}$$

The compressive load on the edge $y = D$ can also be given by a Fourier series as follows

$$\sigma_{yy}(x,0) = \sigma_0 = I_0 + \sum_{m=1}^{\infty} I_m \cos \alpha x \quad \text{for } x = 0.1L \text{ to } 0.9L \tag{4.9a}$$

Here

$$\begin{aligned}
I_0 &= \frac{1}{L} \left[\int_{L/10}^{9L/10} \sigma_0 dx \right] \\
&= \frac{\sigma_0}{L} \left[\frac{9L}{10} - \frac{L}{10} \right] \\
&= \frac{4\sigma_0}{5} \tag{4.9b}
\end{aligned}$$

$$\begin{aligned}
I_m &= \frac{2}{L} \left[\int_{L/10}^{9L/10} \sigma_0 \cos \alpha x dx \right] \\
&= \frac{2\sigma_0}{L} \left[\frac{\sin \alpha x}{\alpha} \right]_{L/10}^{9L/10} \\
&= \frac{2\sigma_0}{\alpha L} \left\{ \sin \left(\frac{9\alpha L}{10} \right) - \sin \left(\frac{\alpha L}{10} \right) \right\} \\
&= \frac{2\sigma_0}{m\pi} \left\{ \sin \left(\frac{9m\pi}{10} \right) - \sin \left(\frac{m\pi}{10} \right) \right\} \tag{4.9c}
\end{aligned}$$

Using boundary condition (v) i.e. $\sigma_{xy}(x,0)=0$ at the edge of $y=0$

$$-\frac{E_1 G_{12}}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \begin{aligned} &(E_1 \alpha^3 + \mu_{12} E_2 r_1^2 \alpha) A_m + (E_1 \alpha^3 + \mu_{12} E_2 r_2^2 \alpha) B_m + \\ &(E_1 e^{r_3 y} \alpha^3 + \mu_{12} E_2 r_3^2 \alpha) C_m + (E_1 e^{r_4 y} \alpha^3 + \mu_{12} E_2 r_4^2 \alpha) D_m \end{aligned} \right\} \sin \alpha x \right] = 0$$

$$-\frac{E_1 G_{12}}{Z_{11}} \left\{ \begin{aligned} &(E_1 \alpha^3 + \mu_{12} E_2 r_1^2 \alpha) A_m + (E_1 \alpha^3 + \mu_{12} E_2 r_2^2 \alpha) B_m + \\ &(E_1 \alpha^3 + \mu_{12} E_2 r_3^2 \alpha) C_m + (E_1 \alpha^3 + \mu_{12} E_2 r_4^2 \alpha) D_m \end{aligned} \right\} = 0 \quad (4.10a)$$

Using boundary condition (vi) i.e. $\sigma_{xy}(x,D)=0$ at the edge of $y=D$

$$-\frac{E_1 G_{12}}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \begin{aligned} &(E_1 e^{r_1 D} \alpha^3 + \mu_{12} E_2 r_1^2 e^{r_1 D} \alpha) A_m + (E_1 e^{r_2 D} \alpha^3 + \mu_{12} E_2 r_2^2 e^{r_2 D} \alpha) B_m + \\ &(E_1 e^{r_3 D} \alpha^3 + \mu_{12} E_2 r_3^2 e^{r_3 D} \alpha) C_m + (E_1 e^{r_4 D} \alpha^3 + \mu_{12} E_2 r_4^2 e^{r_4 D} \alpha) D_m \end{aligned} \right\} \sin \alpha x \right] = 0$$

$$-\frac{E_1 G_{12}}{Z_{11}} \left\{ \begin{aligned} &(E_1 e^{r_1 D} \alpha^3 + \mu_{12} E_2 r_1^2 e^{r_1 D} \alpha) A_m + (E_1 e^{r_2 D} \alpha^3 + \mu_{12} E_2 r_2^2 e^{r_2 D} \alpha) B_m + \\ &(E_1 e^{r_3 D} \alpha^3 + \mu_{12} E_2 r_3^2 e^{r_3 D} \alpha) C_m + (E_1 e^{r_4 D} \alpha^3 + \mu_{12} E_2 r_4^2 e^{r_4 D} \alpha) D_m \end{aligned} \right\} = 0 \quad (4.10b)$$

Using boundary condition (vii) i.e. $\sigma_{yy}(x,0)=4\sigma_0$ at the edge of $y=0$

$$-\frac{E_1 E_2}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 \alpha^2 - E_1 r_1 \alpha^2 + G_{12} r_1^3) A_m + \\ &(\mu_{12} G_{12} r_2 \alpha^2 - E_1 r_2 \alpha^2 + G_{12} r_2^3) B_m + \\ &(\mu_{12} G_{12} r_3 \alpha^2 - E_1 r_3 \alpha^2 + G_{12} r_3^3) C_m + \\ &(\mu_{12} G_{12} r_4 \alpha^2 - E_1 r_4 \alpha^2 + G_{12} r_4^3) D_m \end{aligned} \right\} \cos \alpha x + 6K G_{12} \right] = \sum_{m=1}^{\infty} E_m \cos \alpha x + E_0$$

Therefore,

$$-\frac{E_1 E_2}{Z_{11}} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 \alpha^2 - E_1 r_1 \alpha^2 + G_{12} r_1^3) A_m + \\ &(\mu_{12} G_{12} r_2 \alpha^2 - E_1 r_2 \alpha^2 + G_{12} r_2^3) B_m + \\ &(\mu_{12} G_{12} r_3 \alpha^2 - E_1 r_3 \alpha^2 + G_{12} r_3^3) C_m + \\ &(\mu_{12} G_{12} r_4 \alpha^2 - E_1 r_4 \alpha^2 + G_{12} r_4^3) D_m \end{aligned} \right\} = E_m \quad (4.10c)$$

Using boundary condition (vii) i.e. $\sigma_{yy}(x,D)=\sigma_0$ at the edge of $y=D$

$$-\frac{E_1 E_2}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 e^{r_1 D} \alpha^2 - E_1 r_1 e^{r_1 D} \alpha^2 + G_{12} r_1^3 e^{r_1 D}) A_m + \\ &(\mu_{12} G_{12} r_2 e^{r_2 D} \alpha^2 - E_1 r_2 e^{r_2 D} \alpha^2 + G_{12} r_2^3 e^{r_2 D}) B_m + \\ &(\mu_{12} G_{12} r_3 e^{r_3 D} \alpha^2 - E_1 r_3 e^{r_3 D} \alpha^2 + G_{12} r_3^3 e^{r_3 D}) C_m + \\ &(\mu_{12} G_{12} r_4 e^{r_4 D} \alpha^2 - E_1 r_4 e^{r_4 D} \alpha^2 + G_{12} r_4^3 e^{r_4 D}) D_m \end{aligned} \right\} \cos \alpha x + 6K G_{12} \right] = \sum_{m=1}^{\infty} I_m \cos \alpha x + I_0$$

Therefore,

$$-\frac{E_1 E_2}{Z_{11}} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 e^{r_1 D} \alpha^2 - E_1 r_1 e^{r_1 D} \alpha^2 + G_{12} r_1^3 e^{r_1 D}) A_m + \\ &(\mu_{12} G_{12} r_2 e^{r_2 D} \alpha^2 - E_1 r_2 e^{r_2 D} \alpha^2 + G_{12} r_2^3 e^{r_2 D}) B_m + \\ &(\mu_{12} G_{12} r_3 e^{r_3 D} \alpha^2 - E_1 r_3 e^{r_3 D} \alpha^2 + G_{12} r_3^3 e^{r_3 D}) C_m + \\ &(\mu_{12} G_{12} r_4 e^{r_4 D} \alpha^2 - E_1 r_4 e^{r_4 D} \alpha^2 + G_{12} r_4^3 e^{r_4 D}) D_m \end{aligned} \right\} = I_m \quad (4.10d)$$

From equations (4.7d) and (4.8) or (4.9)

$$-\frac{6E_1 E_2 K G_{12}}{Z_{11}} = E_0 = I_0 = \frac{4\sigma_0}{5}$$

$$K = -\frac{2Z_{11}\sigma_0}{15E_1 E_2 G_{12}} \quad (4.11)$$

The simultaneous equations (4.10a), (4.10b), (4.10c) and (4.10d) can be realized in a simplified matrix form for solution of the unknown terms of arbitrary constants like A_m , B_m , C_m and D_m as follows:

$$\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ H_1 & H_2 & H_3 & H_4 \\ R_1 & R_2 & R_3 & R_4 \\ S_1 & S_2 & S_3 & S_4 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{E}_m \\ \bar{I}_m \end{bmatrix} \quad (4.12)$$

where

$$\left. \begin{aligned} F_i &= E_1 \alpha^3 + \mu_{12} E_2 r_i^2 \alpha \\ H_i &= E_1 e^{r_i D} \alpha^3 + \mu_{12} E_2 r_i^2 e^{r_i D} \alpha \\ R_i &= \mu_{12} G_{12} r_i \alpha^2 - E_1 r_i \alpha^2 + G_{12} r_i^3 \\ S_i &= \mu_{12} G_{12} r_i e^{r_i D} \alpha^2 - E_1 r_i e^{r_i D} \alpha^2 + G_{12} r_i^3 e^{r_i D} \end{aligned} \right\} \quad i=1, 2, 3, 4$$

$$\bar{E}_m = -\frac{Z_{11} E_m}{E_1 E_2}$$

$$\bar{I}_m = -\frac{Z_{11} I_m}{E_1 E_2}$$

$$Z_{11} = \mu_{12} E_1 E_2 + G_{12} (E_1 - \mu_{12}^2 E_2)$$

Either the matrix (4.12) or the four algebraic simultaneous equations (4.10a), (4.10b), (4.10c) and (4.10d) are solved using E_m and I_m values from Eq. of four unknowns, namely, A_m , B_m , C_m and D_m , are obtained, equations of (4.7) are then used for subsequent finding of stress and displacement components at various points of the beam.

4.2 Results of Composite Deep Beam

The analytical solutions of displacement and stress components are obtained using displacement potential function for various aspect ratios (L/D) of the guided deep beam taking glass-epoxy, boron-epoxy and graphite-epoxy material into consideration. The properties of the composite materials considered for solutions are as follows:

Table 4.1 Properties of composites used in the present study [36], [39] and [40]

Composites	E_1 (MPa)	E_2 (MPa)	G_{12} (MPa)	μ_{12}	μ_{21}
Glass Epoxy	43000	8900	4500	0.27	0.0559
Boron Epoxy	282900	24200	10400	0.27	0.0231
Graphite Epoxy	648000	24800	11000	0.25	0.0096

At first the result of a guided composite beam of glass epoxy material having aspect ratio three ($L/D = 3$) and the uniform loading parameter, $\sigma_0 = 40$ N/mm on the top edge is presented in sequence of axial displacement (u_x), lateral displacement (u_y), bending stress (σ_{xx}), normal stress (σ_{yy}) and shearing stress (σ_{xy}). The results of the present approach have then been verified with the help of finite element method. Thereafter the effects of change of aspect ratio and mechanical properties on the elastic fields are observed from the solution of displacement potential approach.

4.2.1 Solution for Displacement Components

Axial displacements (u_x) are found to be of almost similar magnitude about the both mid-horizontal and mid-vertical planes but acting in opposite directions. The values of u_x for sections $0 < x/L < 0.5$ are negative at the lower portion and positive at upper portion of the beam. The maximum magnitude of $u_x/L = -0.000709$ and $u_x/L = +0.000709$ are observed on bottom fibre at the sections of $x/L = 0.1$ and $x/L = 0.9$ respectively. Axial displacements are found to be zero at the mid section of span and at the lateral guided boundaries [Fig. 4.2(a)]

and (b)]. Zero value of u_x at the guided ends verifies the boundary condition of those edges of the beam.

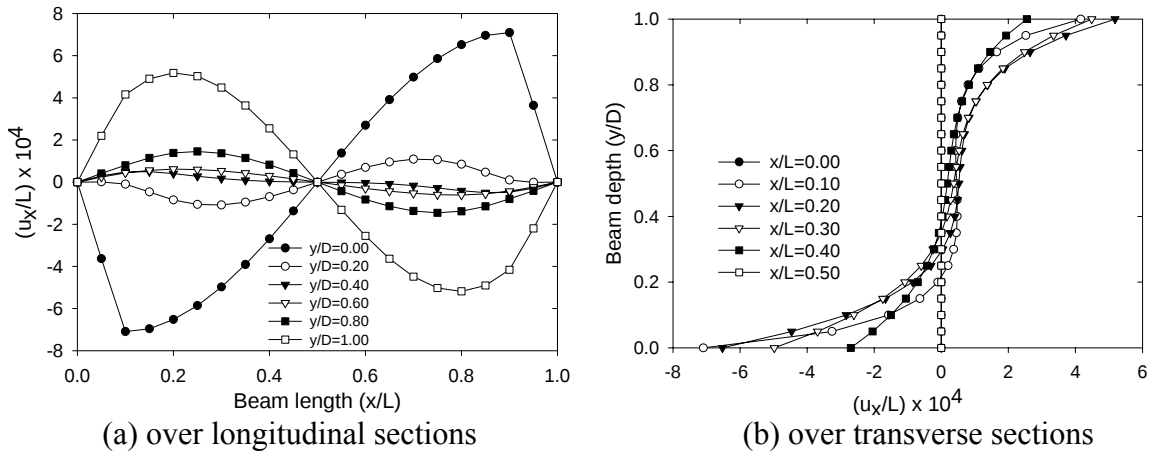


Fig. 4.2 Normalized axial displacement for the Glass-Epoxy guided beam, $L/D = 3$:
(a) over longitudinal sections, (b) over transverse sections

Lateral displacements (u_y) are found to take positive value near the two lateral ends and negative in the region $0.2 < x/L < 0.8$, because of the loading distribution at top edge and support pattern at the bottom boundary [Fig. 4.3(a) and (b)]. Moreover, there is no restriction on the lateral displacement other than the loading and reaction to bring the beam in equilibrium state. The u_y results are in confirmation to the physical condition of the beam being pushed the corners up and forced down at the mid-span region. The normalized values of positive and negative maximum lateral displacements are $u_y/D = 0.003503$ and $u_y/D = -0.00264$ respectively. The positive maximum value is observed at the two ends on the lowest fibre and the negative maximum value is found on the top fibre at the mid section of the span.

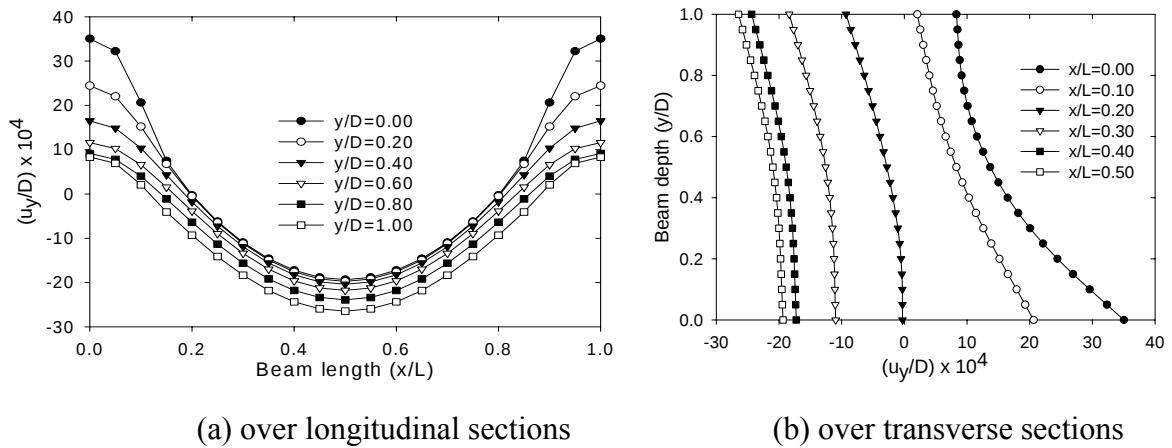


Fig. 4.3 Normalized lateral displacement for the Glass-Epoxy guided beam, $L/D = 3$:
(a) over longitudinal sections, (b) over transverse sections

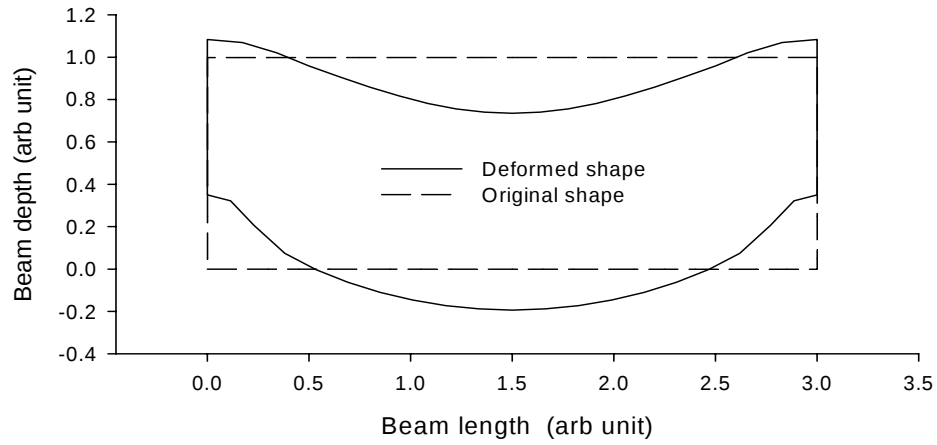


Fig. 4.4 Deformed shape of the Glass-Epoxy beam, $L/D = 3$ (magnification factor $\times 500$)

Fig. 4.4 presents the deformed shape of the orthotropic beam along with its original shape at the magnification of 500 times of displacement. The guided ends have gone up and at the same time centre region of the beam have gone down. The deformation of the top edge is gradual through out the length of beam with the uniformly distributed loading. The bottom edge is also deformed uniformly except a pointed indentation at the supports. However, the overall vertical sliding type deformation is in excellent agreement with the physical condition of applied loading and support of the guided beam.

4.2.2 Solution for Stress Components

Bending stress distribution is observed non-linear over the whole span [Fig. 4.5(a) and (b)]. This non-linearity increases towards the guided ends. The stress (σ_{xx}) maximizes at the top and bottom edges of the beam but carries opposite sign. The maximum normalized values at the top and bottom fibre are 4.825 and -9.025 respectively. Near the guided ends, σ_{xx} is positive for the upper half and negative for the lower half of the beam, but the opposite is observed for sections in the region of $0.1 < x/L < 0.9$.

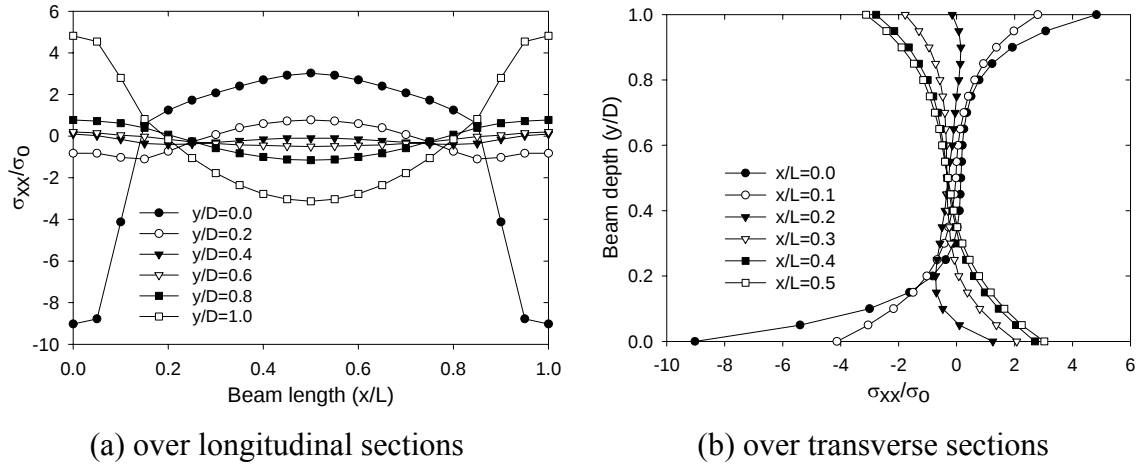


Fig. 4.5 Normalized bending stress for the Glass-Epoxy guided beam, $L/D = 3$:
(a) over longitudinal sections, (b) over transverse sections

As it appears from Fig. 4.6, the lateral stress (σ_{yy}) is of higher concentration around the position of supports on the bottom edge. The uniform load is distributed over 80% of beam length (from $x/L = 0.1$ to $x/L = 0.9$) and support reaction is distributed over 20% of beam length ($x/L = 0.0$ to 0.1 and $x/L = 0.9$ to 1.0). As per this physical condition, reaction is four times of the load intensity. From the solution of displacement potential approach it is understandable that the normalized value of the lateral stress varies from zero to unity at the top layer in the loaded region and it is four at the bottom layer of the support region. This is an intact agreement with the applied loading as well as boundary condition for the glass epoxy orthotropic material.

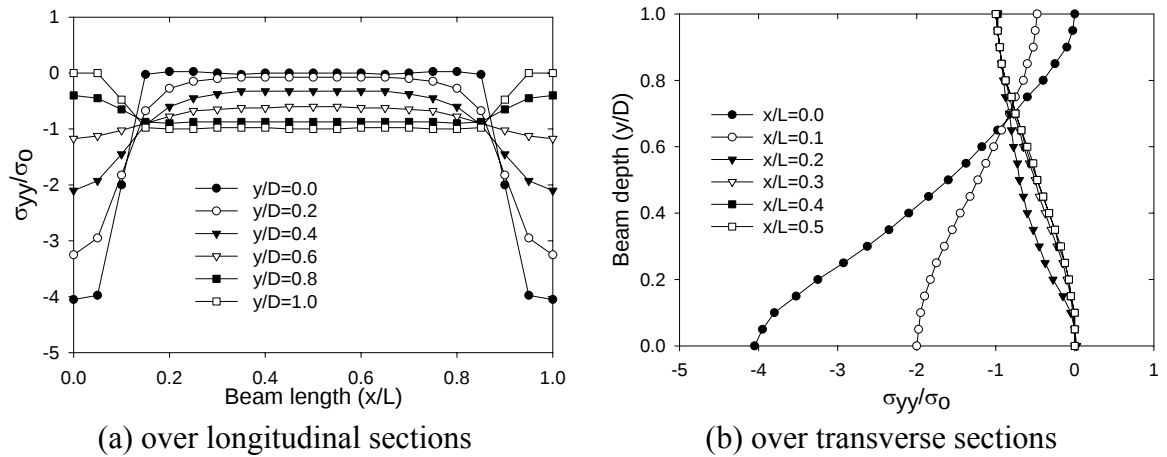


Fig. 4.6 Normalized lateral stress for the Glass-Epoxy guided beam, $L/D = 3$:
(a) over longitudinal sections, (b) over transverse sections

Four edges and mid-span section of the guided deep beam are found free from shearing stress [Fig. 4.7(a) and (b)]. The distribution of shearing stress (σ_{xy}) is anti-symmetric in two sides about the mid-span of the beam [Fig. 4.7(a)]. Fig. 4.7 (b) conforms that the vertical distribution is similar to the standard parabolic profile except that at sections $x/L=0.1$ and 0.9 , *i.e.*, the termination point of loading as shown in Fig. 4.1. The maximum magnitude of shear stress is observed at $y/D=0.05$ of sections $x/L=0.1$ (normalized value = - 1.8) and $x/L=0.9$ (normalized value = 1.8).

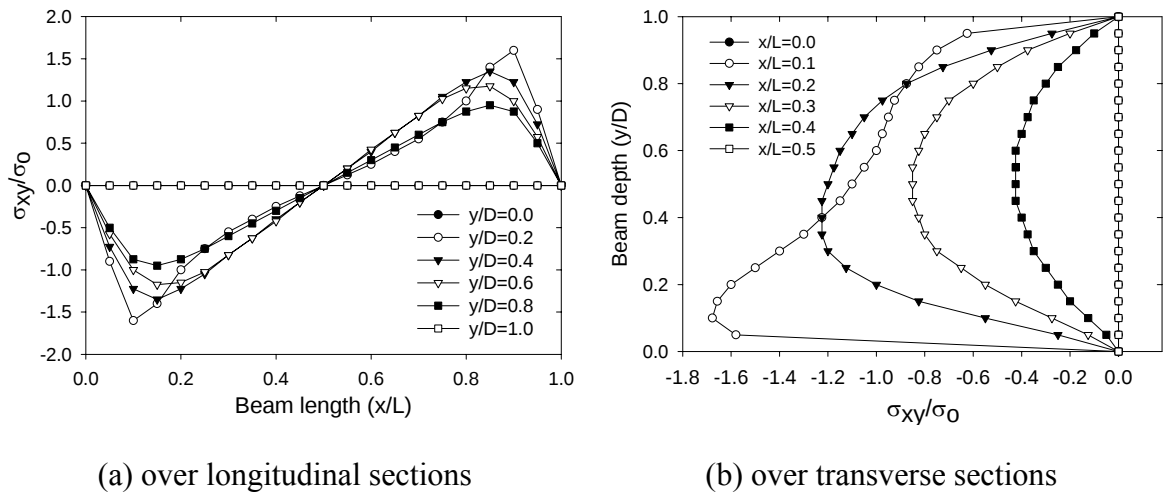


Fig. 4.7 Normalized shearing stress for the Glass-Epoxy guided beam, $L/D = 3$:
(a) over longitudinal sections, (b) over transverse sections

4.3 Verification of Present Results of Composite Materials

The orthotropic guided beam of the present problem cannot be solved by the classical beam theory. However, for the mere sake of comparison and observe the degree of accuracy of displacement potential solution, the elementary as well as stress function solutions are obtained for the unguided deep beam. Since the distribution of reaction at the bottom surface cannot be addressed appropriately using the elementary theory, the beam is considered here to be simply supported taking the resultant of the reaction forces. The solutions for the bending and shear stress are obtained using the following expressions of elementary theory, which are similar to that of isotropic except the elastic modulus E is replaced by E_I .

$$u_y(x) = \frac{\sigma_0}{2E_I} \left(\frac{L}{D}\right)^3 \left[2\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3 - 1 \right] x \quad (4.13)$$

$$\begin{aligned} \sigma_{xx}(x) &= \frac{24\sigma_0}{5} \left(\frac{L}{D}\right)^2 \left(\frac{x}{L} - \frac{1}{20}\right) \left(\frac{1}{2} - \frac{y}{D}\right) && \text{for } x/L < 0.1 \text{ and } > 0.9 \\ &= 6\sigma_0 \left(\frac{L}{D}\right)^2 \left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{1}{25} \right] \left(\frac{1}{2} - \frac{y}{D}\right) && \text{for } x/L = 0.1 \text{ to } 0.9 \end{aligned} \quad (4.14)$$

$$\sigma_{xy}(x) = 6\sigma_0 \left(\frac{L}{D}\right) \left[\frac{1}{4} - \left(\frac{y}{D} - \frac{1}{2}\right)^2 \right] \left(\frac{1}{2} - \frac{x}{L}\right) \quad (4.15)$$

FEM solutions are obtained using the standard facilities of the commercial software ANSYS. For that the relevant boundary conditions used are the same as those used in the analytical solution. Four noded rectangular orthotropic plane elements are used to construct the corresponding mesh network of the beam. All the elements are of same size and their distribution is kept uniform all over the domain. The total number of finite elements used to construct the element mesh network for the present problem is 400 (20 x 20).

The comparison of the ψ -solution with the solution of the elementary theory and FEM is presented in Fig. 4.8, 4.9 and 4.10. The deflection (lateral displacement at $y/D=0.5$) observed in ψ -solution and FEM can give very closer results, where both the ends are having positive values and at the mid-span it is negative. As a result the zero deflection is observed at the two points away from the supports. But the classical beam theory does not predict the positive lateral displacement properly due to its basic consideration to address the problem of simply supported beam. Consequently the deflection magnitude at the mid-

span obtained using beam theory is more than that of ψ -solution as well as FEM [Fig. 4.8a]. If the results of ψ -solution and FEM are translated to have zero deflection at the centre points of supports, the deflection of the beam can then be found as shown in Fig. 4.8b. Now the maximum magnitude of deflection is found to be more in ψ -solution and FEM than beam theory solution.

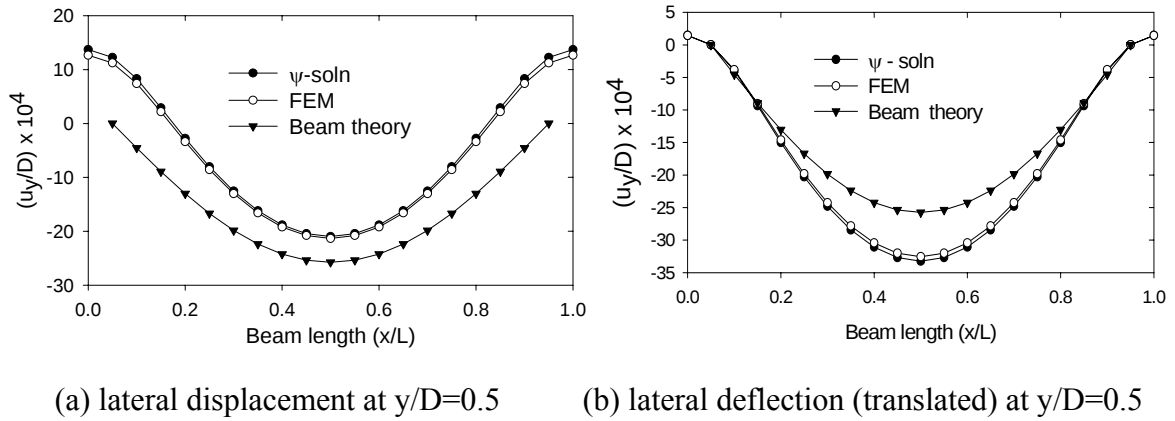


Fig. 4.8 Comparison of lateral deflection at $y/D=0.5$ of Glass-Epoxy deep beam, $L/D=3$

From Eq. (4.14) and (4.15) it is understandable that the bending and shear stress in classical beam theory do not depend on the material properties or fibre configurations. They are only function of applied load and dimensions. As such the result obtained here is the same to the stresses of isotropic materials. The bending stress distribution obtained by the elementary theory gives a linear variation over the beam depth. Moreover, it is observed that the bending stress at the end is under predicted and at the mid-span is over predicted by the elementary theory from those of the ψ -solution and FEM solution [Fig. 4.9(a) and (b)].

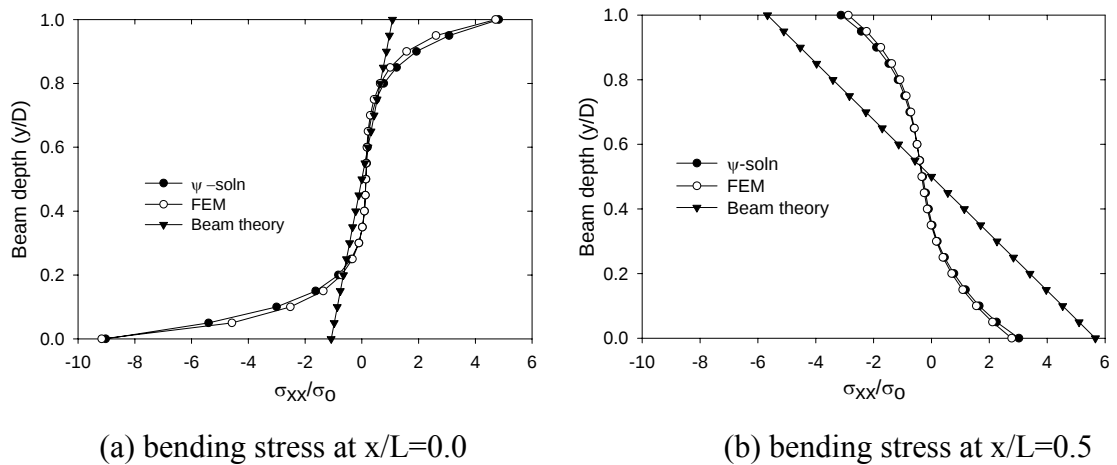


Fig. 4.9 Comparison of bending stress of Glass-Epoxy deep beam, $L/D=3$:
(a) at $x/L=0.0$ (b) at $x/L=0.5$

The lateral stress distribution is not obtainable by the elementary theory. Therefore, Fig. 4.10(a) and (b) represent only ψ -solution and FEM solution for $x/L=0.0$ and 0.1 . It is observed that the lateral stress distribution patterns are similar in both the solutions. At the end of the beam, upper side values coincides each other but a very little deviation is in the lower side of the beam. At the point of load transition ($x/L=0.1$) the FEM solution gives little over estimation than the ψ -solution.

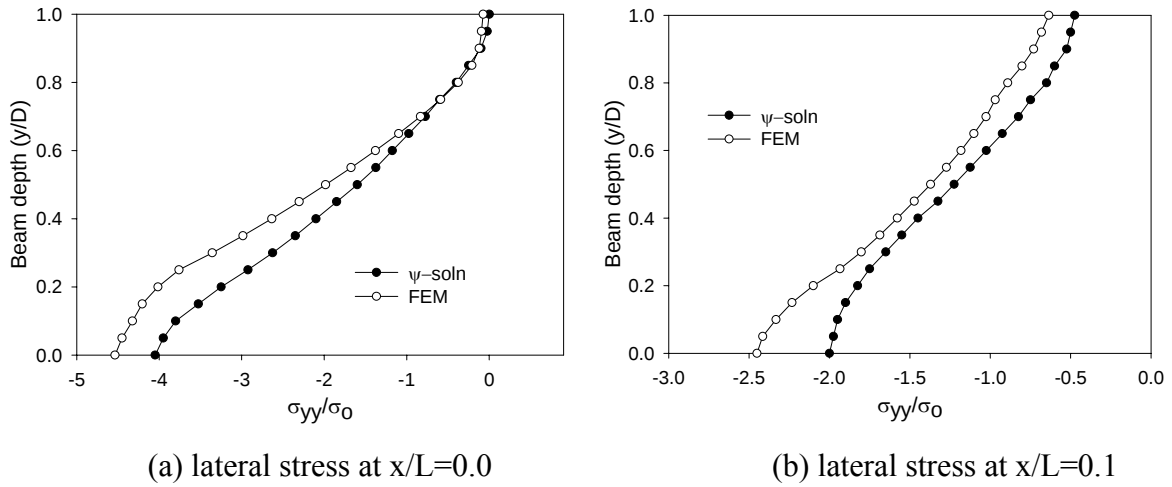


Fig. 4.10 Comparison of lateral stress of Glass-Epoxy deep beam, $L/D=3$:
(a) at $x/L=0.0$ (b) at $x/L=0.1$

The shear stress distribution pattern obtained using classical beam theory is of parabolic type at all the transverse sections of the beam. Distributions of two sections for instance $x/L = 0.0$ and 0.25 are illustrated in Fig. 4.11 (a) and (b). The shear stress magnitudes obtained by the three approaches are found to be close to one another at the section of $x/L=0.25$. But the solutions differ at section $x/L = 0.1$, where the loadings and reactions on the two (top and bottom) boundaries terminate. The parabolic pattern is not observed at this load termination section in the solutions of displacement potential approach. Finite element method also goes side by side to the ψ -solution, but not the beam theory. The ψ -solution and FEM solution depict that the reaction locations are having shear stress concentration in the lower part near support instead of mid longitudinal section.

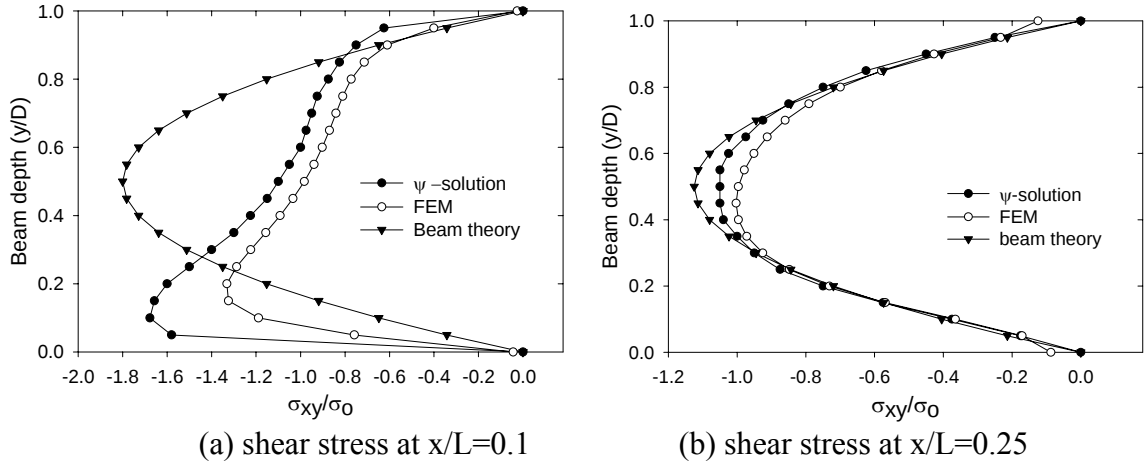


Fig. 4.11 Comparison of shear stress of Glass-Epoxy deep beam, $L/D=3$:
(a) at $x/L=0.1$ (b) at $x/L=0.25$

4.4 Effect of Aspect Ratio on Stresses

The bending stresses are observed for various L/D ratios from 1 to 4 at different sections and seen that the nonlinearity of bending stress gets reduced with the increase of length depth ratio [Fig. 4.12(a) and (b)]. But it does not become linear as observed for isotropic case described in chapter 3 even at the length depth ratio of ten. The maximum magnitude of bending stress is increased with the increase of L/D ratio. The effect of L/D ratio variation is also observed using classical beam theory and FEM for the same physical condition. It appears from table 4.2 that the maximum bending stresses observed in ψ -solution and FEM solution are quite closer, but the results obtained through classical beam theory are different.

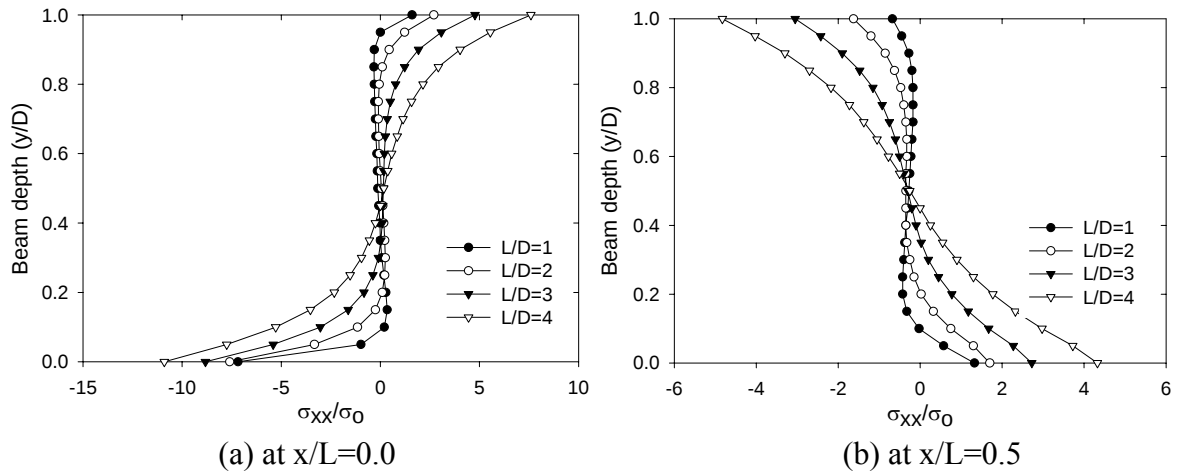


Fig. 4.12 Bending stress distribution of the Glass-Epoxy beam for different aspect ratio: (a) at $x/L=0.0$, (b) at $x/L=0.5$

Table 4.2 Comparison of Maximum Normalized Bending Stress at $x/L=0.5$ of the Glass-Epoxy beam

L/D ratio	Simple theory	ψ -solution		FEM	
	y/D=0 and 1	y/D=0	y/D= 1	y/D=0	y/D=1
1	± 0.63	1.33	-0.67	1.55	-0.62
2	± 2.52	1.70	-1.62	1.87	-1.52
3	± 5.67	2.73	-3.05	2.77	-2.88
4	± 10.08	4.32	-4.82	4.24	-4.54

The lateral stress distributions for $L/D=1$ to 4 at sections from guide end ($x/L=0.0$) and mid-span ($x/L= 0.5$) are shown in Fig. 4.13 (a) and (b) respectively. The lateral stresses for all four aspect ratios are found to follow the loading pattern very correctly. With the increase of aspect ratio the nonlinearity reduces, but still a few degree nonlinearity remains even with higher length depth ratio (like $L/D=4$). This nonlinearity is more significant at the guided ends. Fig. 4.13 (a) and (b) also indicates that the distribution pattern of lateral stress changes while the change of aspect ratio. The overall effect of aspect ratio is less with respect to sections moving away from the guide. The comparison between ψ -solution and FEM solution is presented in table 4.3 that the maximum lateral stresses observed at the middle of the span are quite closer, but the results differ a little at the ends of the beam.

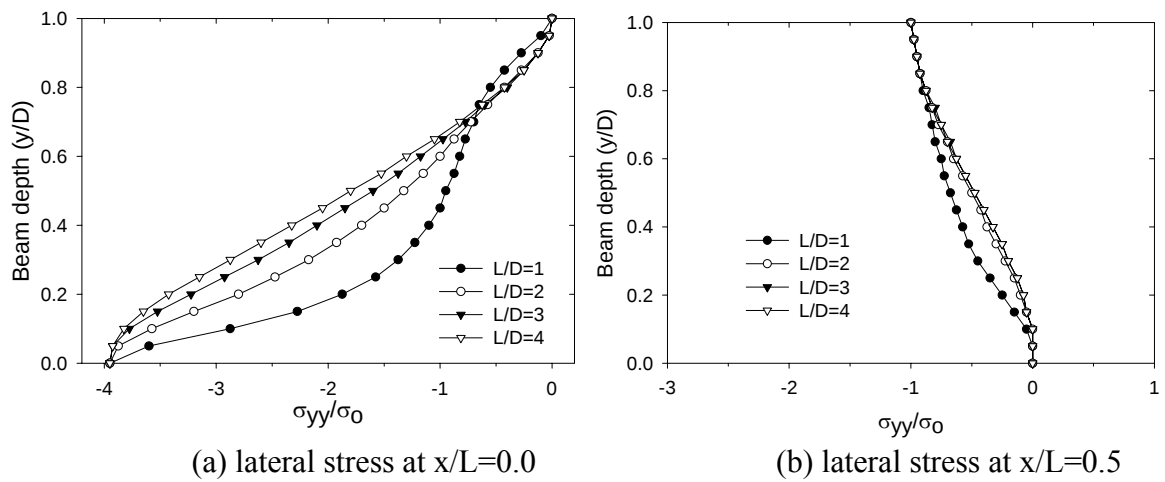


Fig. 4.13 Lateral stress distribution of the Glass-Epoxy beam for different aspect ratio: (a) at $x/L=0.0$, (b) at $x/L=0.5$

Table 4.3 Comparison of Maximum Normalized Lateral Stress at $x/L=0.0$ and 0.5

L/D	at $x/L=0.0$ ($y/D=0$)			at $x/L=0.5$ ($y/D=1$)		
ratio	Simple theory	ψ -soln	FEM	Simple theory	ψ -soln	FEM
1	-	-4.00	-4.46	-	-1.00	-1.01
2	-	-4.00	-4.46	-	-1.00	-1.01
3	-	-4.00	-4.45	-	-1.00	-1.01
4	-	-4.00	-4.45	-	-1.00	-1.01

At the guided ends and at the mid span, where bending stresses are observed maximum, shearing stress is zero for all L/D ratios. Fig. 4.14 (a) is showing the distribution of shearing stress at $x/L=0.1$, where it is observed to be not following parabolic pattern. The shearing stress distribution pattern at $x/L=0.4$ is becoming towards the parabolic shape except for $L/D=1$, which is fairly different from others. With the rise of length depth ratio parabolic pattern of shear stress distribution is experienced along with higher magnitude of maximum shear stress [Fig. 4.14 (b)]. The effect of L/D ratio variation is observed in simple beam theory, ψ -solution and FEM is shown in table 4.4, where again solution in displacement potential approach is nearer to that of FEM but not the classical beam theory.

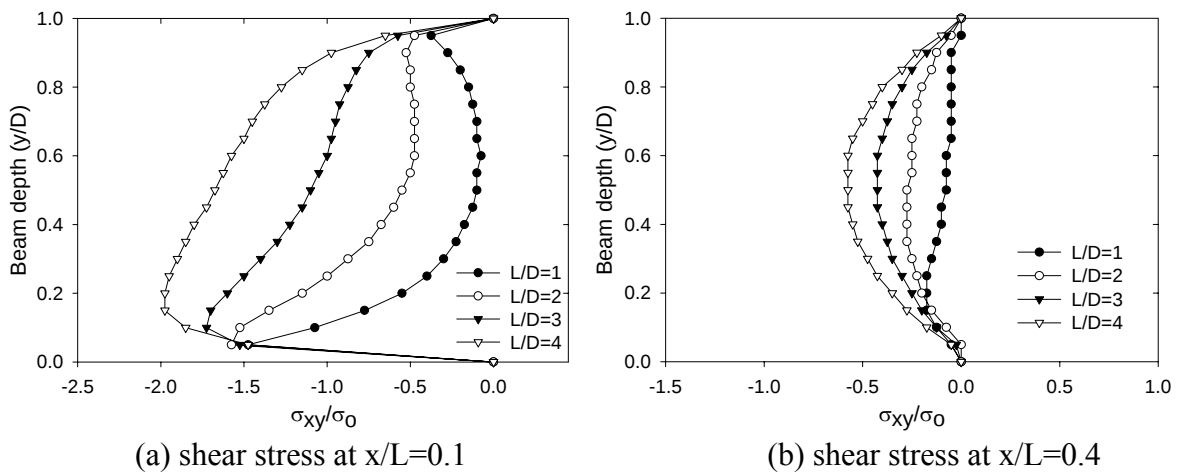
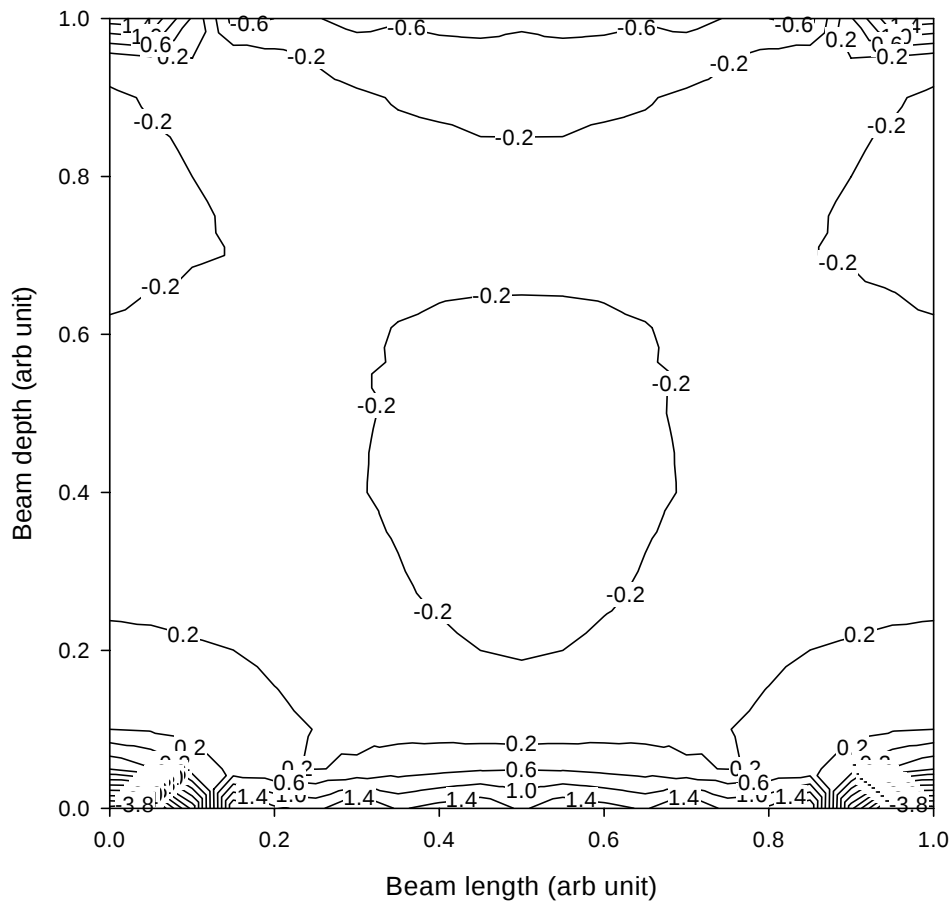


Fig. 4.14 Shearing stress distribution of glass epoxy beam for different aspect ratio, $L/D=1$ to 4 : (a) at $x/L=0.1$, (b) at $x/L=0.4$

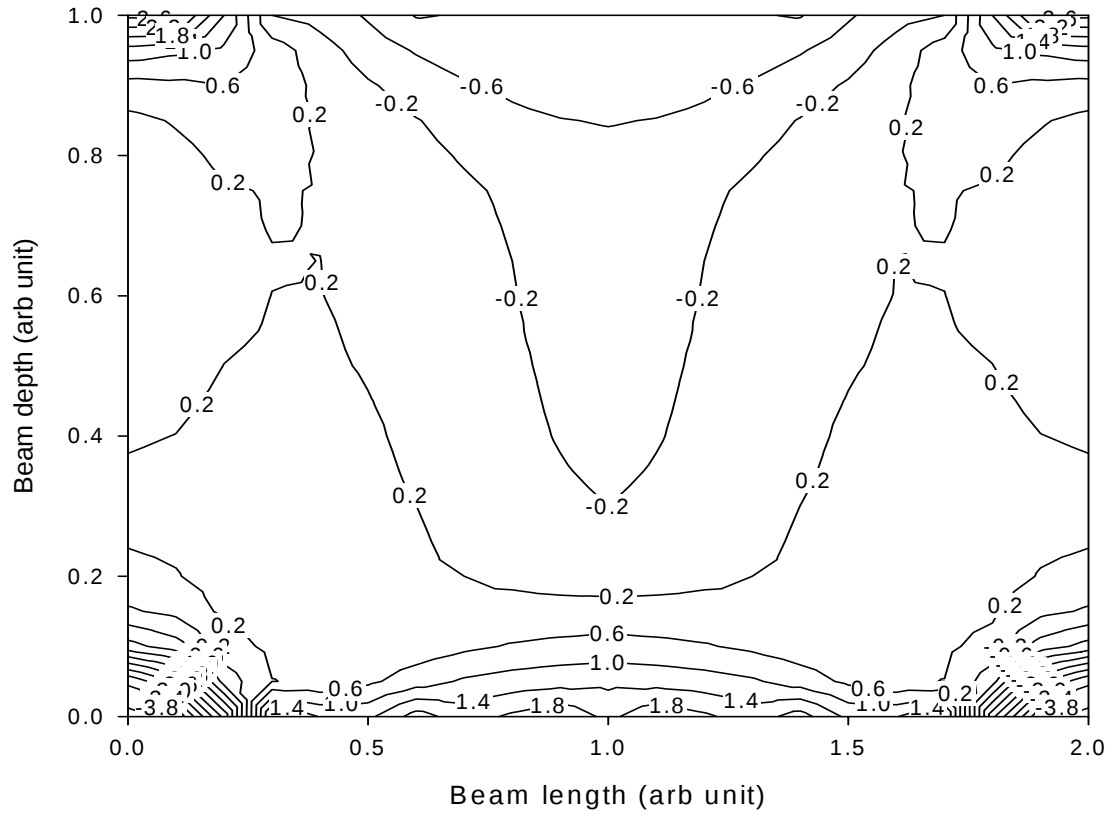
Table 4.4 Comparison of Maximum Normalized Shear Stress at $x/L=0.1$ and $x/L=0.25$

L/D ratio	Maximum Normalized Shearing Stress at $x/L=0.1$			Maximum Normalized Shearing Stress at $x/L=0.25$		
	Simple theory	ψ -soln	FEM	Simple theory	ψ -soln	FEM
1	-0.6	-1.4	-0.98	-0.38	-0.55	-0.56
2	-1.2	-1.57	-1.17	-0.75	-0.70	-0.69
3	-1.8	-1.72	-1.33	-1.12	-1.05	-1.01
4	-2.4	-1.97	-1.63	-1.50	-1.45	-1.37

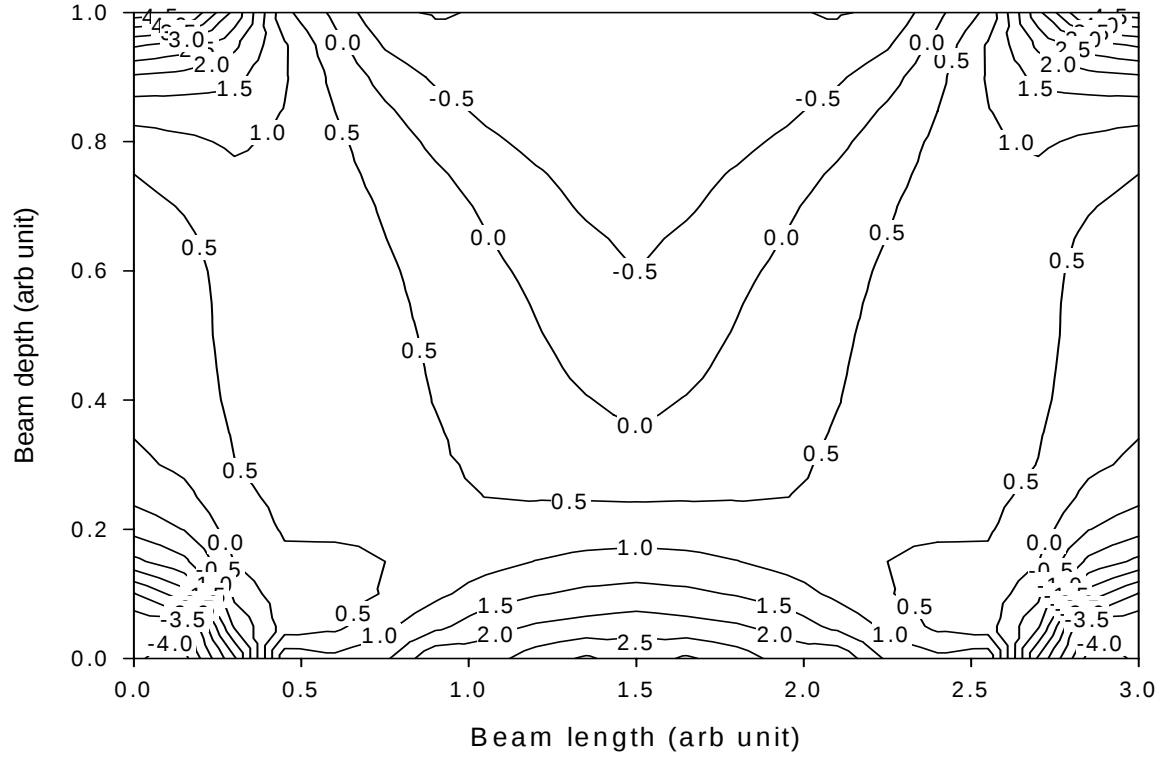
The principal stress distribution is observed from the results of ψ -solution for different aspect ratios. Figs. 4.15 (a), (b) and (c) are presenting the contours of normalised maximum principal stress for aspect ratios one, two and three respectively. Here the highest stress concentration is observed at the guided ends.



(a) Maximum principal stress for $L/D=1$



(b) Maximum principal stress for $L/D=2$



(c) Maximum principal stress for $L/D=3$

Fig. 4.15 Contour of maximum principal stress distribution of the orthotropic beam of glass epoxy material: (a) for $L/D=1$, (b) for $L/D=2$, (c) for $L/D=3$

4.5 Effect of Material Properties on Elastic Fields

The solutions of orthotropic guided beam are obtained for three composites such as glass epoxy, boron epoxy and graphite epoxy materials taking the properties mentioned earlier in table 4.1. Here the results of displacement as well as stress variations at various sections of those beams are presented. Fig. 4.16(a) and (b) represent the axial and lateral displacement at the section of $x/L=0.25$ and $x/L=0.5$, where their magnitudes are maximum over all other sections. Both the axial and lateral displacements are of significantly higher magnitude in case of glass epoxy than those of boron epoxy or graphite epoxy materials. The normalized maximum values of axial displacement for glass-epoxy, boron-epoxy and graphite-epoxy materials are 0.000583, 0.00013 and 0.000086 respectively and the lateral displacements are 0.00264, 0.00098 and 0.000904 respectively.

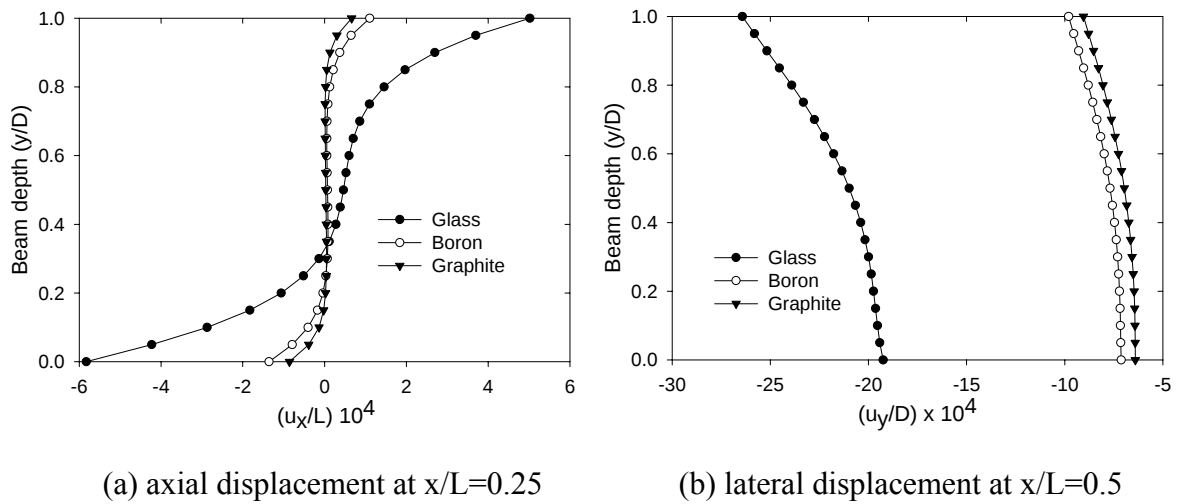


Fig. 4.16 Displacement of the beam ($L/D=3$) for glass-epoxy, boron-epoxy and graphite-epoxy materials : (a) axial at $x/L=0.25$, (b) lateral at $x/L=0.5$

The variation of bending stress distribution is observed at the top and bottom edges of the beam [Figs. 4.17]. At the $y/D=0.5$ line or so, it is not that much significant as to the guided region. The effect of material properties on bending stress is more visible at the guided ends [Fig. 4.17 (a)]. As a whole, the concentration of bending stress is found to be more in those materials which have higher elastic modulus. The maximum normalised bending stress at the support points of guided ends of graphite, boron and glass epoxy composites are 18.95, 13.825 and 8.825 respectively.

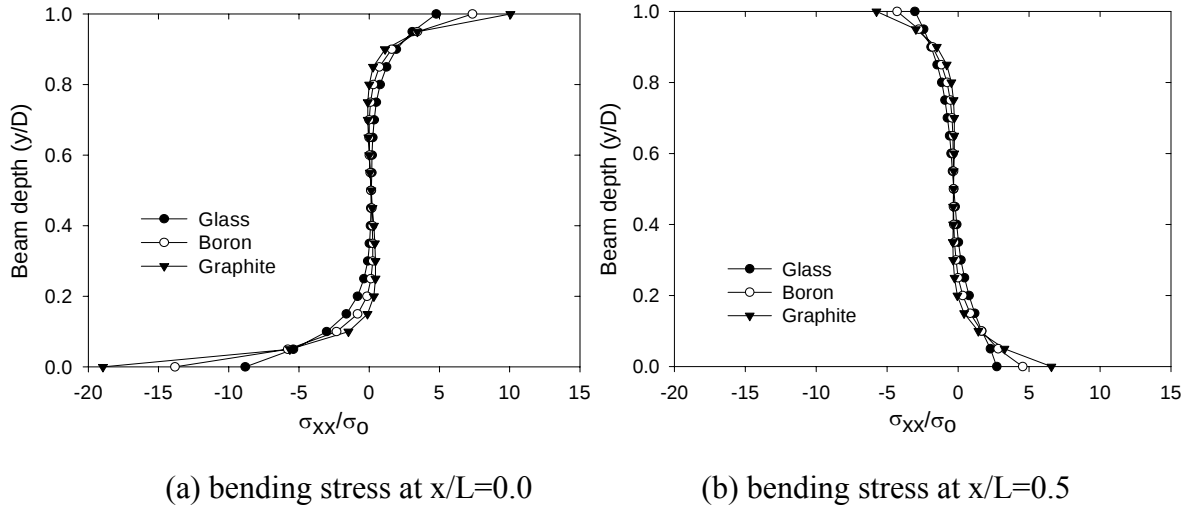


Fig. 4.17 Bending stress of the beam ($L/D=3$) for glass-epoxy, boron-epoxy and graphite-epoxy materials: (a) at $x/L=0.0$, (b) at $x/L=0.5$

The shearing stress distribution observed is not the same for different materials, which was free from the material properties in the classical beam theory. Figs. 4.18 (a) and (b) indicate that shearing stress distributions are not taking the proper parabolic pattern at any section of the beam for the current L/D ratio of three. At the load transition sections ($x/L=0.1$ and 0.9) the distribution pattern is rather unique type, where the maximum magnitude is nearer the bottom fibre (at $y/D=0.05$) instead of centre line ($y/D=0.5$) of the beam for all three materials [Fig. 4.18 (a)]. The maximum normalised bending stress at the load transition sections of graphite, boron and glass epoxy composites are 2.125, 1.975 and 1.725 respectively. However, the shearing stress parabolic distribution pattern is observed nearer to the mid-span for very high L/D ratios.

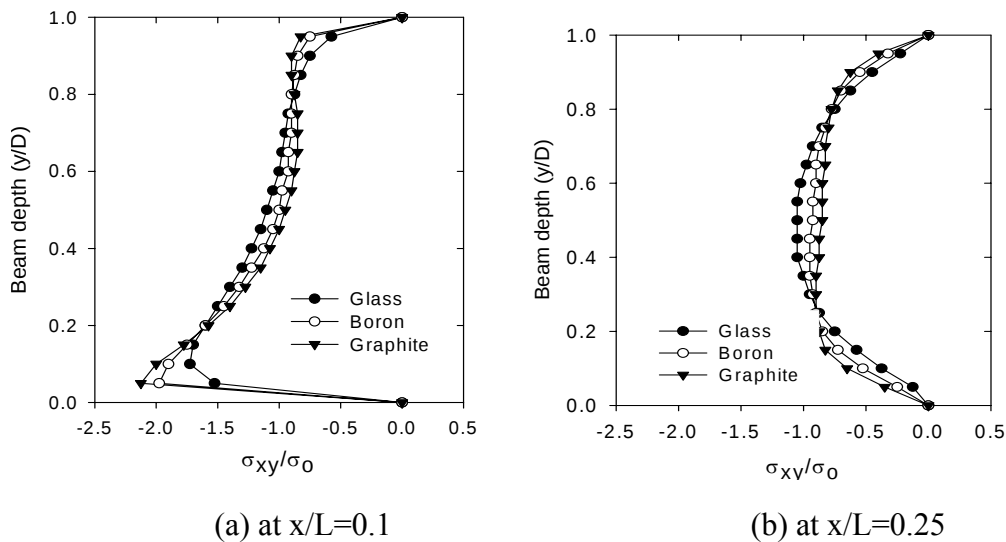


Fig. 4.18 Shear stress of the beam ($L/D=3$) for glass-epoxy, boron-epoxy and graphite-epoxy materials: (a) at $x/L=0.1$, (b) at $x/L=0.25$

4.6 Analysis of Elastic Fields in Orthotropic Relatively Slender Beams

The ψ -solution results are observed for higher aspect ratios to perceive the slenderness effect of the beam and find out the rationality of its use over the century old and widely used beam theory. With the rise of slenderness ratio the deflection magnitudes are found to be increased in both the approaches. But the inherent setback of the beam theory consideration is the assumption of zero deflection at resultant point of support. Basically the equilibrium condition may not allow such situation all the time. In the present problem zero deflection is observed somewhere inner to the support locations. Fig. 4.19 indicates that the maximum magnitudes of deflection observed through potential solutions are little less than those of beam theory. It is because of different locations of zero deflection in two solutions.

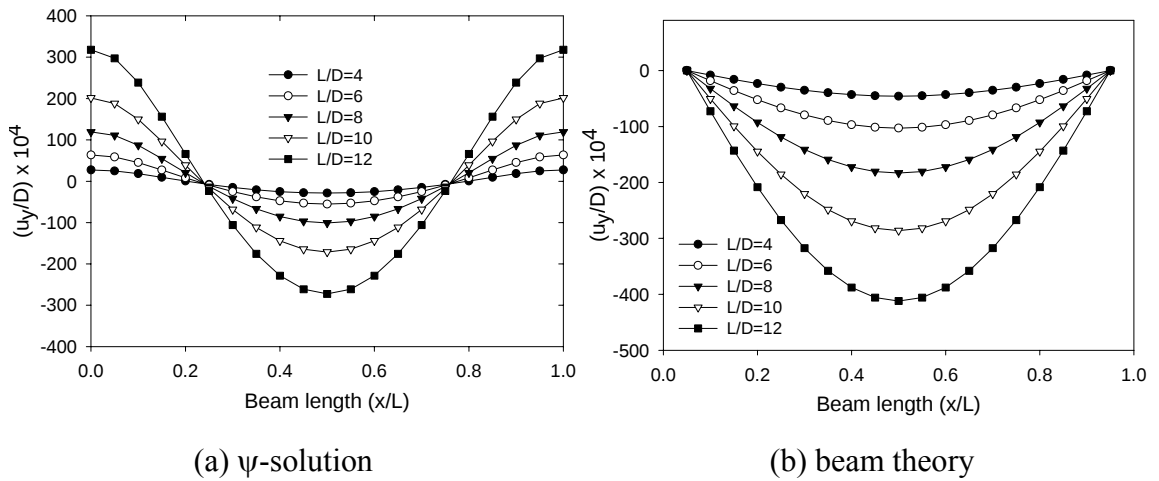
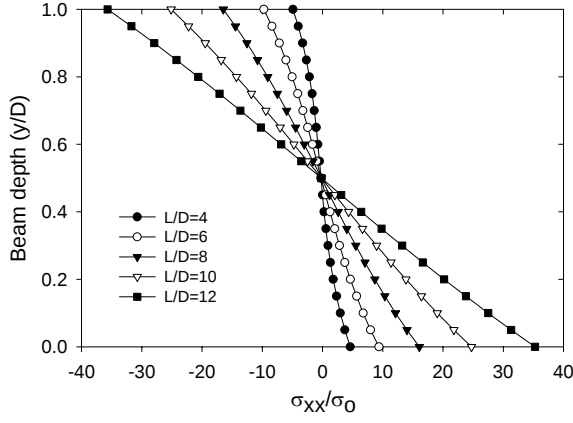
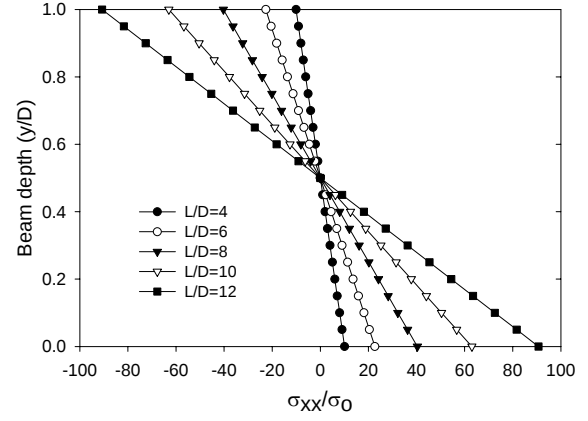


Fig. 4.19 Lateral displacement of the glass-epoxy beams, (at $y/D=0.5$): (a) displacement potential solution, (b) classical beam theory

The nonlinearity of bending stress gets reduced with the increase of length depth ratio [Fig. 4.20(a) and (b)] and it becomes linear while the aspect ratio reaches to ten. However there is difference in magnitudes. The maximum magnitude of bending stress is increased with the increase of aspect ratio in both the solutions but the potential solution gives lesser magnitude than that of beam theory.

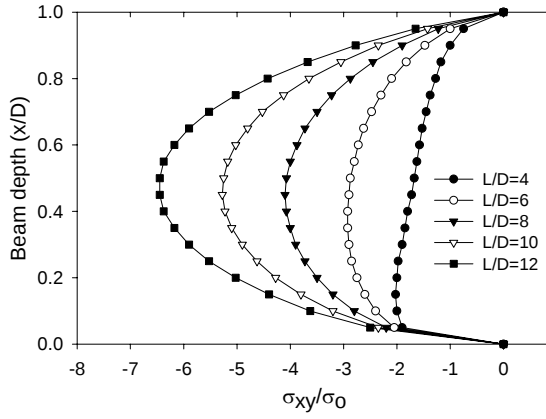


(a) ψ -solution

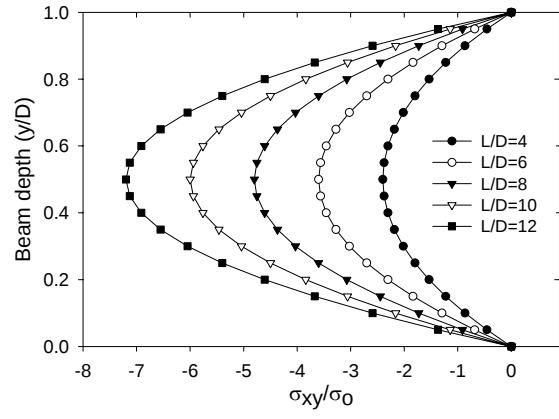


(b) beam theory

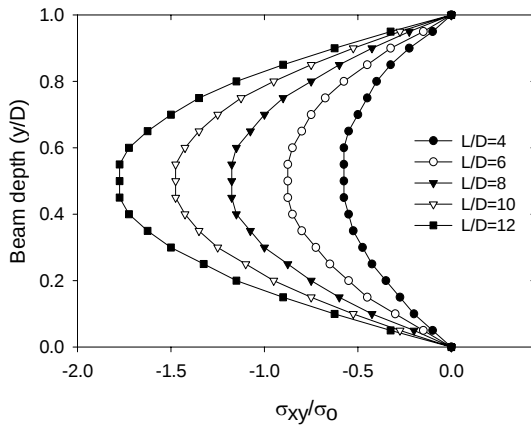
Fig. 4.20 Bending stress distribution of the glass-epoxy beams, (at $x/L=0.5$):
(a) displacement potential solution, (b) classical beam theory



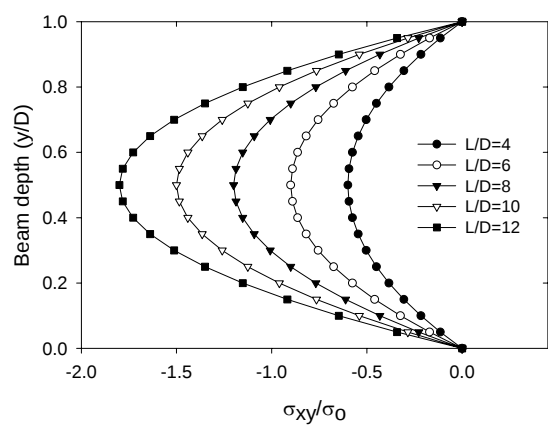
(a) ψ - solution (at $x/L=0.1$)



(b) beam theory (at $x/L=0.1$)



(c) ψ - solution (at $x/L=0.4$)



(d) beam theory (at $x/L=0.4$)

Fig. 4.21 Shear stress distribution of the glass-epoxy beams: (a) ψ - solution ($x/L=0.1$),
(b) beam theory ($x/L=0.1$), (c) ψ - solution ($x/L=0.4$), (d) beam theory ($x/L=0.4$)

The shear stress distribution for higher aspect ratios, i.e., for slender beams are almost similar in potential solution as well as beam theory. Shear stresses are found to similar both in terms of magnitudes and patterns. Fig. 4.21 (a) and (b) illustrate the shear stress distribution at $x/L=0.1$ in potential solution and beam theory respectively and Fig. 4.21 (c) and (d) for the same at $x/L=0.4$.

4.7 Summary

Analytical solution of the elastic field of a guided deep composite beam of orthotropic material is investigated using displacement potential approach confirming all the physical conditions. The solution of a glass epoxy beam for L/D ratio 3 is observed at first and the results are found to be more rational and improved in comparison to classical beam theory. The effects of changing aspect ratio are found to be significant and imperative to be considered for better design. With the rise of aspect ratio the non-linearity of stress components is highly influenced and reduced, but it is not reaching a state of linear condition up to the aspect ratio of eight. Thereafter the bending stress distribution is becoming almost linear and at the aspect ratio of ten it is linear. In all case the guided ends and load transition sections are identified to be the most critical region in terms of stresses. While the material of the beam is changed from glass-epoxy to boron-epoxy or graphite-epoxy, the range of maximum magnitude as well as the non-linearity is found to be changed.

CHAPTER 5

COMPOSITE DEEP BEAM UNDER THREE POINT BENDING

5.1 Guided Composite Beam under Symmetric Three Point Bending (Case-I)

5.1.1 Problem Articulation

A rectangular section deep orthotropic composite beam of length L , width D and thickness W (considered to be unity for plane stress condition) is subjected to a point loading at the mid-span its top boundary and simply supporting at its bottom two corners. By this symmetric loading arrangement the beam is in equilibrium as shown in figure 5.1. Both lateral edges of the beam i.e. at $x=0$ and $x=L$ are roller guided so that the axial displacements are restrained, but the lateral displacements are allowed to have any value.

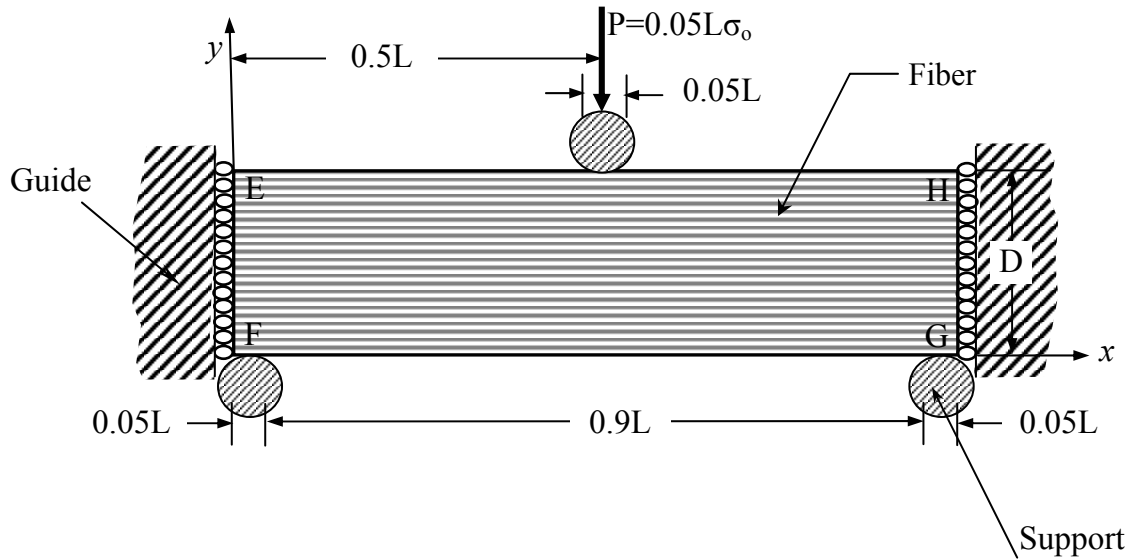


Fig. 5.1 Geometry and loading (symmetric) of orthotropic beam for three point bending

5.1.2 Boundary Conditions

The physical conditions at different boundaries of the beam are expressed mathematically as follows:

(i) $u_x = 0$ at the edge of $x = 0$

- (ii) $u_x = 0$ at the edge of $x = L$
- (iii) $\sigma_{xy}(0, y) = 0$ at the edge of $x = 0$
- (iv) $\sigma_{xy}(L, y) = 0$ at the edge of $x = L$
- (v) $\sigma_{xy}(x, 0) = 0$ at the edge of $y = 0$
- (vi) $\sigma_{xy}(x, D) = 0$ at the edge of $y = D$
- (vii) The lateral stress at the edge of $y = D$ is related to the applied load for the three point bending. Since the point load is actually acting over a certain area of the beam, for instance it can be considered for the length of $x=0.475L$ to $0.525L$.
Again it is considered that the load intensity is σ_0 . Therefore, the magnitude of point load, $P= 0.05L\sigma_0$. Then $\sigma_{yy}(x, D) = \sigma_0$ for $x=0.475L$ to $0.525L$
- (viii) Similarly, the lateral stress at the edge of $y = 0$ is related to the reactions at the support. In this case $\sigma_{yy}(x, 0) = \sigma_0 / 2$ for $x=0$ to $0.05L$ and $0.95L$ to L

5.1.3 Analytical Solution

Mathematical model here is the partial differential equation derived from the equations of equilibrium and equations of compatibility based on Displacement Potential Function $\psi(x, y)$ obtained from Eq. (2.28) as follows.

$$E_1 G_{12} \frac{\partial^4 \psi}{\partial x^4} + E_2 (E_1 - 2\mu_{12} G_{12}) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_2 G_{12} \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (5.1)$$

In this case the displacement and stress components are also obtained from Eq. (2.29) as follows:

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y} \quad (5.2a)$$

$$u_y(x, y) = -\frac{1}{Z_{11}} \left[E_1^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12} (E_1 - \mu_{12}^2 E_2) \frac{\partial^2 \psi}{\partial y^2} \right] \quad (5.2b)$$

$$\sigma_{xx}(x, y) = \frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial y^3} \right] \quad (5.2c)$$

$$\sigma_{yy}(x, y) = \frac{E_1 E_2}{Z_{11}} \left[(\mu_{12} G_{12} - E_1) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12} \frac{\partial^3 \psi}{\partial y^3} \right] \quad (5.2d)$$

$$\sigma_{xy}(x, y) = -\frac{E_1 G_{12}}{Z_{11}} \left[E_1 \frac{\partial^3 \psi}{\partial x^3} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \quad (5.2e)$$

$$\text{Where } Z_{11} = \mu_{12} E_1 E_2 + G_{12} (E_1 - \mu_{12}^2 E_2)$$

The displacement potential function ψ should be varied with cosine function of x in order to meet the requirement of physical conditions of the two opposing guided ends are automatically satisfied i.e. automatic satisfaction of boundary conditions from (i) to (iv). At the same time the expression for ψ should also be compatible to the distribution of load. Considering all these factors the expression for ψ may be approximated as follows:

$$\psi = \sum_{m=1}^{\infty} Y_m \cos \alpha x + Ky^3 \quad (5.3)$$

where $Y_m = f(y)$, $\alpha = \frac{m\pi}{L}$ and $K = \text{Arbitrary constant}$

Using the derivatives of equation (5.3) in equation (5.1) it is found that

$$Y_m''' - \left(\frac{E_1}{G_{12}} - 2\mu_{12} \right) Y_m'' \alpha^2 + \frac{E_1}{E_2} Y_m \alpha^4 = 0 \quad (5.4)$$

The general solution of the above 4th order ordinary differential equation can be as

$$Y_m = A_m e^{r_1 y} + B_m e^{r_2 y} + C_m e^{r_3 y} + D_m e^{r_4 y} \quad (5.5)$$

where A_m , B_m , C_m and D_m are arbitrary constants

and r_1 , r_2 , r_3 and r_4 are the roots of following equation

$$r^4 - \left(\frac{E_1}{G_{12}} - 2\mu_{12} \right) \alpha^2 r^2 + \frac{E_1}{E_2} \alpha^4 = 0$$

Thus

$$r_1, r_2 = \frac{\alpha}{\sqrt{2}} \left[\left(\frac{E_1}{G_2} - 2\mu_{12} \right) \pm \sqrt{\left(\frac{E_1}{G_2} - 2\mu_{12} \right)^2 - 4 \frac{E_1}{E_2}} \right]^{\frac{1}{2}} \quad (5.6a)$$

$$r_3, r_4 = -\frac{\alpha}{\sqrt{2}} \left[\left(\frac{E_1}{G_2} - 2\mu_{12} \right) \pm \sqrt{\left(\frac{E_1}{G_2} - 2\mu_{12} \right)^2 - 4 \frac{E_1}{E_2}} \right]^{\frac{1}{2}} \quad (5.6b)$$

Now using equation (5.3) and (5.5) in the expressions for the components of displacement and stress [5.2]

$$u_x(x, y) = -\sum_{m=1}^{\infty} (A_m r_1 e^{r_1 y} + B_m r_2 e^{r_2 y} + C_m r_3 e^{r_3 y} + D_m r_4 e^{r_4 y}) \alpha \sin \alpha x \quad (5.7a)$$

$$u_y(x, y) = \frac{1}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \left(E_1^2 \alpha^2 - E_1 G_{12} r_1^2 + \mu_{12}^2 E_2 G_{12} r_1^2 \right) e^{r_1 y} A_m + \left(E_1^2 \alpha^2 - E_1 G_{12} r_2^2 + \mu_{12}^2 E_2 G_{12} r_2^2 \right) e^{r_2 y} B_m + \right. \right. \\ \left. \left. \left(E_1^2 \alpha^2 - E_1 G_{12} r_3^2 + \mu_{12}^2 E_2 G_{12} r_3^2 \right) e^{r_3 y} C_m + \left(E_1^2 \alpha^2 - E_1 G_{12} r_4^2 + \mu_{12}^2 E_2 G_{12} r_4^2 \right) e^{r_4 y} D_m \right\} \right] \cos \alpha x - (E_1 G_{12} - \mu_{12}^2 E_2 G_{12}) 6K y \quad (5.7b)$$

$$\sigma_{xx}(x, y) = -\frac{E_1 G_{12}}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \left(E_1 r_1 e^{r_1 y} \alpha^2 + \mu_{12} E_2 r_1^3 e^{r_1 y} \right) A_m + \left(E_1 r_2 e^{r_2 y} \alpha^2 + \mu_{12} E_2 r_2^3 e^{r_2 y} \right) B_m + \right. \right. \\ \left. \left. \left(E_1 r_3 e^{r_3 y} \alpha^2 + \mu_{12} E_2 r_3^3 e^{r_3 y} \right) C_m + \left(E_1 r_4 e^{r_4 y} \alpha^2 + \mu_{12} E_2 r_4^3 e^{r_4 y} \right) D_m \right\} \right] \cos \alpha x + 6K \mu_{12} E_2 \quad (5.7c)$$

$$\sigma_{yy}(x, y) = -\frac{E_1 E_2}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \left(\mu_{12} G_{12} r_1 e^{r_1 y} \alpha^2 - E_1 r_1 e^{r_1 y} \alpha^2 + G_{12} r_1^3 e^{r_1 y} \right) A_m + \right. \right. \\ \left. \left(\mu_{12} G_{12} r_2 e^{r_2 y} \alpha^2 - E_1 r_2 e^{r_2 y} \alpha^2 + G_{12} r_2^3 e^{r_2 y} \right) B_m + \right. \\ \left. \left(\mu_{12} G_{12} r_3 e^{r_3 y} \alpha^2 - E_1 r_3 e^{r_3 y} \alpha^2 + G_{12} r_3^3 e^{r_3 y} \right) C_m + \right. \\ \left. \left(\mu_{12} G_{12} r_4 e^{r_4 y} \alpha^2 - E_1 r_4 e^{r_4 y} \alpha^2 + G_{12} r_4^3 e^{r_4 y} \right) D_m \right\} \cos \alpha x + 6K G_{12} \quad (5.7d)$$

$$\sigma_{xy}(x, y) = -\frac{E_1 G_{12}}{Z_{11}} \left[\sum_{m=1}^{\infty} \left\{ \left(E_1 e^{r_1 y} \alpha^3 + \mu_{12} E_2 r_1^2 e^{r_1 y} \alpha \right) A_m + \left(E_1 e^{r_2 y} \alpha^3 + \mu_{12} E_2 r_2^2 e^{r_2 y} \alpha \right) B_m + \right. \right. \\ \left. \left(E_1 e^{r_3 y} \alpha^3 + \mu_{12} E_2 r_3^2 e^{r_3 y} \alpha \right) C_m + \left(E_1 e^{r_4 y} \alpha^3 + \mu_{12} E_2 r_4^2 e^{r_4 y} \alpha \right) D_m \right\} \sin \alpha x \quad (5.7e)$$

From the expressions of (5.7a) and (5.7e) it is evident that the boundary conditions (i) to (iv) are automatically satisfied. Thus the next requirement is to satisfy the remaining boundary conditions of (v) to (viii).

Now the compressive load exerted at the mid-span on the edge $y=0$ of the beam may be considered as acting over at least some length of the beam, for instance $x=0.475L$ to $0.525L$. It may then be taken as Fourier series in the following manner:

$$\sigma_{yy}(x, 0) = \sigma_0 = I_0 + \sum_{m=1}^{\infty} I_m \cos \alpha x \quad \text{for } x=0.475L \text{ to } 0.525L$$

Here

$$I_0 = \frac{1}{L} \left[\int_{19L/40}^{21L/40} \sigma_0 dx \right] = \frac{\sigma_0}{20} \quad (5.8a)$$

$$\begin{aligned}
I_m &= \frac{2}{L} \left[\int_{19L/40}^{21L/40} \sigma_0 \cos \alpha x dx \right] \\
&= \frac{2\sigma_0}{\alpha L} \left\{ \sin\left(\frac{21\alpha L}{40}\right) - \sin\left(\frac{19\alpha L}{40}\right) \right\} \\
&= \frac{2\sigma_0}{m\pi} \left\{ \sin\left(\frac{21m\pi}{40}\right) - \sin\left(\frac{19m\pi}{40}\right) \right\}
\end{aligned} \tag{5.8b}$$

The reaction load at the support on the edge $y = D$ can also be given by a Fourier series as follows

$$\sigma_{yy}(x,0) = \sigma_0 / 2 = E_0 + \sum_{m=1}^{\infty} E_m \cos \alpha x \quad \text{for } x=0 \text{ to } 0.05L \text{ and } 0.95L \text{ to } L$$

Here

$$E_0 = \frac{1}{L} \left[\int_0^{L/20} \sigma_0 / 2 dx + \int_{19L/20}^L \sigma_0 / 2 dx \right] = \frac{\sigma_0}{20} \tag{5.9a}$$

$$\begin{aligned}
E_m &= \frac{2}{L} \left[\int_0^{L/20} \frac{\sigma_0}{2} \cos \alpha x dx + \int_{19L/20}^L \frac{\sigma_0}{2} \cos \alpha x dx \right] \\
&= \frac{\sigma_0}{\alpha L} \left\{ \sin\left(\frac{\alpha L}{20}\right) - 0 + \sin(\alpha L) - \sin\left(\frac{19\alpha L}{20}\right) \right\} \\
&= \frac{\sigma_0}{m\pi} \left\{ \sin\left(\frac{m\pi}{20}\right) + \sin(m\pi) - \sin\left(\frac{19m\pi}{20}\right) \right\}
\end{aligned} \tag{5.9b}$$

Using boundary conditions (v) $\sigma_{xy}(x,0)=0$ at the edge of $y=0$, (vi) $\sigma_{xy}(x,D)=0$ at the edge of $y=D$, (vii) $\sigma_{yy}(x,0)=\sigma_0/2$ at the edge of $y=0$ and (viii) $\sigma_{yy}(x,D)=\sigma_0$ at the edge of $y=D$

$$-\frac{E_1 G_{12}}{Z_{11}} \left\{ (E_1 \alpha^3 + \mu_{12} E_2 r_1^2 \alpha) A_m + (E_1 \alpha^3 + \mu_{12} E_2 r_2^2 \alpha) B_m + \right. \\ \left. (E_1 \alpha^3 + \mu_{12} E_2 r_3^2 \alpha) C_m + (E_1 \alpha^3 + \mu_{12} E_2 r_4^2 \alpha) D_m \right\} = 0 \tag{5.10a}$$

$$-\frac{E_1 G_{12}}{Z_{11}} \left\{ (E_1 e^{r_1 D} \alpha^3 + \mu_{12} E_2 r_1^2 e^{r_1 D} \alpha) A_m + (E_1 e^{r_2 D} \alpha^3 + \mu_{12} E_2 r_2^2 e^{r_2 D} \alpha) B_m + \right. \\ \left. (E_1 e^{r_3 D} \alpha^3 + \mu_{12} E_2 r_3^2 e^{r_3 D} \alpha) C_m + (E_1 e^{r_4 D} \alpha^3 + \mu_{12} E_2 r_4^2 e^{r_4 D} \alpha) D_m \right\} = 0 \tag{5.10b}$$

Again using boundary conditions

$$-\frac{E_1 E_2}{Z_{11}} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 \alpha^2 - E_1 r_1 \alpha^2 + G_{12} r_1^3) A_m + \\ &(\mu_{12} G_{12} r_2 \alpha^2 - E_1 r_2 \alpha^2 + G_{12} r_2^3) B_m + \\ &(\mu_{12} G_{12} r_3 \alpha^2 - E_1 r_3 \alpha^2 + G_{12} r_3^3) C_m + \\ &(\mu_{12} G_{12} r_4 \alpha^2 - E_1 r_4 \alpha^2 + G_{12} r_4^3) D_m \end{aligned} \right\} = E_m \quad (5.10c)$$

$$-\frac{E_1 E_2}{Z_{11}} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 e^{r_1 D} \alpha^2 - E_1 r_1 e^{r_1 D} \alpha^2 + G_{12} r_1^3 e^{r_1 D}) A_m + \\ &(\mu_{12} G_{12} r_2 e^{r_2 D} \alpha^2 - E_1 r_2 e^{r_2 D} \alpha^2 + G_{12} r_2^3 e^{r_2 D}) B_m + \\ &(\mu_{12} G_{12} r_3 e^{r_3 D} \alpha^2 - E_1 r_3 e^{r_3 D} \alpha^2 + G_{12} r_3^3 e^{r_3 D}) C_m + \\ &(\mu_{12} G_{12} r_4 e^{r_4 D} \alpha^2 - E_1 r_4 e^{r_4 D} \alpha^2 + G_{12} r_4^3 e^{r_4 D}) D_m \end{aligned} \right\} = I_m \quad (5.10d)$$

From equations (5.7d) and (5.8a) or (5.9a)

$$-\frac{6E_1 E_2 K G_{12}}{Z_{11}} = E_0 = I_0 = \frac{\sigma_0}{20}$$

$$K = -\frac{Z_{11} \sigma_0}{120 E_1 E_2 G_{12}} \quad (5.11)$$

The four algebraic equations (5.10a), (5.10b), (5.10c) and (5.10d) are solved simultaneously and the values of four unknowns, such as A_m , B_m , C_m and D_m , are obtained. Therefore, stress and displacement components at various points of the beam can be obtained using equations of (5.7).

5.1.4 Results of Three Point Bending

The analytical solutions of displacement and stress components are obtained for various aspect ratios (L/D) of the beam. At first the result of a guided composite beam of glass epoxy material having aspect ratio three and the uniform loading parameter = 40 N/mm is presented in sequence of axial displacement (u_x), lateral displacement (u_y), bending stress (σ_{xx}), normal stress (σ_{yy}) and shearing stress (σ_{xy}). Thereafter a verification of the present solution is made comparing the results of displacement potential approach with that of FEM.

i. Solution of Displacements

Axial displacements (u_x) are found to be zero at the mid section of span and at the lateral guided boundaries [Fig. 5.2(a) and (b)]. Zero value of u_x at the guided ends verifies the boundary condition of those edges of the beam. Axial displacements distribution is found skew-symmetric about the mid-span of the beam. The values of u_x for sections $0 < x/L < 0.5$ are negative at the lower portion and positive at upper portion of the beam. The maximum magnitudes of $u_x/L = \pm 0.000058$ are observed on bottom fibre at the sections of $x/L = 0.2$ and $x/L = 0.8$ respectively.

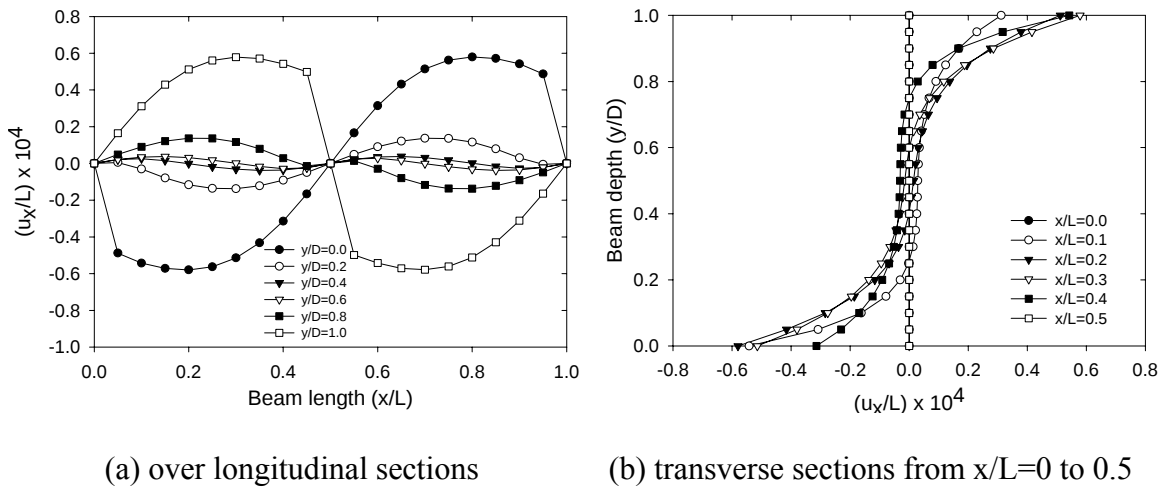


Fig. 5.2 Normalized axial displacement for three point bending, $L/D = 3$:
(a) over longitudinal sections, (b) over transverse sections

Lateral displacements (u_y) are found to take positive value near the two guided lateral ends and negative in the region $0.25 < x/L < 0.75$, on account of loading at top edge and balanced by two bottom corner support [Fig. 5.3(a) and (b)]. There is no restriction on the lateral displacement other than the loading and reaction to bring the beam in equilibrium state. The u_y results are in confirmation to the physical condition of the beam. As a result the beam is being pushed up at the corners and forced down at the mid-span region. The normalized values of positive and negative maximum lateral displacements are $u_y/D = 0.000352$ and $u_y/D = -0.000506$ respectively. The maximum magnitude is observed on the topmost fibre at the mid-span.

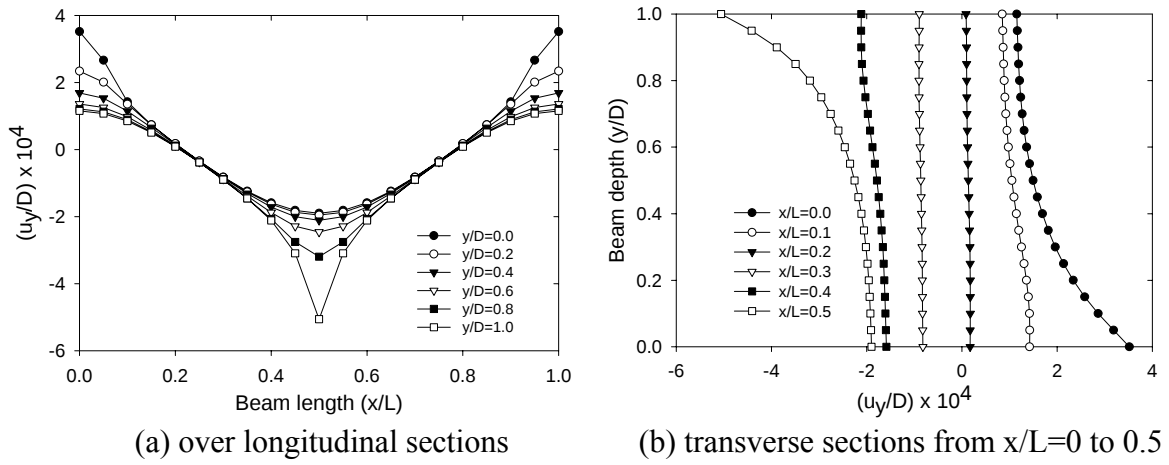


Fig. 5.3 Normalized lateral displacement distribution for three point bending, $L/D = 3$:
(a) over longitudinal sections, (b) over transverse sections

Fig. 5.4 illustrates the deformed shape of the orthotropic simply supported guided beam subjected to a point load at the mid-span. Here the deformation is shown along with its original shape at the magnification of 500 times of displacement. The guided ends have gone up and at the same time centre region of the beam have gone down. The deformation of the top edge at the location of load is like a notch. The bottom edge is deformed almost uniformly. The vertical sliding at the guided region and down word deformation at the mid-span is in proper agreement with the physical condition of applied loading and support of the guided beam.

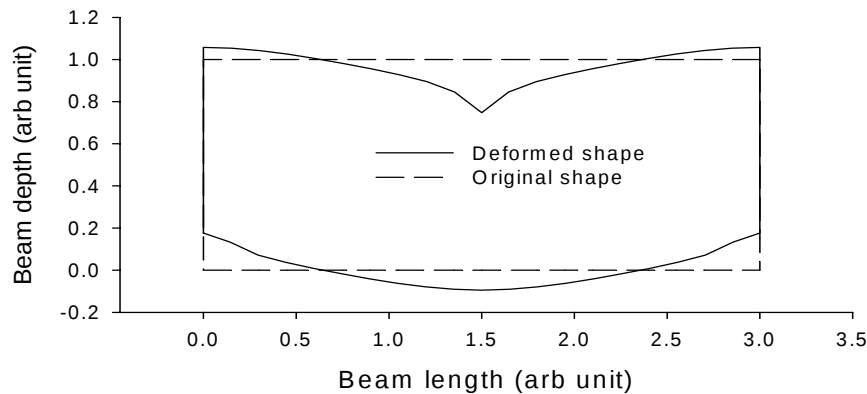


Fig. 5.4 Deformation under three point bending, $L/D = 3$ (magnification factor $\times 500$)

ii. Solution of Stresses

Bending stress distribution is observed non-linear over the whole span [Fig. 5.5(a) and (b)]. The stress (σ_{xx}) maximizes at the mid-span top fibre of the beam where the point type load is acting. The next locations of bending stress concentration are the two bottom corners and the bottom fibre at mid-span of the beam. The maximum magnitude of normalized bending stress on the top fibre at mid-span is a 2.13.

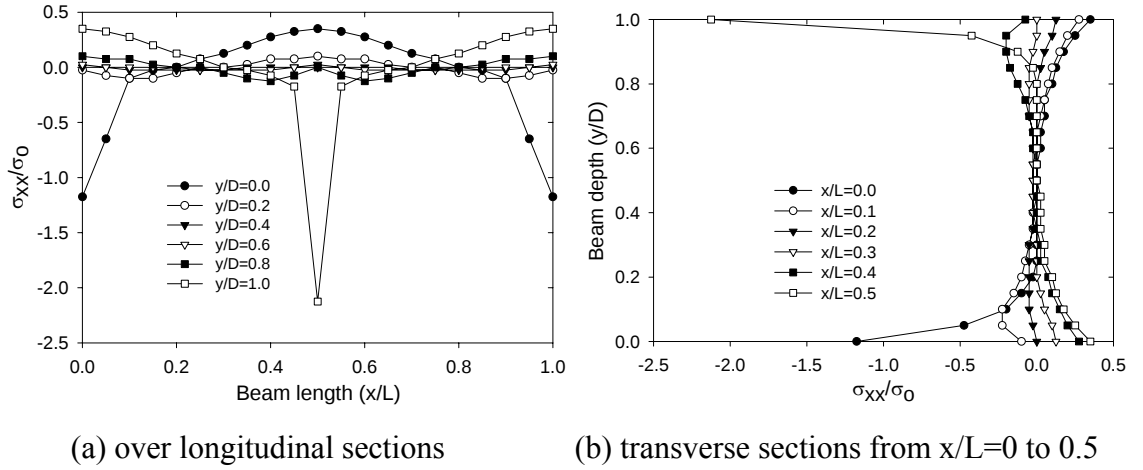


Fig. 5.5 Distribution of bending stress of a glass epoxy beam for three point bending, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

Lateral stress (σ_{yy}) concentrations are observed at the topmost fibre of mid-span section and bottom two corners. From Fig. 5.6 it is understandable that each reaction is half of the load and the normalized value of the lateral stress varies from zero to about unity at the topmost layer in the loaded region and it is almost half at the bottom layer of the support region, which confirms the physical condition of the problem.

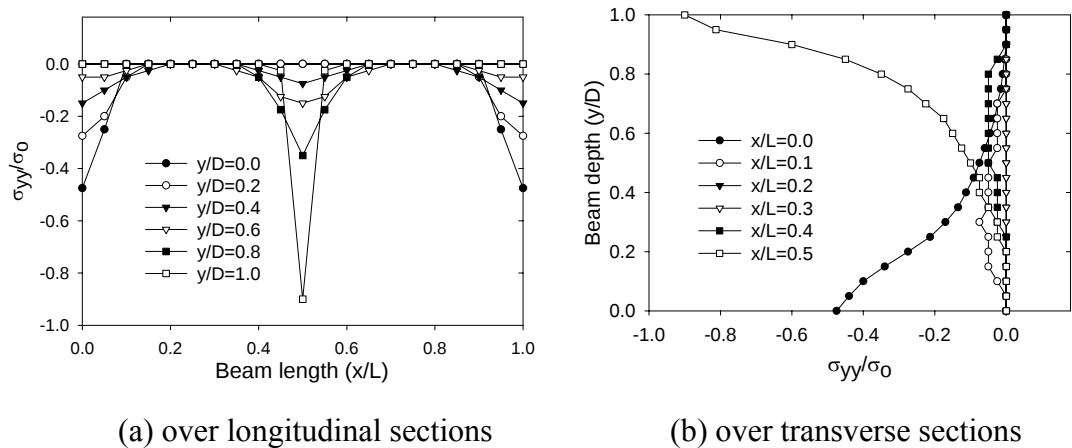


Fig. 5.6 Distribution of lateral stress of a glass epoxy beam for three point bending, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

All four edges and mid-span section of the guided deep beam are found free from shearing stress [Fig. 5.7(a) and (b)]. The distribution of shearing stress (σ_{xy}) for point loading is anti-symmetric in two sides about the mid-span of the beam like distributed load [Fig. 5.7(a)]. The maximum concentration of shearing stress is observed near the bottom corners at the supports and the point of shear stress concentration is at the top edge where the termination of loading takes place. The normalised maximum magnitude of shear stress is ± 0.2 at sections $x/L=0.05$ and 0.95 on $y/D=0.05$.

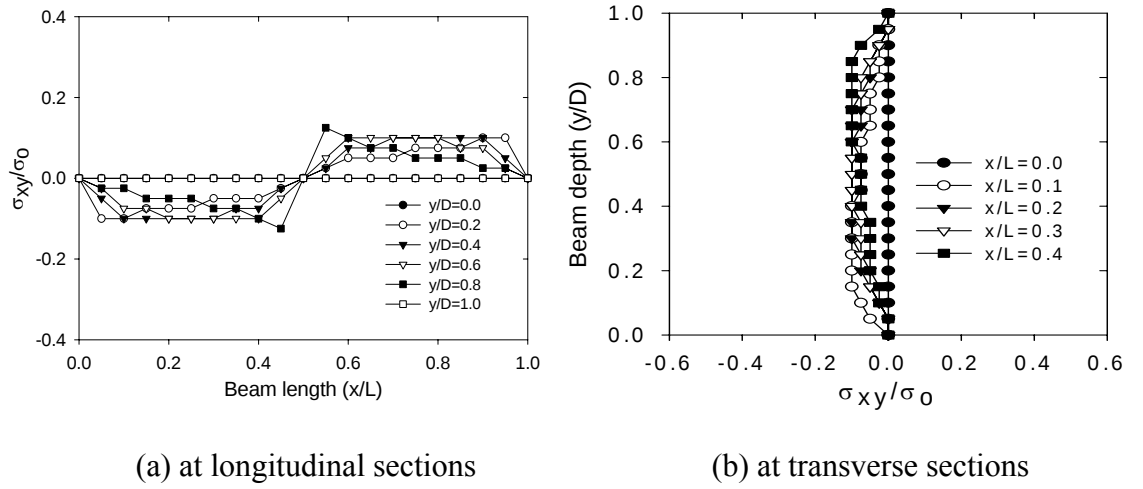


Fig. 5.7 Distribution of shearing stress of a glass epoxy beam for three point bending, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

5.2 Analytical Solution for Guided Composite Beam under Asymmetric Three Point Bending (Case-II)

5.2.1 Problem Articulation

A rectangular section deep orthotropic composite beam of length L , width D and thickness W (considered to be unity for plane stress condition) is subjected to a asymmetric point loading at the section of $x/L=0.6$ of its top boundary and simply supported at its bottom two corners. With this loading arrangement the beam is to be in equilibrium condition as shown in figure 5.8. Both lateral edges of the beam i.e. at $x=0$ and $x=L$ are roller guided so that the axial displacements are restrained, but the beam can have lateral displacements of any magnitude.

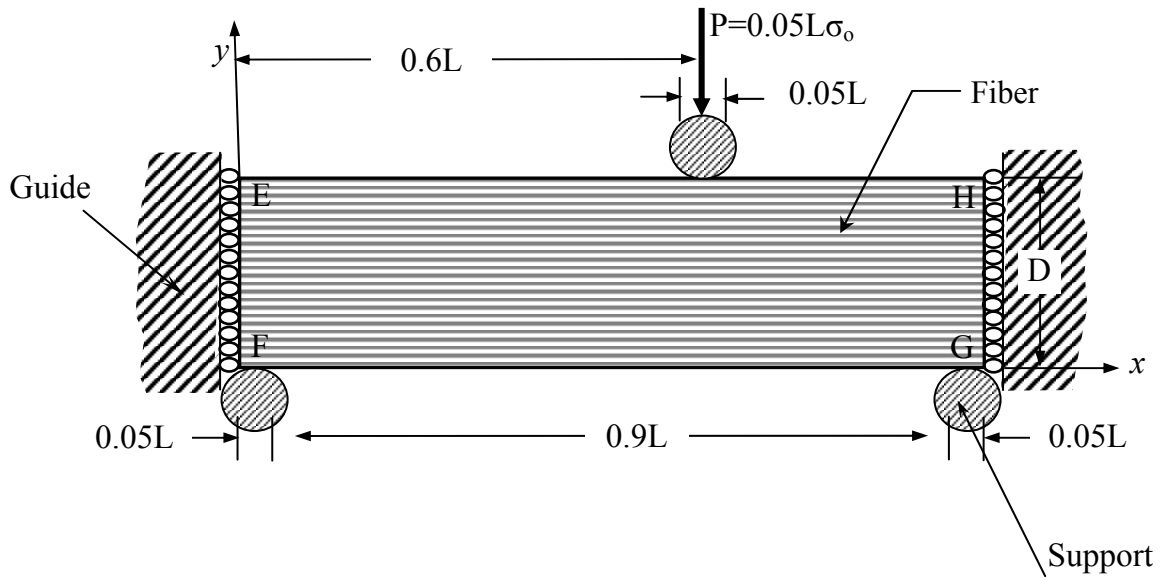


Fig. 5.8 Geometry and loading (asymmetric) of orthotropic beam

5.2.2 Boundary Conditions

The physical conditions at different boundaries of the beam are similar to that of section 5.1.2 except the conditions supporting end, EH. Since the lateral stress at this edge is in relation to the reaction and the shear stress is zero, the boundary conditions would be here as follows:

$$\sigma_{xy}(x, D) = 0 \text{ for } 0 \leq x \leq L \quad \text{and} \quad \sigma_{yy}(x, D) = \sigma_0 \text{ for } x = 0.575L \text{ to } 0.625L$$

5.2.3 Analytical Solution

Mathematical model here is as same as Eq. (5.1) and the components of displacement and stress are also obtained from Eq. (5.2a) to (5.2e). Considering the physical conditions of the problem ψ -function may be approximated as same as to Eq. (5.3).

$$\psi = \sum_{m=1}^{\infty} Y_m \cos \alpha x + Ky^3 \quad ; \text{ where } Y_m = f(y), \quad \alpha = \frac{m\pi}{L} \text{ and } K = \text{Arbitrary constant}$$

Using the derivatives of above expression in similar way for problem 5.1 (symmetric three point bending) the 4th order ordinary differential equation and its general solution can be found. But the difference remains with the top and bottom boundary conditions.

Now the compressive load exerted at the supports located on the edge $y = 0$ of the beam may be taken as Fourier series in the following manner:

$$\sigma_{yy}(x,0) = \sigma_0 / 2 = E_0 + \sum_{m=1}^{\infty} E_m \cos \alpha x \quad \text{for } x=0 \text{ to } 0.05L \text{ and } 0.95L \text{ to } L$$

Here

$$E_0 = \frac{1}{L} \left[\int_0^{L/20} \sigma_0 / 2 dx + \int_{19L/20}^L \sigma_0 / 2 dx \right] = \frac{\sigma_0}{20} \quad (5.12a)$$

$$\begin{aligned} E_m &= \frac{2}{L} \left[\int_0^{L/20} \frac{\sigma_0}{2} \cos \alpha x dx + \int_{19L/20}^L \frac{\sigma_0}{2} \cos \alpha x dx \right] \\ &= \frac{\sigma_0}{\alpha L} \left\{ \sin\left(\frac{\alpha L}{20}\right) - 0 + \sin(\alpha L) - \sin\left(\frac{19\alpha L}{20}\right) \right\} \\ &= \frac{\sigma_0}{m\pi} \left\{ \sin\left(\frac{m\pi}{20}\right) + \sin(m\pi) - \sin\left(\frac{19m\pi}{20}\right) \right\} \end{aligned} \quad (5.12b)$$

The load acting at the section of $x/L=0.6$ on the edge $y = D$ can also be given by a Fourier series as follows

$$\sigma_{yy}(x,0) = \sigma_0 = I_0 + \sum_{m=1}^{\infty} I_m \cos \alpha x \quad \text{for } x= 0.475L \text{ to } 0.525L$$

Here

$$I_0 = \frac{1}{L} \left[\int_{23L/40}^{25L/40} \sigma_0 dx \right] = \frac{\sigma_0}{20} \quad (5.13a)$$

$$\begin{aligned}
I_m &= \frac{2}{L} \left[\int_{23L/40}^{25L/40} \sigma_0 \cos \alpha x dx \right] \\
&= \frac{2\sigma_0}{\alpha L} \left\{ \sin\left(\frac{25\alpha L}{40}\right) - \sin\left(\frac{23\alpha L}{40}\right) \right\} \\
&= \frac{2\sigma_0}{m\pi} \left\{ \sin\left(\frac{25m\pi}{40}\right) - \sin\left(\frac{23m\pi}{40}\right) \right\}
\end{aligned} \tag{5.13b}$$

Using boundary conditions $\sigma_{xy}(x,0)=0$ and $\sigma_{yy}(x,0)=\sigma_0/2$ for the edge of $y=0$ and $\sigma_{xy}(x,D)=0$ and $\sigma_{yy}(x,D)=\sigma_0$ at the edge of $y=D$

$$-\frac{E_1 G_{12}}{Z_{11}} \left\{ (E_1 \alpha^3 + \mu_{12} E_2 r_1^2 \alpha) A_m + (E_1 \alpha^3 + \mu_{12} E_2 r_2^2 \alpha) B_m + \right. \tag{5.14a}$$

$$\left. - \frac{E_1 G_{12}}{Z_{11}} \left\{ (E_1 e^{r_1^D} \alpha^3 + \mu_{12} E_2 r_1^2 e^{r_1^D} \alpha) A_m + (E_1 e^{r_2^D} \alpha^3 + \mu_{12} E_2 r_2^2 e^{r_2^D} \alpha) B_m + \right. \right. \tag{5.14b}$$

$$\left. \left. - \frac{E_1 E_2}{Z_{11}} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 \alpha^2 - E_1 r_1 \alpha^2 + G_{12} r_1^3) A_m + \\ &(\mu_{12} G_{12} r_2 \alpha^2 - E_1 r_2 \alpha^2 + G_{12} r_2^3) B_m + \\ &(\mu_{12} G_{12} r_3 \alpha^2 - E_1 r_3 \alpha^2 + G_{12} r_3^3) C_m + \\ &(\mu_{12} G_{12} r_4 \alpha^2 - E_1 r_4 \alpha^2 + G_{12} r_4^3) D_m \end{aligned} \right\} = E_m \right. \right. \tag{5.14c}$$

$$\left. - \frac{E_1 E_2}{Z_{11}} \left\{ \begin{aligned} &(\mu_{12} G_{12} r_1 e^{r_1^D} \alpha^2 - E_1 r_1 e^{r_1^D} \alpha^2 + G_{12} r_1^3 e^{r_1^D}) A_m + \\ &(\mu_{12} G_{12} r_2 e^{r_2^D} \alpha^2 - E_1 r_2 e^{r_2^D} \alpha^2 + G_{12} r_2^3 e^{r_2^D}) B_m + \\ &(\mu_{12} G_{12} r_3 e^{r_3^D} \alpha^2 - E_1 r_3 e^{r_3^D} \alpha^2 + G_{12} r_3^3 e^{r_3^D}) C_m + \\ &(\mu_{12} G_{12} r_4 e^{r_4^D} \alpha^2 - E_1 r_4 e^{r_4^D} \alpha^2 + G_{12} r_4^3 e^{r_4^D}) D_m \end{aligned} \right\} = I_m \right. \tag{5.14d}$$

From equations (5.7d) and (5.12a) or (5.13a)

$$-\frac{6E_1 E_2 K G_{12}}{Z_{11}} = E_0 = I_0 = \frac{\sigma_0}{20}$$

$$K = -\frac{Z_{11} \sigma_0}{120 E_1 E_2 G_{12}} \tag{5.15}$$

The four algebraic equations (5.14a), (5.14b), (5.14c) and (5.14d) are solved simultaneously and the values of four unknowns, such as A_m , B_m , C_m and D_m , are obtained. Therefore, stress and displacement components at various points of the beam can be obtained using expressions of Eqs. (5.7a) to (5.7e).

5.2.4 Results of Asymmetric Three Point Bending

The analytical solutions of displacement and stress components are obtained for a glass epoxy beam subjected to asymmetric point loading [Fig. 5.8] on the top edge at $x/L=0.6$ having various aspect ratios. The solution of such a composite orthotropic beam with aspect ratio three ($L/D=3$) taking the concentration of applied point loading = 40 N/mm is presented for axial displacement (u_x), lateral displacement (u_y), bending stress (σ_{xx}), normal stress (σ_{yy}) and shearing stress (σ_{xy}) in succession.

i. Solution of Displacements

Fig. 5.9 (a) and (b) illustrate that the guided edges of the deep composite beam do not contain any axial displacement u_x , which validates the proper satisfaction of boundary condition of those edges. Axial displacements distribution takes its skew pattern about the point of loading. The values of u_x are negative at the lower portion and positive at upper portion of the beam for sections $0 < x/L < 0.6$ and vice versa for the remaining part of $x/L=0.6$ to 1.0. The maximum magnitudes of $u_x/L = \pm 0.0000599$ are observed only on bottom fibre at the section of $x/L = 0.25$.

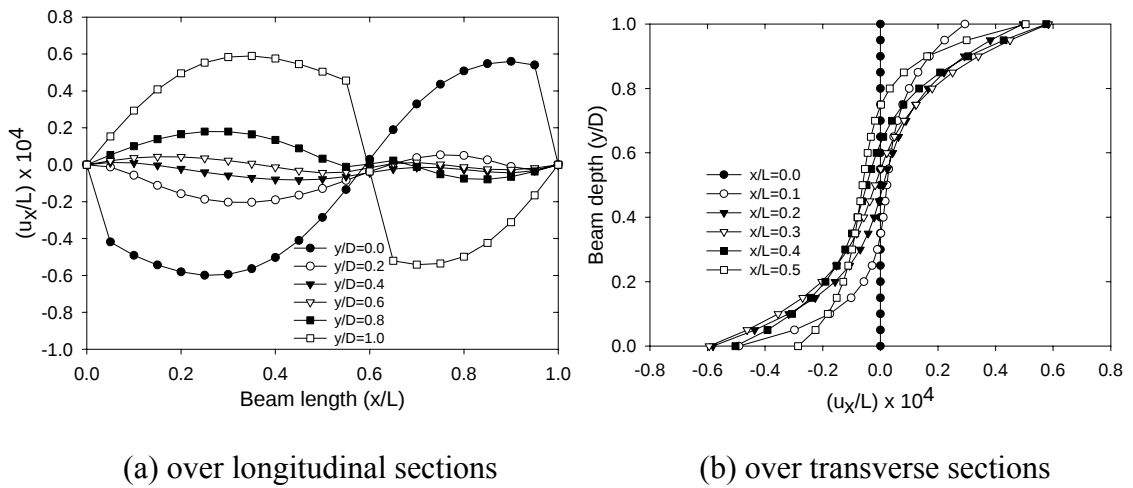


Fig. 5.9 Normalized axial displacement glass-epoxy beam with asymmetric point loading, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Lateral displacements (u_y) are found to follow the loading direction and take the negative values in the region $0.3 < x/L < 0.85$, but positive near the two guided lateral ends [Fig. 5.10(a) and (b)]. The u_y results are in confirmation to the physical condition of having no restriction on the lateral boundaries of the beam. As such it is seen that the

lowest fibres of the beam is pressed up at the highest order in the corners. At the same time the loaded region of the beam is strained down. The normalized maximum values of positive and negative lateral displacements are $u_y/D=0.000343$ and $u_y/D= - 0.000501$ respectively. The maximum magnitude is observed on the topmost fibre at the point of loading.

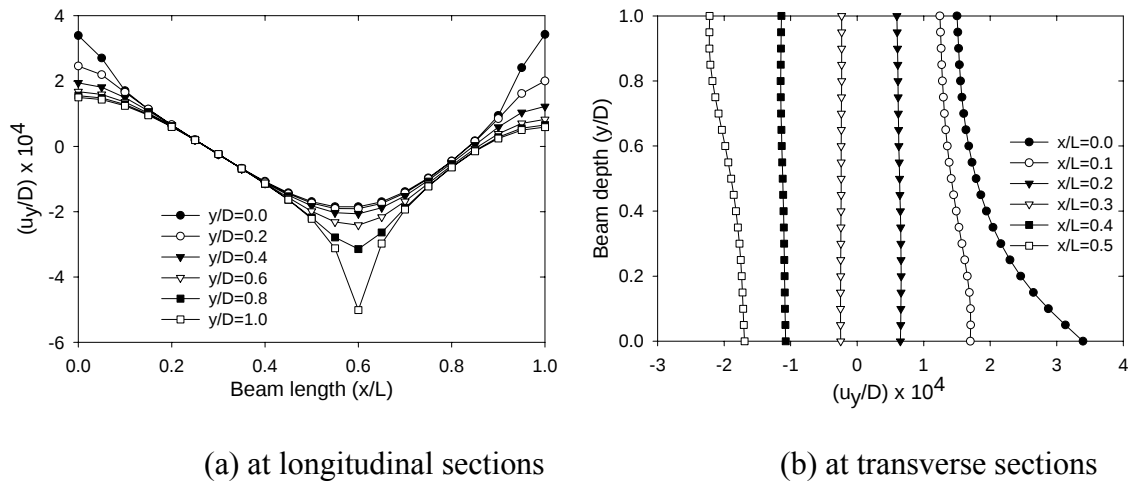


Fig. 5.10 Normalized lateral displacement glass-epoxy beam with asymmetric point loading, $L/D = 3$: (a) over longitudinal sections, (b) over transverse sections

Fig. 5.11 demonstrates the deformed shape of the orthotropic simply supported guided beam subjected to an asymmetric point load at $x/L=0.6$. Here the deformation is shown along with its original shape at the magnification of 500 times of displacement. The two bottom corners of guided ends have been compressed up and at the same time top two corners have not displaced the same. It results the reduction of depth at the corners from the original shape of the beam. Other than the guided ends the beam has moved downwards. The top edge at the location of load takes a sharp deformation, which commensurate the lateral stress at that point. The deformation at the bottom edge of the beam is almost uniform over the length. The upward sliding at the guided region and downward deformation in between is an agreement with the physical condition of applied loading and support of the guided beam. The overall state of deformed shape illustrates the rationality of accuracy in the present solution using displacement function.

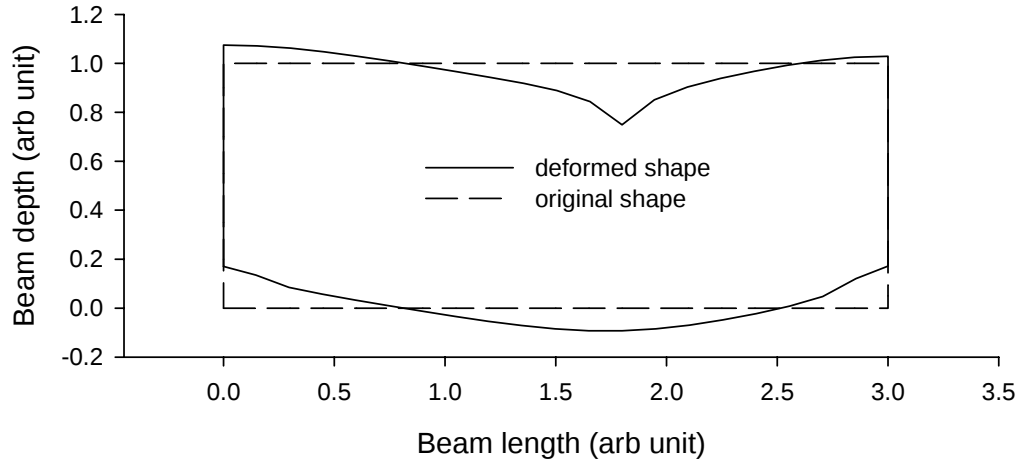


Fig. 5.11 Deformed shape of the glass-epoxy beam with asymmetric three point bending, $L/D = 3$ (magnification factor $\times 500$)

ii. Solution of Stresses

Bending stress distribution is observed non-linear over the whole span towards both the axes [Fig. 5.12(a) and (b)]. The non-linearity pattern takes a change from guide to load. On bottom fibre, the bending stress is with negative values at the two guides and it becomes positive near the region of loading. For the top fibre it is reversed. The highest concentration of bending stress (σ_{xx}) is seen on the top fibre at the point of loading. The next location of bending stress concentration is the bottom corner near to the load of the beam. The value of maximum normalized bending stress is 2.08.

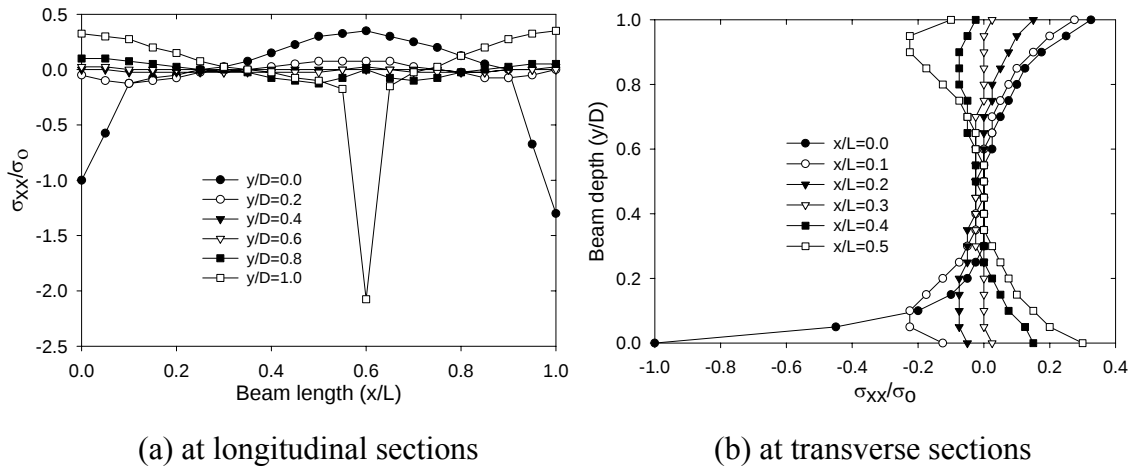


Fig. 5.12 Distribution of bending stress of a glass epoxy beam under asymmetric three point loading, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

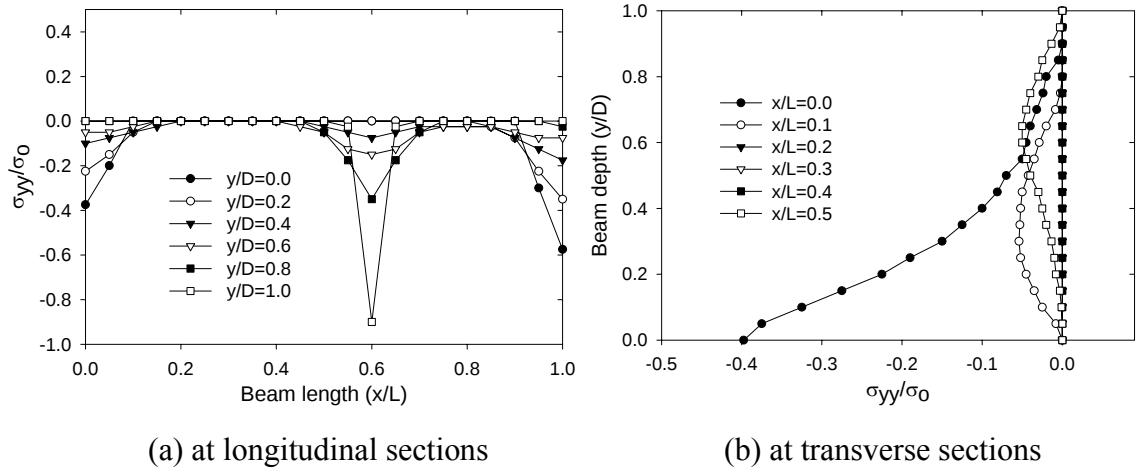


Fig. 5.13 Distribution of lateral stress of a glass epoxy beam under asymmetric three point loading, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

The highest concentration of lateral stress (σ_{yy}) is observed at the topmost fibre at the point of loading ($x/L=0.6$). The next points are bottom corners nearer the supports and bottom fibre at the load. Fig. 5.13(a) illustrates that reactions at the supports are rational with the position of loading. The normalized value of the lateral stress is completely zero from $x/L=0.2$ to 0.4 for all fibres of the beam. At the top fibre there is no lateral stress other than the point of loading, where it takes the normalised value about unity.

Fig. 4.14(a) illustrates that all four edges and mid-span section of the guided deep beam are found to be free from shearing stress. The distribution of shearing stress (σ_{xy}) for asymmetric point loading at $x/L=0.6$ is also asymmetric. The magnitude is higher in the shorter distant part with respect to the load than the longer part of the beam. The maximum magnitude of shear stress is observed at $y/D=0.05$ of section $x/L=0.95$ (normalized value = 0.225). It is noticeable from Fig. 5.14(a) that the shear stress distribution pattern is not of parabolic shape as well known to every one. It is rather different for different sections of the beam. It gives the food for thought to look into further finding considering different loading conditions.

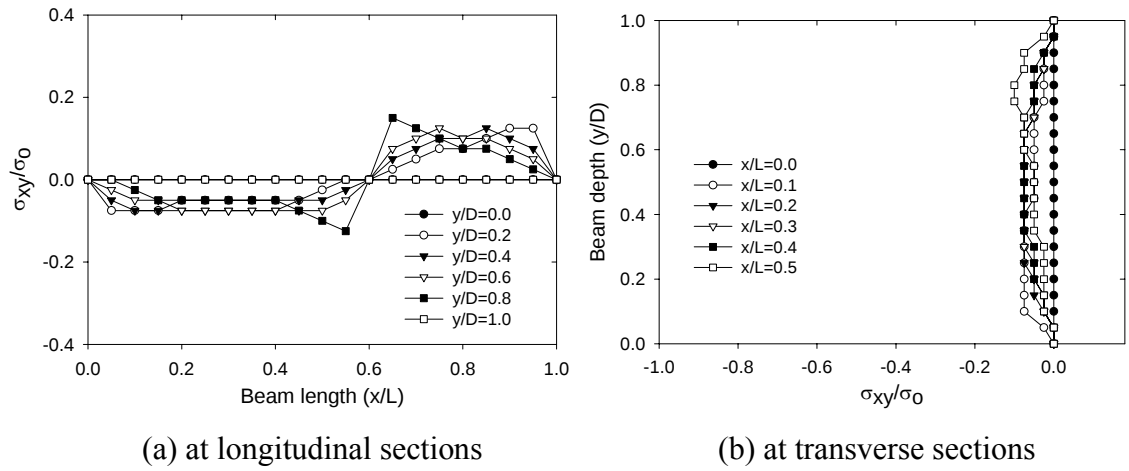
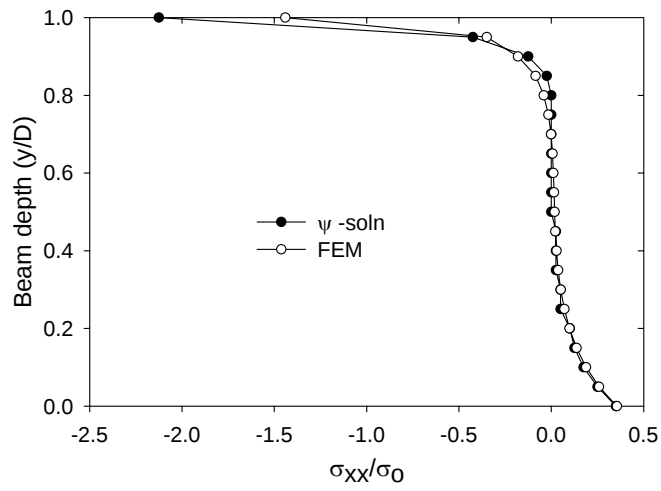


Fig. 5.14 Distribution of shearing stress of a glass epoxy beam under asymmetric three point loading, $L/D=3$: (a) over longitudinal sections, (b) over transverse sections

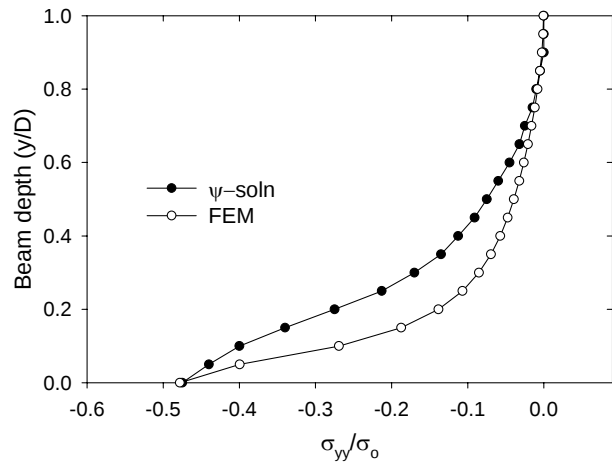
5.3 Verification of ψ – solution for Three Point Bending

The solution for three point bending of the guided beam obtained using displacement potential formulation is verified with the results of the FEM. FEM solutions are obtained using the standard facilities of the commercial software ANSYS. For that the relevant boundary conditions used are the same as those used in the analytical solution. Four noded rectangular plane elements are used to construct the corresponding mesh network of the beam. All the elements are of same size and their distribution is kept uniform all over the domain. The total number of finite elements used to construct the element mesh network for the present problem is 400 (20 x 20).

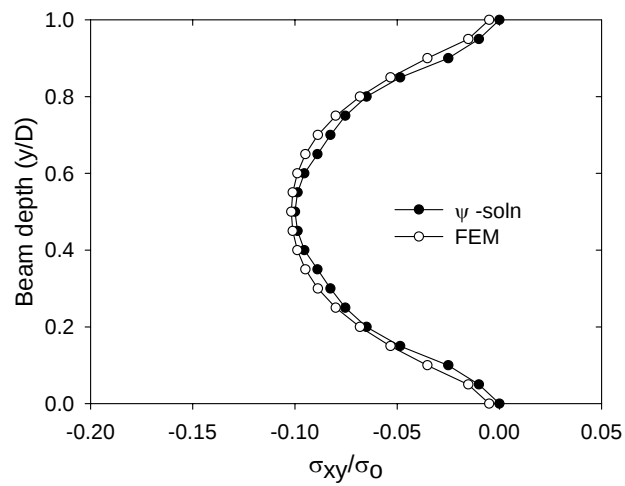
The distributions of bending stress at $x/L=0.5$, lateral stress at $x/L=0.5$ and shearing stress at $x/L=0.25$ are investigated of a glass-epoxy beam ($L/D=3$) under symmetric three point bending for comparison purpose using displacement potential and FEM. The obtained results are presented in Fig 5.15(a), (b) and (c) respectively



(a) bending stress at $x/L=0.5$



(b) lateral stress at $x/L=0.0$



(c) shearing stress at $x/L=0.25$

Fig. 5.15 Comparison of distribution of stresses of glass-epoxy beam for three point bending, $L/D=3$: (a) bending stress at $x/L=0.5$, (b) lateral stress at $x/L=0.0$ (c) shearing stress at $x/L=0.25$

The bending stress distribution pattern at the mid-span ($x/L=0.5$) is very close to each other. The magnitude is exactly the same for lower half (from $y/D=0.0$ to $y/D=0.7$) of the beam [Fig 5.15(a)]. However, a very little difference is observed at the top edge in two methods. While the lateral stress is investigated at $x/L=0$, the magnitudes at the top and bottom edges are the same in both the methods [Fig 5.15(b)]. However, a deviation is observed in the region of $y/D=0.1$ to 0.7 due to lower resolution. Further reasons for this deviation may be investigated taking different loading configuration. The shearing stress distribution pattern at $x/L=0.25$ is found with very good agreement in the both solutions. However, the potential displacement solution estimates little higher but negligible magnitude [Fig 5.15(c)].

To have more confirmation about the performance of displacement potential function on three point bending, the solution for an isotropic material (steel having $E=209$ GPa and $\mu=0.3$) is also obtained and compared with FEM solution in a similar manner done for orthotropic material. The corresponding results of bending and shear distributions for the case of isotropic material are shown in figures 5.16 (a) and (b) respectively. Here too the result is found with good rationality and consistency. As such the displacement potential approach can be considered as a good tool for the solution of guided beam with various loading conditions.

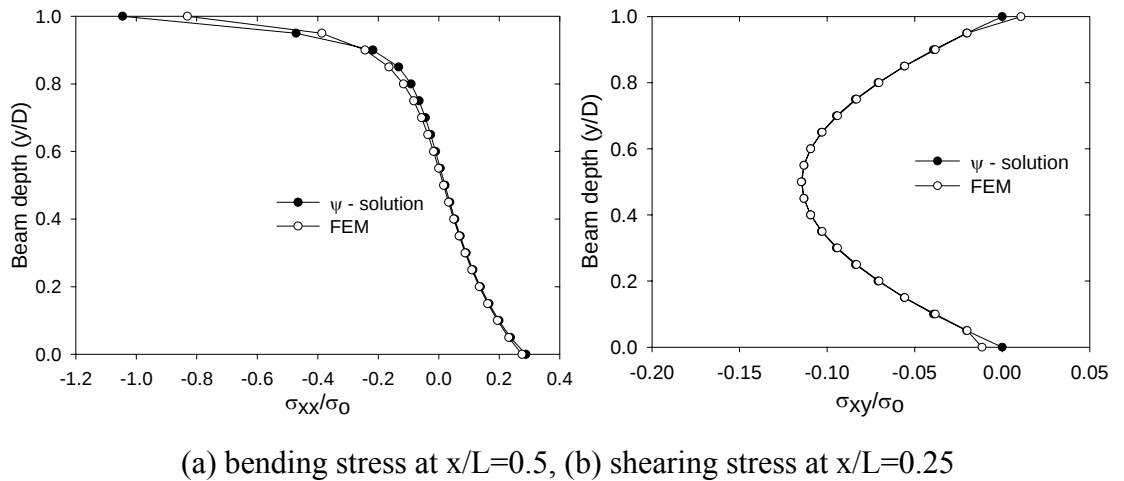


Fig. 5.16 Comparison of distribution of stresses of steel beam for three point bending, $L/D=3$: (a) bending stress at $x/L=0.5$, (b) shearing stress at $x/L=0.25$

5.4 Summary

Analytical solution using displacement potential approach for the elastic fields of a guided deep composite beam of orthotropic material under three point bending is explored satisfying all the physical conditions of the beam appropriately. The solution of a glass epoxy beam for the aspect ratio three is observed for symmetric loading as well as asymmetric loading (taking load at $x/L=0.6$). Both the solutions indicate that the displacement potential approach can address three point bending of composite materials effectively. Displacement and stress components are of the similar maximum values also. The stress concentration is investigated and found to be at the near location of point load. The results obtained using displacement potential solutions are then compared with that of finite element solution. The stress analysis observed from the FEM confirms the reliable results of displacement potential solution. To this end the solution of isotropic material is also obtained and compared with the solution of FEM, which enhances the validity of the displacement potential approach in relation to its utility for addressing deep beams under point loading.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusion

The central objective of this research is to develop the analytical solutions for guided isotropic and orthotropic deep beams. Keeping this central intention in mind the literature has been surveyed and found that lot of work have been done in the field elastic analysis but none is for deep beams with guided ends. The speciality of the guided ends is the mixed mode of boundary conditions. Basically, the guided ends provide the freedom of lateral displacement but not the axial one. At this scenario the necessity of imposing boundary restraints is essential. But it is neither practicable using the classical Bernoulli-Euler beam theory nor with the Airy's stress function for the solution of guided beam. It may be mentioned that the stress function can handle boundary conditions only in terms of load or stress. So, the work has to be started with the displacement potential formulation, which is defined in terms of space variables x and y with the expressions of two displacement components. The displacement potential can take care of the boundary conditions of both displacements and stresses appropriately. The bi-harmonic partial differential equation of a single variable has been conceived to be reached as to the 4th order differential equation of stress function. Interestingly, the differential equations for both orthotropic and isotropic materials are found to be in line with the expected pattern. This gives possible easiness of solution approach.

Based on the definition of ψ -function, analytical solutions of the elastic fields for guided isotropic and orthotropic deep beams subjected to uniformly distributed load over a certain length and point load are developed and the results are investigated. In the investigation of solutions, the distributions of stress components are found to be different at different locations. The guided ends of the beam are identified to be the most critical sections in terms of stresses. The load transition section is also found to be different from loaded as well as unloaded sections of the beam. It displays a unique behaviour, which necessitates special attention from the designer of any machine parts. While the support reactions are in

the pattern of parabolic form, bending and lateral stress concentrations are found to be less than that of a uniformly distributed support and the effect is just local.

The comparative study has been carried out to ascertain the reliability and credibility of solutions obtained through displacement potential function and observed its superiority for the solution of guided deep beam. In some cases tools for comparison is the classical Bernoulli-Euler beam theory, some cases Airy's stress function wherever suitable and FEM for all the cases. It reveals that the classical theory of bending is simply inadequate to predict the stress and displacements fields of the guided beam. For the sake of comparison, simply supported beam with the distributed load is taken into account as sample for classical theory and stress function. Comparison with FEM gives the proximity of present approach towards it for each and every solution.

While the aspect ratio is varied the remarkable variation of non-linearity in both bending and shearing stress distribution is observed. FEM confirms these variations but not the classical Bernoulli- Euler beam theory. With smaller aspect ratio the maximum magnitude of shear stress is not found at the centreline of the beam as observed for higher aspect ratio. In beam theory bending stress distribution is always found linear and shear stress parabolic, which do not reflect the actuality of the problem for short beam.

The normalized maximum deflection, bending stress and shear stress found through present solution of isotropic material (steel) with aspect ratio three are 2.94×10^{-4} , 5.24 and 1.51 respectively; where as these found by FEM are 2.32×10^{-4} , 5.35 and 1.27 respectively. It indicates that there is a possibility of keeping less allowance based on FEM solution, which may cause the non-operability of the machine due to higher deflection in reality. Again the underestimation of maximum shear stress may compromise with the reliability and life span of the machine parts. On the other hand normalized maximum deflection, bending stress and shear stress found through present solution of orthotropic material (Glass-Epoxy) with aspect ratio three are 33.22×10^{-4} , 9.02 and 1.8 respectively. The deviations of these values with corresponding FEM results are 2.05, 18.29 and 4.44 percent respectively. Here, the underestimation of bending stress may cause the break down of machine part if the design is developed on the basis of FEM results.

The solution of unguided simply supported deep beam with uniformly distributed load is done using a special technique of displacement potential approach taking the St. Venant's principle into account. Analytical solution developed for the unguided deep beam is compared with the similar solution obtained using Airy's stress function. It reveals that the present approach gives satisfactory results. So, the displacement potential solution for an unguided beam may be considered useful as similar to stress function solution.

As a whole the result of present analysis is expected to be of real use for reliable and economic design of such beams. The present study thus removes one of the major limitations of the literature related to beam analysis.

6.2 Recommendations

The displacement potential function developed for the guided deep beam is a new avenue for the structural analysis. The method has been investigated and instituted as capable to deal isotropic and orthotropic guided beams effectively for uniformly distributed load. The present approach needs to be expanded for the analytical solution of anisotropic materials. It also requires to be investigated for a variety of loading configuration in order to have its wide range of adoptability and versatility.

The current method for the solution of unguided deep beam may also be extended further. More works would provide more authenticity for onward application of ψ -function in the solution of unguided beams with various loading conditions. It is, therefore, recommended to have continuation of the ongoing works for the utilization of displacement potential approach for both guided as well as unguided beams of various configurations in future.

REFERENCES

1. Airy, G.B., British Association for the Advancement of Science Report, 1862.
2. Timoshenko, S.P. and Goodier, J. N., Theory of Elasticity, -3rd Ed., McGraw Hill Book Co., New York, 1970.
3. Borg, S. F., Fundamentals of Engineering Elasticity, World Scientific, Singapore, 2nd ed., 1990.
4. Rankine, Applied Mechanics, 14th ed., pp. 344, 1895.
5. Grashof, Elastizitt Festigkcit, 2nd ed., 1878.
6. Levy, M., Compt. Rend, Vol. 126, pp. 1235, 1898.
7. Scewald, F., Abhandl, Aerodynamic Inst. Tech. Hochschule, Achen, vol.7, pp. 11, 1927.
8. Ribiere, M. C., Sur divers cas de la flexon des primes rectangles, thesis, Bordeaux, 1889.
9. Filon, L.N.G., Phil, trans., Series A, vol. 201, pp. 63,1903.
10. Bleich, F., Bauingenieur, vol. 4, pp. 255, 1923.
11. Beyer, K., Die Statik in Esienbetonban, 2d ed, pp723, 1934.
12. Ribiere, Compt, Rend. Vol. 132. pp. 315, 1901.
13. Flamant, Compt. Rend, Paris, vol. 114. pp. 1465, 1892.
14. Stokes, G.G., Mathematical and Physical Papers, vol. 5, p.238, 1935.
15. Conway, H. D., Chow, L., and Morgan, G.W., “Analysis of deep beams,” Journal of Applied Mechanics, Trans ASME 18(2), 163–172, 1951.
16. Chow, L., Conway, H. D., and Winter, G., “Stresses in Deep Beams”, Trans ASCE, Paper No. 2557, 1952.
17. Conway, H. D., and Ithaca, N. Y., “Some problems of orthotropic plane stress,” Journal of Applied Mechanics, Trans ASME, 52, 72–76, 1953.
18. Chapel, R. E., and Smith, H. W., “Finite-Difference solutions for plane stresses,” AIAA Journal 6, 1156–1157, 1968.
19. Parker, D. F., “The role of Saint-Venant’s Solutions in rod and beam theories,” Journal of Applied Mechanics, Trans ASME 46, 861–866, 1979.

20. Horgan, C. O., and Knowels, J. K., "Recent developments concerning Saint-Venant's Principle," *Advances in Applied Mechanics*, 23, 179–269, 1983.
21. Horgan, C. O., and Simmonds, J. G., "Saint-Venant End Effects in Composite Structures," *Composite Engineering*, Vol 4, No 3, 279–286, 1994.
22. John Wiley & Sons Inc, *Handbook of Experimental Stress Analysis*, New York, 1950.
23. Frocht, M.M., *Photoelasticity*, John Wiley & Sons Inc, New York, vol.2, 1948.
24. Uddin, M. W., "Finite Difference Solution of Two-dimensional Elastic Problems with mixed Boundary Conditions", M. Sc. Thesis, Carleton University, Canada, 1966.
25. Durelli, A. J., and Ranganayakamma, B., "Parametric solution of stresses in beams," *Journal of Engineering Mechanics* 115(2), 401–414, 1989.
26. Rehfield, L. W. and Murthy, L. N., "Toward a new engineering theory of bending: Fundamentals", *AIAA Journal*, vol. 20. pp.693-699, 1982.
27. Murty A. V. K., "Towards a consistent beam theory", *AIAA Journal*, vol. 22. pp.811-816, 1984.
28. Suzuki S., *Stress analysis of short beams*, *AIAA Journal*, vol.24. pp.1396-1398, 1986.
29. Hardy S. J. and Pipelzadeh M. K., "Static analysis of short beams, *Journal of strain Analysis*", vol. 26, No. 1, pp.15-29, 1991.
30. Ahmed, S. R., Idris, A. B. M., and Uddin, M. W., "Numerical solution of both ends fixed deep beams," *Computers & Structures* 61(1), 21–29, 1996.
31. Ahmed, S. R., Khan, M. R., Islam, K. M. S., and Uddin, M. W., "Investigation of stresses at the fixed end of deep cantilever beams," *Computers & Structures* 69, 329–338, 1998.
32. Ahmed S.R., Idris A.B.M. and Uddin M.W., "An alternative method for numerical Solution of mixed boundary value elastic problems". *Journal of Wave-Material Interaction*, vol.14, No.1-2, pp.12-25, 1999.
33. Ahmed, S. R., Hossain, M. Z. and Uddin, M.W., "A general mathematical formulation for finite-difference solution of mixed-boundary-value problems of anisotropic materials," *Computers & Structures*, 83, 35–51, 2005.
34. Akanda, M. A. S., Ahmed, S. R. and Uddin, M. W., "Stress analysis of gear teeth using displacement potential function and finite differences," *International Journal for Numerical Methods in Engineering*, 53, 1629–1640, 2002.
35. Ahmed, S. R., Nath, S. K. D., and Uddin, M.W., "Optimum shapes of tire treads for avoiding lateral slippage between tires and roads," *International Journal for Numerical Methods in Engineering*, 64, 729–750, 2005.

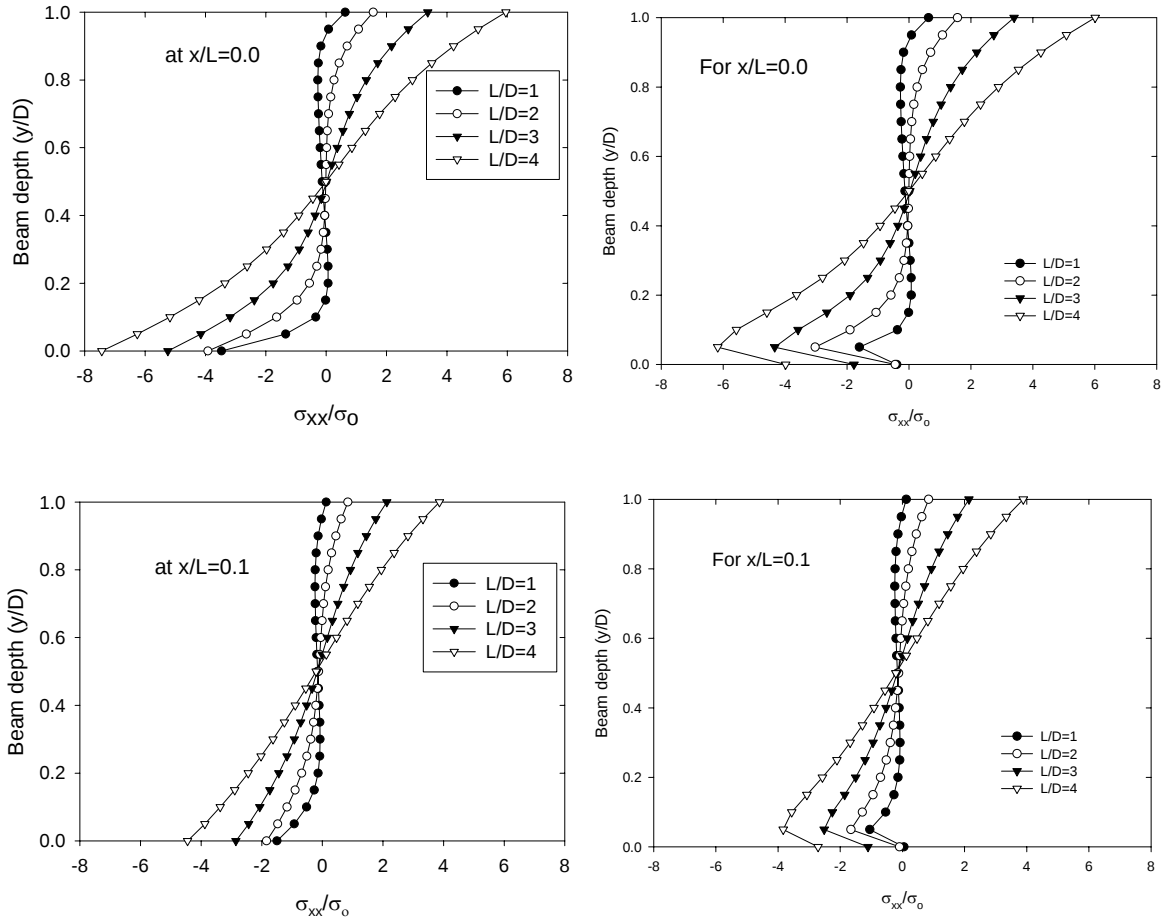
36. Nath, S. K. D., Ahmed, S. R., and Afsar, A. M., “Displacement potential solution of short stiffened flat composite bars under axial loadings,” *International Journal of Applied Mechanics and Engineering* 11(3), 557–575, 2006.
37. Nath, S. K. D., Afsar, A. M., and Ahmed, S. R., “Displacement potential approach to solution of stiffened orthotropic composite panels under uniaxial tensile load,” *Journal of Aerospace Engineering, Part G, IMechE*, 221(5), 896–881, 2007.
38. Mukhopadhyay, M., “*Mechanics of Composite Materials and Structures*,” Universities Press, Hyderabad, pp46, 2004.
39. Jones, R. M., “*Mechanics of Composite Materials*,” Taylor Francis, Philadelphia, 1999.
40. Kaw, A. K., “*Mechanics of Composite Materials*,” CRC Press LLC, 1997.

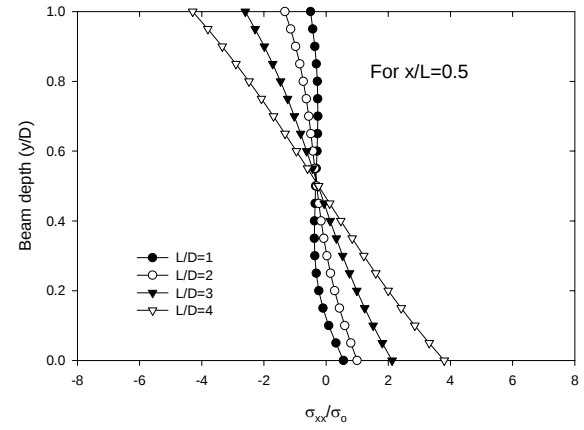
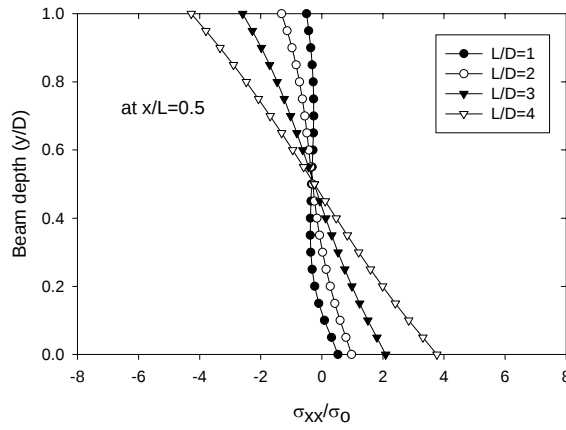
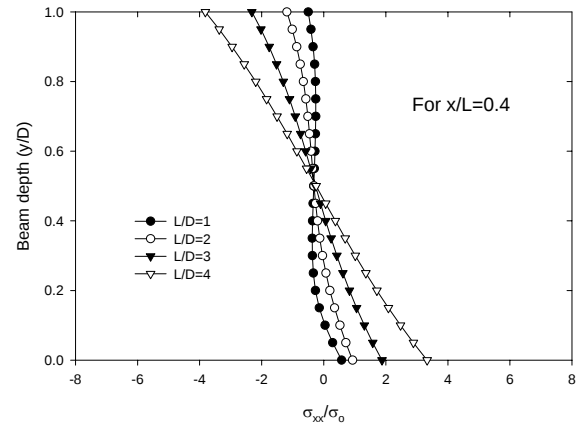
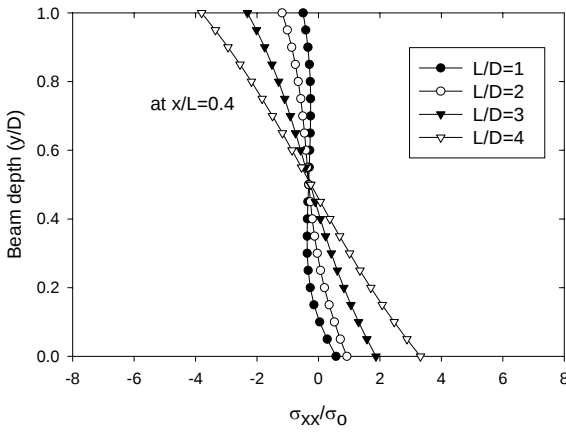
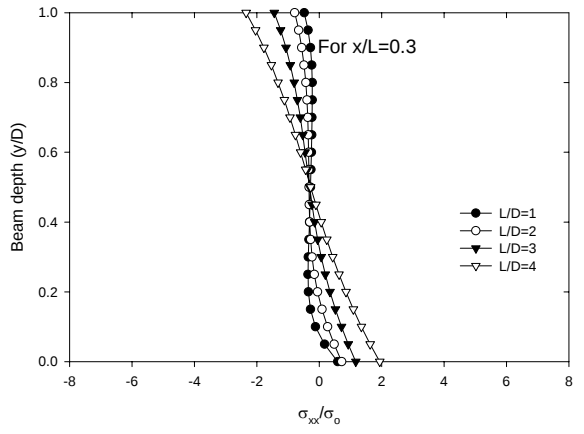
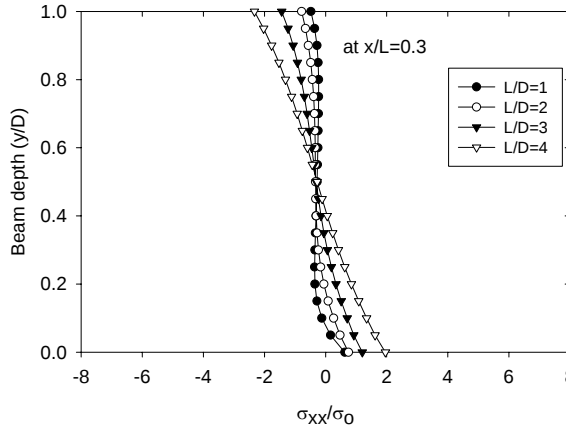
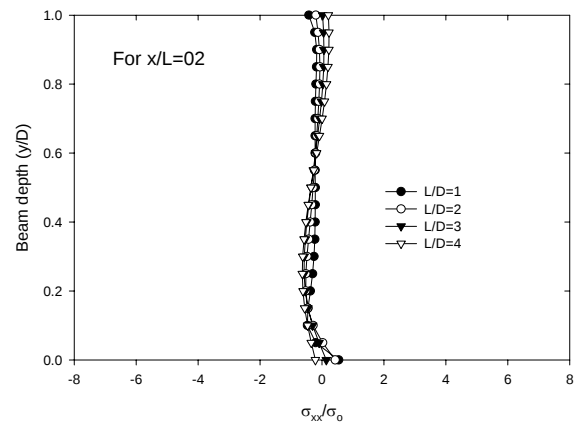
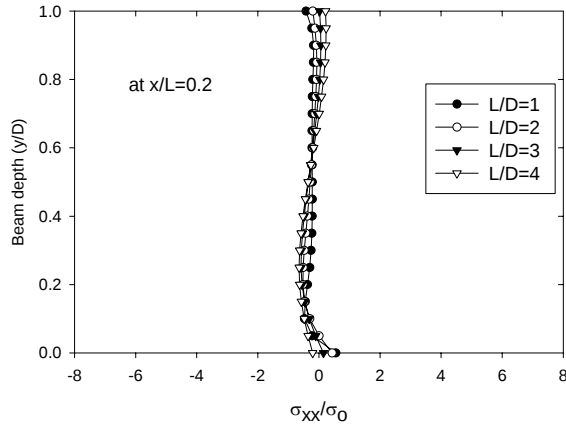
APPENDIX - A

Distribution of Stress Components for Different Aspect Ratios

Graphs presented here are of an isotropic beam with uniformly distributed load and uniformly distributed supports vs parabolic supports. The bending, lateral and shearing stress distribution are presented in sequence at order of sections starting from the left section of the beam. Pictures at the left are for uniform supports and right for parabolic supports.

Bending stresses (σ_{xx}) distribution: For the skew-symmetric configuration the graphs for the sections from $x/L=0.0$ to 0.5 are shown.





Lateral stresses (σ_{yy}) distribution: For the symmetric configuration the graphs for the sections from $x/L=0.0$ to 0.5 are shown.

