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Modulation instability analysis, and characterize time-dependent variable coefficient solutions in electromagnetic transmission and biological field

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ABSTRACT

This study integrates the telegraph model with the traveling wave coefficient. This model characterizes planar random motion of particles in fluid flow, pressure waves from pulsatile blood flow in arteries, electrical impulses in nerve and muscle cell axons, and electromagnetic waves in superconducting media. Using the efficient and well-established Enhanced Modified Simple Equation (EMSE) method, we investigate solitary wave solution with the variable coefficient of the telegraph model. To investigate the variable coefficient solitary wave solution, we apply a transformation variable $\eta = h(t)x + g(t)$, which is dependent on the time variable function. The diverse functions of $h(t)$ and $g(t)$ yield many novel and exclusive solutions. Numerical solutions are plotted in 3D, density, and 2D. instanton soliton, kinky periodic wave, lump wave, kink wave, anti-kink wave, double periodic wave, diverse types periodic wave, lump with bell wave periodic wave, periodic lump wave, etc. Solitary waves explain complex wave phenomena through nonlinear effects like self-interaction and dispersion. Due to the classical nonlinear telegraph model's wave dynamics, they are used in brain function and communication system design. We conclude by testing the governing model's modulation instability. Dispersive and nonlinear processes modulating the stable state cause instability in high-order nonlinear equations. Examining the wave movement role and modulation instability analysis to assess solution stability shows that all solutions are accurate and stable. The computational difficulties and results demonstrate the approaches' clarity, effectiveness, and simplicity, suggesting they can be applied to evolutionary dynamic and static nonlinear equations in computational physics, other real-world situations, and numerous academic disciplines.

1. Introduction

Nonlinear evolution equations are fundamental in understanding complex dynamics across numerous scientific domains. They are pivotal in describing phenomena where interactions between variables lead to nonlinear behavior, which often manifests as chaotic or emergent phenomena. These equations underpin the study of quantum mechanics[1,2] and electromagnetism to fluid dynamics[3,4] finance[5] and beyond.[6–8] One of their key significances lies in elucidating pattern formation, where intricate structures arise from nonlinear interactions, such as spiral waves in chemical reactions or coherent structures in turbulent flows. Additionally, nonlinear evolution equations provide insights into soliton phenomena, where localized waves maintain their shape and propagate without dispersion. These solitons find applications in diverse fields of mathematical physics and engineering. It is difficult to construct exact soliton solutions of diverse nonlinear evolution equations. A great deal of research has been done to develop new methods of solving nonlinear PDEs, including unified technique[6], exp-function method,[7] Lie symmetry analysis,[8] NMSE technique, [9] separable of variables,[10] extended f-expansion technique,[11] improved modified extended tanh method,[12] expanded exp(ϕ)-expansion method, generalized Kudryashov method, and modified Khater method,[13] Hirota bilinear scheme,[14] EMSE technique,[15] SE technique,[16] exp(ϕ)-expansion and NMK method,[17] Bifurcation analysis,[18] three-wave method,[19] cubic B-spline method and

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finite difference technique,[20] MSE technique,[21] darboux transformation,[22] modified expansion schemes,[23] EMSE technique,[24] ESGEE technique,[25] Hirota bilinear,[26] multiple Exp-function method,[27,28] $exp(-K(\rho))$ − expansion technique and rational $tan(K(\rho))$ − expansion technique,[29] $tan(\Phi(\rho)/2)$ − expansion technique,[30] and so on. In addition, powerful computer software packages such as Maple are available to aid in the solution of nonlinear PDEs. The motivation of this work is to explore the variable coefficient solitary wave solution of second-order nonlinear telegraph models. This model is useful in the evolution of dominant genes and propagation of nerve pulses, the evolution of the set of Duffing oscillators, and so forth. The classical nonlinear telegraph model is considered as[31]:

$$F_{tt} + a_1 F_t = a_2 F_{xx} + \alpha F^3 + \beta F^2 + \gamma F, \tag{1}$$

where, a_2, a_1, α, β, and γ are unknown constants. Eq. (5) is referred to as a second-order nonlinear telegraph equation. If we set $\beta = \gamma$, $a_2 = \alpha^2$, $\gamma = -\beta$, $a_1 = \alpha = 0$, then Eq. (5) is represented by the Klein–Gordon model,[32] If we set $\gamma = -m^2$, $a_2 = 1$, $\alpha = -\lambda$, $a_1 = \beta = 0$, then the Eq. (5) is represented the phi4 model.[33,34]

The Telegraph equation in Eq. (1) can be utilized to clarify an extensive number of phenomena in a variety of real-world scenarios, like the planar random motion of a particle in a flow of fluid, the conveyance of electrical signals through muscular and nerve cell axons, the transmission of electromagnetic radiation in superconductive media, and the transmission of pressure waves in pulsatile blood flow in arteries.[35–37] Recently, diverse techniques have been used to solve the telegraph model in different forms, including, Alexei I.[33] seek exact solutions to these equations by the direct method of symmetry reductions from telegraph model, Hind Al-badrani et al.[38] applied ADM and Modified ADM is used to obtain one soliton solution to the nonlinear telegraph equation, Araujo et al.[39] applied the elliptic regularization techniques to nonlinear telegraph equations, Hyunsoo Kima et al.[40] explored exact travelling wave solutions of the nonlinear partial differential equations, Jang[41] used a new (functional) iterative procedure for the constructing of the (numerical) solutions of a general nonlinear telegraph equation. In this article, the enhanced MSE technique[42,43] for cubic telegraph model to investigate the variable coefficient solutions. MSE method, developed for years, the EMSE method is one of them, is one of the most algebraic, effective, and direct methods that have no predefined solution. The present work has the advantage of solving the telegraph equation by using the EMSE method. The EMSE technique is studied in Section 2; the variable coefficient solution of the telegraph model has been calculated in Section 3; Section 4 is dedicated to the effect of diverse variable coefficient function, with 3D density, and 2D plot; in Section 5, the modulation instability is analyzed of the studied model; in Section 6, the study method is compared with others method; to wind up the argument, finally, the conclusion is in position at the end.

2. Enhanced modified simple equation method

In this section, a general explanation of the enhanced modified simple equation technique is provided. We take into consideration NLPDEs when writing the enhanced modified simple equation technique[42,43]:

$$H(\wp, \wp_t, \wp\wp_x, \wp_{xx}, \cdots\cdots) = 0, \tag{2}$$

where, $g(x, t)$ wave function. To get the variable coefficient solution, we assume the wave variable $\wp(x, t) = F(\xi)$; $\xi = h(t)x + g(t)$, where $h(t)$ and $g(t)$ are differentiable functions of time.

Now, we calculate the ordinary form of the Eq. (2) by using wave variable ξ as

$$R(F, (\dot{h}(t)x + \dot{g}(t))F', h(t)F', h(t)^2 F'', \cdots\cdots) = 0, \tag{3}$$

where, $\cdot \equiv d/dt$ and $' \equiv d/d\xi$.

Now, the solution of Eq. (3) is considered as

$$F(\xi) = \sum_{k=0}^{m} p_k(t) \left(\frac{M'(\xi)}{M(\xi)}\right); p_n \neq 0, \tag{4}$$

where, p_k is a function of the unknown t and m is the balance number. The highest linear and non-linear terms can be balanced to determine the balance number m.

Now, we insert solution Eq. (4), its derivative, and different form into Eq. (3), and then we get the following types of polynomials

$$P(\xi) = d_1 \frac{x}{\psi(\xi)} + d_2 \frac{x}{(\psi(\xi))^2} + \ldots + C_0 \frac{1}{\psi(\xi)^0} + C_1 \frac{1}{\psi(\xi)} + C_2 \frac{1}{(\psi(\xi))^2} + \ldots + C_m \frac{1}{(\psi(\xi))^m}.$$

If we set $d_1 = d_2 = \ldots = C_0 = C_1 = C_2 = \ldots = C_m = 0$, then we get a system of equations. If we solve the obtained system of equations, then the values of $p_k(t)$, $h(t)$, $M(\xi)$ and $g(t)$ are acquired. Finally, if we substituted the values of $p_k(t)$, $h(t)$, $M(\xi)$, and $g(t)$ into Eq. (4), then the obtained solutions are done.

3. Application of enhanced form of modified simple equation method

In this section, the EMSE scheme applies to find the time-dependent exact solution for the telegraph equation. Now, we consider the telegraph equation

$$F_{tt} + a_1 F_t = a_2 F_{xx} + \alpha F^3 + \beta F^2 + \gamma F, \tag{5}$$

where, a_1, a_2, α, β, and γ are unknown constants. Let us consider $F(x, t) = F(\eta)$; $\eta = h(t)x + g(t)$ to convert Eq. (5) into the ordinary form

$$-(\dot{h}x + \dot{g})^2 F'' - a_1(\dot{h}x + \dot{g})F' + a_2 h^2 F'' + \alpha F^3 + \beta F^2 + \gamma F = 0. \tag{6}$$

According to the EMSE method, the solution is

$$F = p_0 + p_1 \frac{M'(\eta)}{M(\eta)}. \tag{7}$$

The value of Eq. (7) inserts into Eq. (6). The following system of equations are obtained as

$$\frac{1}{M^0}: \ \alpha p_0^3 + \ \beta p_0^2 + \gamma p_0 = 0, \tag{8}$$

$$\frac{1}{M}: -\dot{g}^2 p_1 M''' - a_1 \dot{g} p_1 M'' + a_2 h^2 p_1 M''' + 3\alpha p_0^2 p_1 M' + 2\beta p_0 p_1 M' + \gamma p_1 M' = 0, \tag{9}$$

$$\frac{1}{M^2}: \ 3\dot{g}^2 p_1 M' M'' + a_1 \dot{g} p_1 M'^2 - 3a_2 h^2 p_1 M' M'' + 3\alpha p_0 p_1^2 M'^2 + \beta p_1^2 M'^2 = 0, \tag{10}$$

$$\frac{1}{M^3}: -2\dot{g}^2 p_1 M'^3 + 2a_2 h^2 p_1 M'^3 + \alpha p_1^3 M'^3 = 0, \tag{11}$$

$$\frac{x}{M}: -2p_1 M''' \dot{h}\dot{g} - a_1 \dot{h} p_1 M'' = \dot{h}, \tag{12}$$

$$\frac{x}{M^2}: 6p_1 \dot{h}\dot{g} M' M'' + p_1 \dot{h} a_1 M'^2 = 0, \tag{13}$$

$$\frac{x^2}{M}: p_1 \dot{h}^2 M''' = 0, \tag{14}$$

$$\frac{x^2}{M^2}: \dot{h}^2 p_1 M' M'' = 0, \tag{15}$$

$$\frac{x^2}{M^3}: \ 2\dot{h}^2 p_1 M'^3 = 0, \tag{16}$$

$$\frac{x}{M^3}: -4p_1\dot{h}\dot{g}M'^3 = 0. \tag{17}$$

From Eq. (8)

$$p_0 = 0,\ \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},\ \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}.$$

From Eq. (11)

$$p_1 = -\sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}},\ \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}.$$

From Eqs. (12)–(17)

$$h = k.$$

From Eq. (10)

$$M' = \frac{3a_2h^2 - 3\dot{g}^2}{(a_1\dot{g} + 3\alpha p_0p_1 + \beta p_1)}M''. \tag{18}$$

From Eq. (9)

$$\frac{M'''}{M''} = \frac{a_1\dot{g}(a_1\dot{g} + 3\alpha p_0p_1 + \beta p_1) - \left(3a_2h^2 - 3\dot{g}^2\right)(3\alpha p_0{}^2 + 2\beta p_0 + \gamma)}{\left(a_2h^2 - \dot{g}^2\right)(a_1\dot{g} + 3\alpha p_0p_1 + \beta p_1)} = \theta.$$

Integrating the above equation

$$M'' = B_0e^{\theta\eta}, \tag{19}$$

where, $\theta = a_1\dot{g}(a_1 \frac{\dot{g} + 3\alpha p_0p_1 + \beta p_1) - (3a_2h^2 - 3\dot{g}^2)(3\alpha p_0{}^2 + 2\beta p_0 + \gamma))}{(a_2h^2 - \dot{g}^2)(a_1\dot{g} + 3\alpha p_0p_1 + \beta p_1)}$.

Insert Eq. (19) into Eq. (17)

$$M' = B_0Qe^{\theta\eta}, \tag{20}$$

where, $Q = \frac{3a_2h^2 - 3\dot{g}^2}{(a_1\dot{g} + 3\alpha p_0p_1 + \beta p_1)}$.

Integrating, we get

$$M = \frac{B_0Qe^{\theta\eta}}{\theta} + B_1. \tag{21}$$

Insert the Eqs. (20) and (21) into Eq. (8)

$$F(x,t) = p_0 + \frac{p_1Qe^{\theta\eta}B_0}{\frac{B_0Qe^{\theta\eta}}{\theta} + B_1}. \tag{22}$$

Case 01. When $p_0 = 0$ and $p_1 = \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}$ then the solution of Eq. (22) is

$$F(x,t) = \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{Qe^{\theta\eta}B_0}{B_0Q/\theta e^{\theta\eta} + B_1}, \tag{23}$$

where, $\eta = kx + g(t)$, and k is an arbitrary constant.

Case 02. When $p_0 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$ and $p_1 = -\sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}$ then the solution of Eq. (23) is

$$F(x,t) = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} - \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{Qe^{\theta\eta}B_0}{B_0Q/\theta e^{\theta\eta} + B_1}. \tag{24}$$

If $\beta^2 > 4\alpha\gamma$ or $\beta = 4\alpha\gamma$, then the solution Eq. (24) becomes a hyperbolic form

$$F(x,t) = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} - \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{\theta QB_0(\cosh(\theta\eta) + \sinh(\theta\eta))}{QB_0(\cosh(\theta\eta) + \sinh(\theta\eta)) + 2\theta B_1}. \tag{25}$$

If $\beta^2 < 4\alpha\gamma$, then the solution Eq. (24) becomes a trigonometric form

$$F(x,t) = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} - \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{\theta QB_0(\cos(\theta\eta) + i\sin(\theta\eta))}{QB_0(\cos(\theta\eta) + i\sin(\theta\eta)) + 2\theta B_1}. \tag{26}$$

where, $\eta = kx + g(t)$, and k is an arbitrary constant.

Case 03. When $p_0 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$ and $p_1 = -\sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}$ then the solution of Eq. (23) is

$$F(x,t) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} - \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{Qe^{\theta\eta}B_0}{B_0Q/\theta e^{\theta\eta} + B_1}. \tag{27}$$

If $\beta^2 > 4\alpha\gamma$ and $\beta = 4\alpha\gamma$, then the solution Eq. (27) becomes a hyperbolic form

$$F(x,t) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} - \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{\theta QB_0(\cosh(\theta\eta) + \sinh(\theta\eta))}{QB_0(\cosh(\theta\eta) + \sinh(\theta\eta)) + 2\theta B_1}. \tag{28}$$

If $\beta^2 < 4\alpha\gamma$, then the solution Eq. (27) becomes a trigonometric form

$$F(x,t) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} - \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{\theta QB_0(\cos(\theta\eta) + i\sin(\theta\eta))}{QB_0(\cos(\theta\eta) + i\sin(\theta\eta)) + 2\theta B_1}, \tag{29}$$

where, $\eta = kx + g(t)$, and k is an arbitrary constant.

Case 04. When $p_0 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$ and $p_1 = \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}$ then the solution of Eq. (23) is

$$F(x,t) = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} + \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{Qe^{\theta\eta}B_0}{B_0Q/\theta e^{\theta\eta} + B_1}. \tag{30}$$

If $\beta^2 > 4\alpha\gamma$ or $\beta = 4\alpha\gamma$, then the solution Eq. (30) becomes a hyperbolic form

$$F(x,t) = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} + \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{\theta QB_0(\cosh(\theta\eta) + \sinh(\theta\eta))}{QB_0(\cosh(\theta\eta) + \sinh(\theta\eta)) + 2\theta B_1}. \tag{31}$$

If $\beta^2 < 4\alpha\gamma$, then the solution Eq. (30) becomes a trigonometric form

$$F(x,t) = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} + \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{\theta QB_0(\cos(\theta\eta) + i\sin(\theta\eta))}{QB_0(\cos(\theta\eta) + i\sin(\theta\eta)) + 2\theta B_1}, \tag{32}$$

where, $\eta = kx + g(t)$, and k is an arbitrary constant.

Case 05. When $p_0 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$ and $p_1 = \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}$ then the solution of Eq. (22) is

$$F(x,t) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} + \sqrt{\frac{2\dot{g}^2 - 2a_2h^2}{\alpha}}\ \frac{Qe^{\theta\eta}B_0}{B_0Q/\theta e^{\theta\eta} + B_1}. \tag{33}$$

If $\beta^2 > 4\alpha\gamma$ or $\beta = 4\alpha\gamma$, then the solution of Eq. (33) becomes a hyperbolic form

$$F(x,t)=\frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}+\sqrt{\frac{2\dot{g}^2-2a_2h^2}{\alpha}}\,\frac{\theta QB_0(\cosh(\theta\eta)+\sinh(\theta\eta))}{QB_0(\cosh(\theta\eta)+\sinh(\theta\eta))+2\theta B_1}. \tag{34}$$

If $\beta^2 < 4\alpha\gamma$, then the solution of Eq. (33) becomes a trigonometric form

$$F(x,t)=\frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}+\sqrt{\frac{2\dot{g}^2-2a_2h^2}{\alpha}}\,\frac{\theta QB_0(\cos(\theta\eta)+i\sin(\theta\eta))}{QB_0(\cos(\theta\eta)+i\sin(\theta\eta))+2\theta B_1}, \tag{35}$$

where, $\eta = kx + g(t)$, and k is an arbitrary constant.

4. Numerical discussion and graphical analysis

In this section, we discuss the numerical form of some obtained analytic solitary wave solutions. Under some conditions, the obtained solutions are expressed as trigonometric, hyperbolic, and exponential soliton solutions. From the numerical solution, we integrated some solitary wave phenomena of the nonlinear telegraph model in electrical engineering and telecommunications systems and the biological field. Solitary waves, stemming from the classical nonlinear telegraph model, have broad applications across physics engineering, and biology. In nonlinear optics, they ensure robust information transmission in optical fibers, resisting dispersion and maintaining their shape and velocity. Furthermore, in biology, they model nerve impulses, offering insights into neuronal communication and neurological disorders. These solitary waves, governed by the intricate interplay of nonlinear effects like self-interaction and dispersion, provide a fundamental framework for understanding complex wave phenomena. Their applications span from designing communication systems to elucidating brain function, demonstrating their significance in various fields, and highlighting the versatility of the classical nonlinear telegraph model in capturing essential aspects of wave dynamics. To confirm the behavior of the obtained solution, we illustrated some phenomena in terms of different natural phenomena such as *sin*, *cos*, *tan*, *tanh*, *sech*, etc. In Figs. 1–12, we display some obtained exact soliton solutions and also show the effect of diverse variable coefficient functions while keeping the remaining parameters constant. The obtained exact soliton solution of the telegraph equation holds significant importance in various fields of physics and engineering. Understanding signal transmission, such as in electrical engineering and telecommunications, requires an understanding of solitons, which are stable, localized waveforms. Their distortion-free propagation allows for effective long-distance data transfer. Under the condition $\beta^2 > 4\alpha\gamma$ or $\beta = 4\alpha\gamma$ then the solution Eq. (22) provides the hyperbolic form of the solutions in Eqs. (25), (28), (31), and (34), and for $\beta^2 < 4\alpha\gamma$ the solution Eq. (22) provides trigonometric solutions in Eqs. (26), (29), (32), and (35). Since g(t) is a time-varying coefficient function both conditions exist in the solutions Eqs. (25), (26), (28), (29), (31), (32), (34), and (35) solution for the time range negative to positive.

For the parameters $a_1 = 0.2,\ a_2 = 2,\ \alpha = 0.2,\ \beta = -0.5, m = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01,\ g = e^{-t^2} + 2$ the solutions of Eq. (23) represented instanton soliton with bright and dark bell shape in Fig. 1. The solution Eq. (23) also represented instanton soliton with a bright bell shape for $\alpha < 0 = -0.2$ in Fig. 2. In Fig. 3, we illustrated kinky periodic soliton solutions for the parameters $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.2, \beta = 0.5, m = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01,\ g = \cot(t)$ from solution Eq. (23). For the parameters $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.2,\ \beta = 0.5, m = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01,\ g = \text{sech}(t)$, the solution Eq. (23) provided lump wave soliton solutions in Fig. 4. Fig. 5 represents the periodic soliton solutions of the Eq. (23) for the parameters $a_1 = 0.2, a_2 = 1,\ \alpha = 0.2,\ \beta = 0.5, m = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01, g = \cot(t)$.

In Fig. 6, the solutions Eq. (30) provided kink-type soliton for the special value of the parameters $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.2,\ \beta = 0.5, \gamma = -0.5, h = 1, B_0 = 0.1,\ B_1 = -0.01, g = e^t$. For the parameters $a_1 = -0.2,\ a_2 = 1,\ \alpha = -0.2,\ \beta = 0.5, \gamma = 0.5, h = 1, B_0 = 0.1,\ B_1 = -0.01, g = e^t$ the solution Eq. (30) represented soliton solutions in Fig. 7. For the special value of the parameters $a_1 = 0.67,\ a_2 = -0.5,\ \alpha = 3, \beta = -0.5, \gamma = -0.5, h = 1, B_0 = 0.1,\ B_1 = -0.01, g = e^t$, the solutions Eq. (30) provided a kinked shape with a singular soliton in Fig. 8.

In Fig. 9, we illustrated the periodic wave solutions of Eq. (33) for the special value of the parameters $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.5,\ \beta = 0.5, \gamma = -0.5, h = 1, B_0 = 0.1,\ B_1 = -0.01, g = \sec(t)$. In Fig. 10, we illustrated kink shape soliton solutions of Eq. (33) for the parameters $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.2,\ \beta = 0.5, m = -0.5, h = 1, B_1 = 0.1,\ B_1 = 1, g = 2t$. The solution Eq. (33) represented the anti-kink shape with periodic wave soliton solutions for the parameters $g = -2t$ in Fig. 11. Fig. 12 represented periodic lump wave soliton solutions of Eq. (33) for $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.5,\ \beta = 0.5, \gamma = -0.5, h = 1, B_0 = 0.1,\ B_1 = -0.01, g = \text{sech}(t)$.

5. Modulation instability analysis

Instability is a frequent occurrence in high-order nonlinear equations; it is brought on by the modulation of the stable state due to the interaction of dispersive and nonlinear processes.[44,45] We will use stability analysis techniques in the following part to determine the modulation instability of the K-G equation.

$$F_{tt} + a_1F_t = a_2F_{xx} + \alpha F^3 + \beta F^2 + \gamma F. \tag{36}$$

In this section, we identify the traveling wave modulation instability in Eq. (36). Pretend the disturbed solution provided by

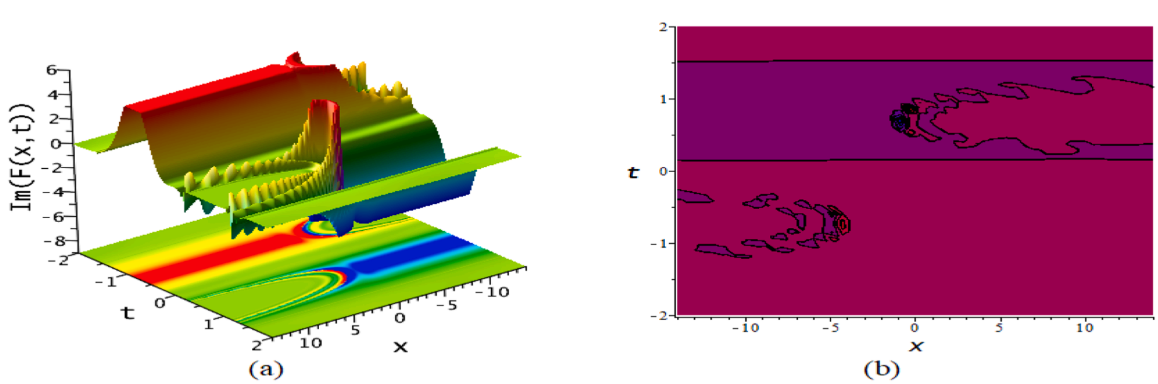

Fig. 1. 3D plot of instanton soliton solutions of the Eq. (23) for the parameters $a_1 = 0.2,\ a_2 = 2,\ \alpha = 0.2,\ \beta = -0.5, \gamma = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01,\ g = e^{-t^2} + 2$.

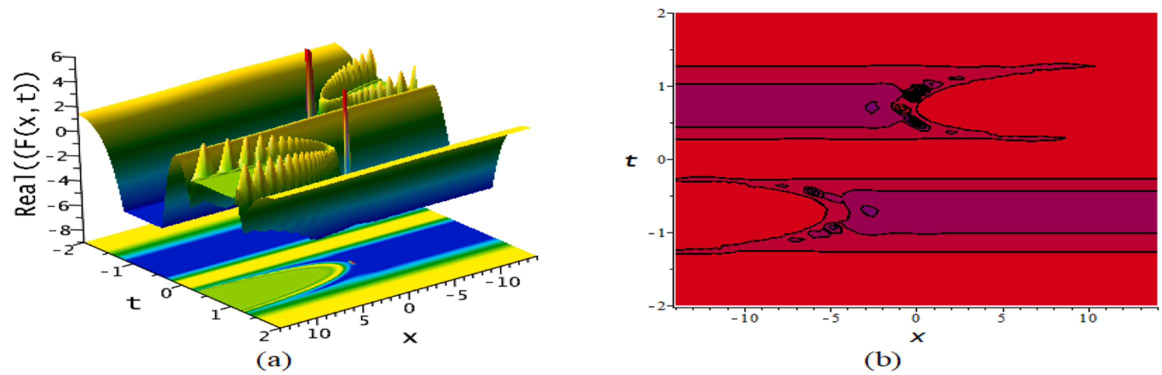

Fig. 2. 3D plot of instanton soliton solutions of the Eq. (23) for the parameters$a_1 = 0.2,\ a_2 = 2,\ \alpha = -0.2,\ \beta = -0.5, \gamma = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01,\ g = e^{-t^2} + 2$.

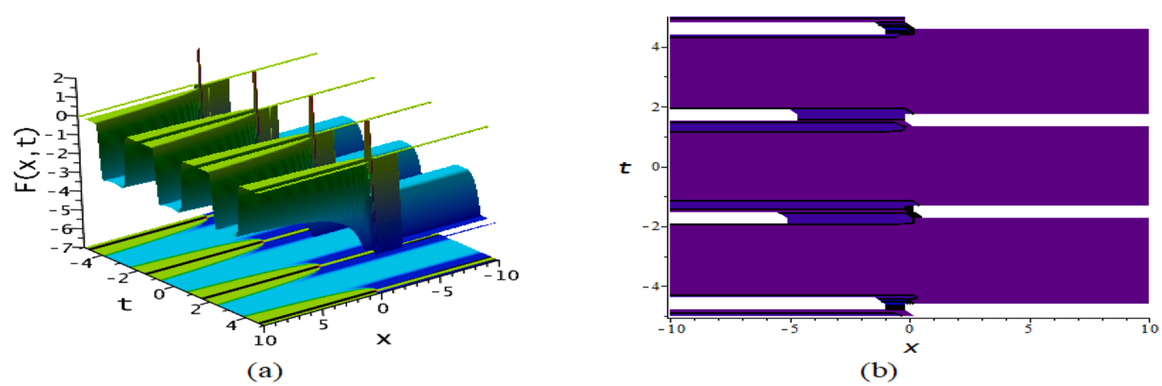

Fig. 3. 3D plot of kinky periodic soliton solutions of Eq. (23) for the parameters $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.2,\ \beta = 0.5, \gamma = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01,\ g = \cot(t)$.

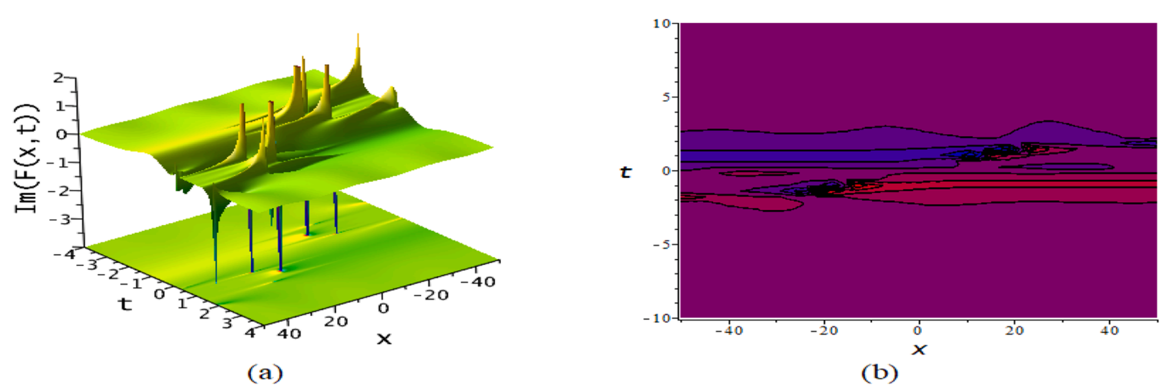

Fig. 4. 3D plot of lump wave soliton solutions of Eq. (23) for the parameters $a_1 = 0.2,\ a_2 = 1,\ \alpha = 0.2,\ \beta = 0.5, \gamma = -0.5, h = 1, B_1 = 0.1,\ B_1 = -0.01,\ g = \mathrm{sech}(t)$.

$$F(x,t) = \rho P(x,t) + \phi. \tag{37}$$

The constant ϕ directs the steady state solution for Eq. (36). By substituting Eq. (37) into Eq. (36), we get

$$\rho P_{tt} + \rho a_1 P_t - \rho a_2 P_{xx} - \alpha(\rho P(x,t) + \phi)^3 - \beta(\rho P(x,t) + \phi)^2 - \gamma(\rho P(x,t) + \phi) = 0.$$

Linearized form in ρ is

$$\rho P_{tt} + \rho a_1 P_t - \rho a_2 P_{xx} - 3\alpha\rho P(x,t)\phi^2 - 2\beta\rho\phi F(x,t) - \gamma\rho F(x,t) = 0. \tag{38}$$

Assume the solution of Eq. (38) stated as

$$P(x,t) = e^{i(kx+\omega t)}. \tag{39}$$

Here, k is the normalized wave number, inserting Eq. (39) into Eq. (38)

$$\rho\omega^2 + i\rho a_1\omega - \rho a_2 k^2 - 3\alpha\rho\phi^2 - 2\beta\rho\phi - \gamma\rho = 0.$$

So, the solution for ω is

$$\omega = -i\rho a_1 \pm \frac{\sqrt{-(\rho a_1)^2 - 4\rho\left(-\rho a_2 k^2 - 3\alpha\rho\phi^2 - 2\beta\rho\phi - \gamma\rho\right)}}{2\rho}. \tag{40}$$

If $-(\rho a_1)^2 - 4\rho(-\rho a_2 k^2 - 3\alpha\rho\phi^2 - 2\beta\rho\phi - \gamma\rho) \geq 0$, the ω is real

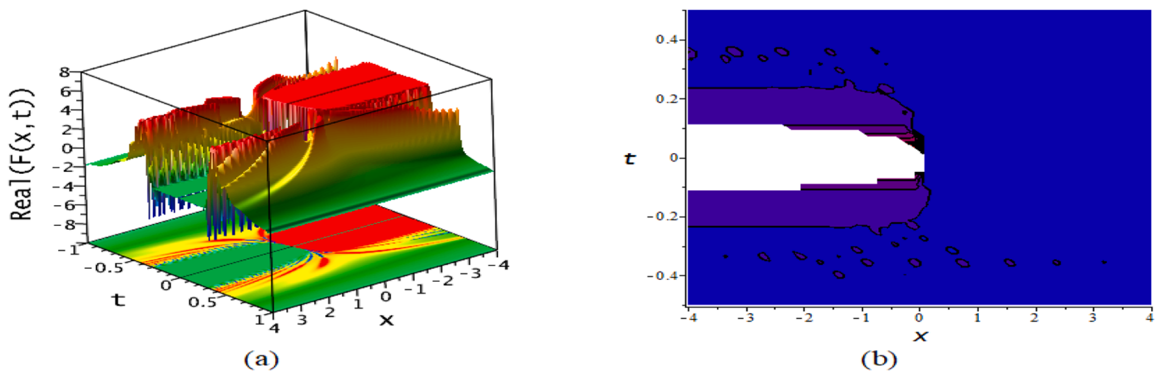

Fig. 5. 3D plot of periodic wave solutions of the Eq. (23) for the parameters $a_1 = 0.2$, $a_2 = 1$, $\alpha = 0.5$, $\beta = 0.5, \gamma = -0.5, h = 1, B_0 = 0.1$, $B_1 = -0.01$, $g = \tanh\ (t)$.

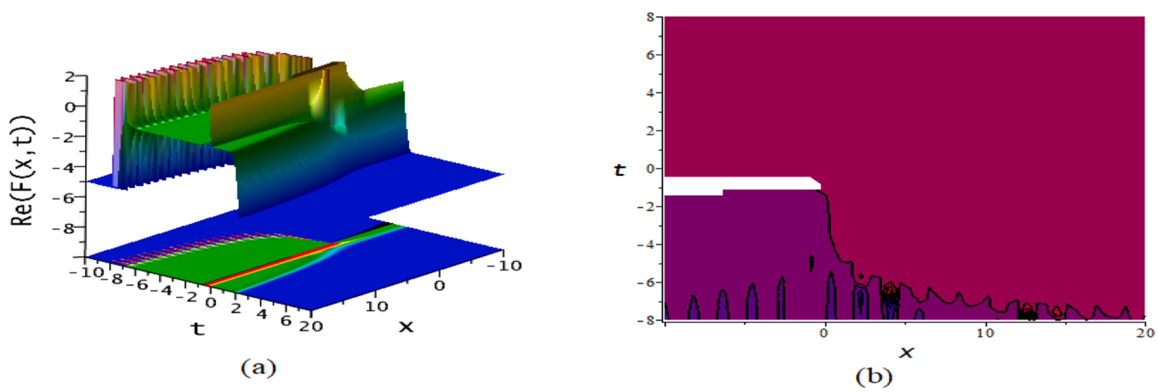

Fig. 6. 3D plot of kink-type solutions of Eq. (30) for the special value of the parameters $a_1 = 0.2$, $a_2 = 1$, $\alpha = 0.2$, $\beta = 0.5, \gamma = -0.5, h = 1, B_0 = 0.1$, $B_1 = -0.01$, $g = e^t$.

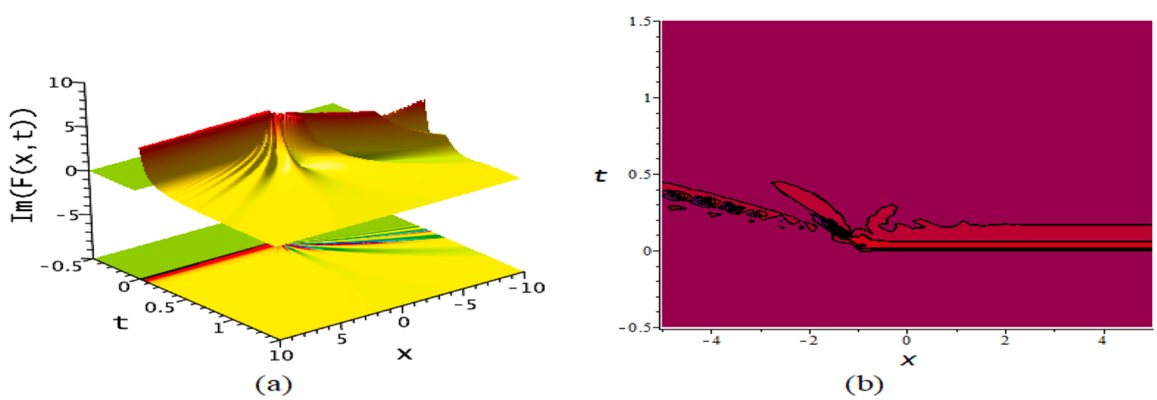

Fig. 7. 3D plot of soliton solutions of the Eq. (30) for the parameters $a_1 = -0.2$, $a_2 = 1$, $\alpha = -0.2$, $\beta = 0.5, \gamma = 0.5, h = 1, B_0 = 0.1$, $B_1 = -0.01, g = e^t$.

therefore, the state of stability is resistant to minor perturbations. However, when the weather, $-(\rho a_1)^2 - 4\rho(-\rho a_2 k^2 - 3\alpha\rho\phi^2 - 2\beta\rho\phi - \gamma\rho) < 0$, and ω are complicated, the disturbance increases exponentially. In Fig. 13, the Gain spectrum of modulation instability (MI) for the variations of different parameter values is manifested, whereas in Figs. 14–16 the 3D plots of the Gain spectrum for several combinations of the parameter values are exhibited

$$H(spec.) = 2Im(\omega) = 2Im\left(-i\rho a_1 \pm \frac{\sqrt{-(\rho a_1)^2 - 4\rho\left(-\rho a_2 k^2 - 3\alpha\rho\phi^2 - 2\beta\rho\phi - \gamma\rho\right)}}{2\rho}\right). \tag{41}$$

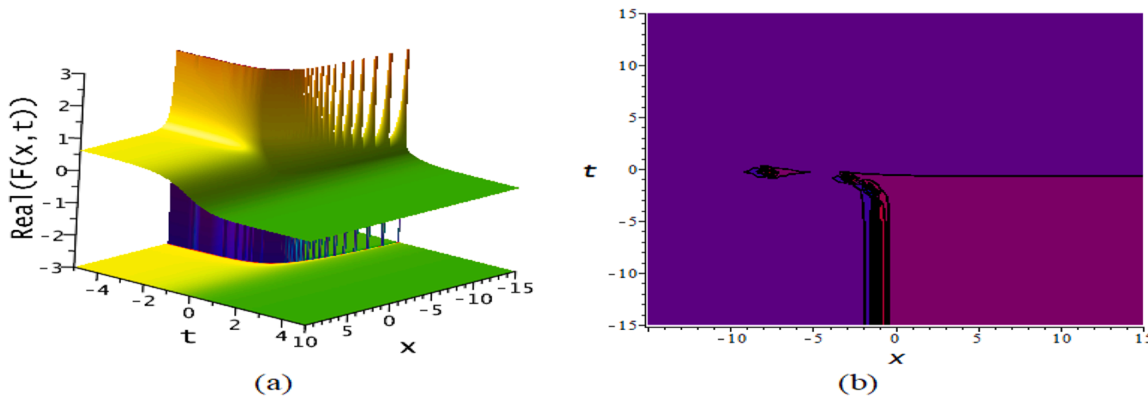

Fig. 8. 3D plot of kink shape with singular soliton solutions of Eq. (30) for the special value of the parameters $a_1 = 0.67$, $a_2 = -0.5$, $\alpha = 3$, $\beta = -0.5$, $\gamma = -0.5$, $h = 1$, $B_0 = 0.1$, $B_1 = -0.01$, $g = e^t$.

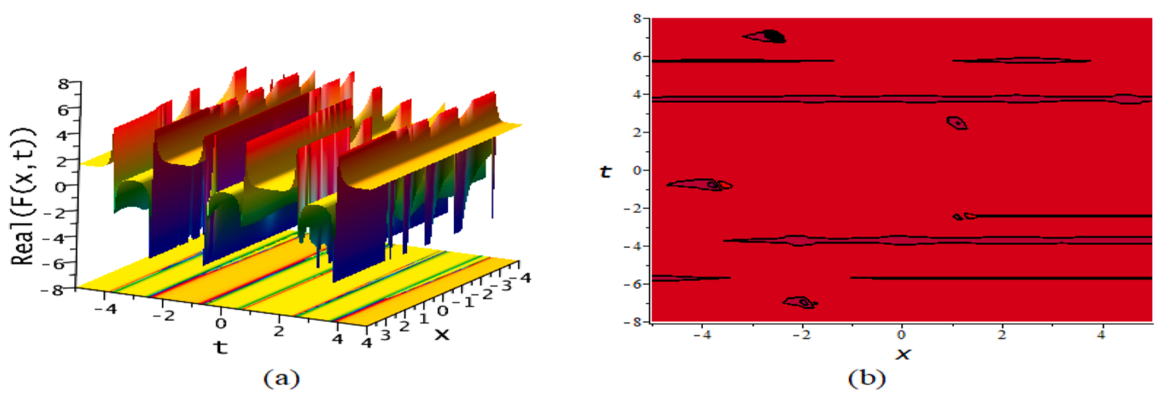

Fig. 9. 3D plot of periodic wave solutions of Eq. (33) for the special value of the parameters $a_1 = 0.2$, $a_2 = 1$, $\alpha = 0.5$, $\beta = 0.5$, $\gamma = -0.5$, $h = 1$, $B_0 = 0.1$, $B_1 = -0.01$, $g = \sec(t)$.

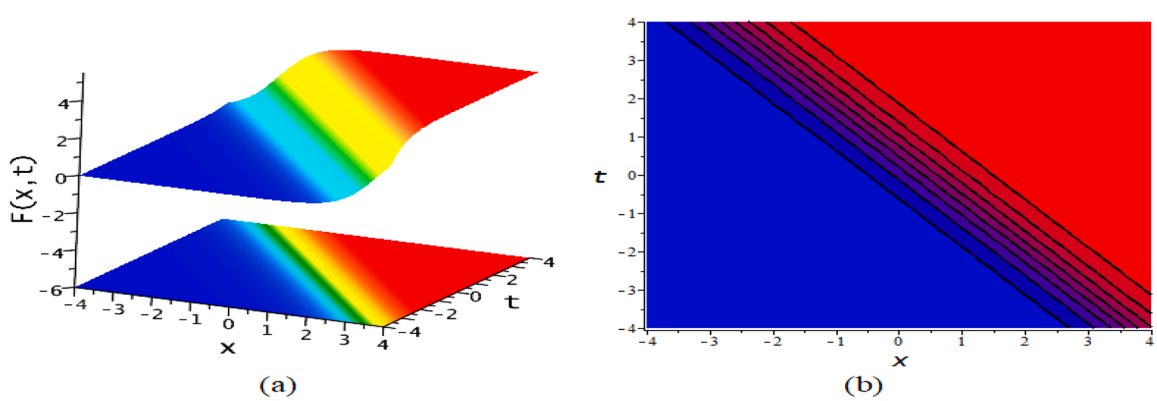

Fig. 10. 3D plot of kink shape soliton solutions of Eq. (33) for the parameters $a_1 = 0.2$, $a_2 = 1$, $\alpha = 0.2$, $\beta = 0.5$, $m = -0.5$, $h = 1$, $B_1 = 0.1$, $B_1 = 1$, $g = 2t$.

6. Comparison with the existing methods

6.1. Comparison with other methods

In this study, we used an Enhanced Modified Simple Equation method[37,38] to construct the exact traveling wave variable from the telegraph model. The main advantage of this technique is to investigate the variable coefficient exact soliton solutions. By using diverse natural variable coefficients, we obtain a more exact traveling wave solution. For this purpose, we use $\eta = h(t)x + g(t)$ traveling wave variable where $h(t)$ and $g(t)$ are the function of timet. If we set the different functions of $h(t)$ and $g(t)$then we get more solutions. This technique has some

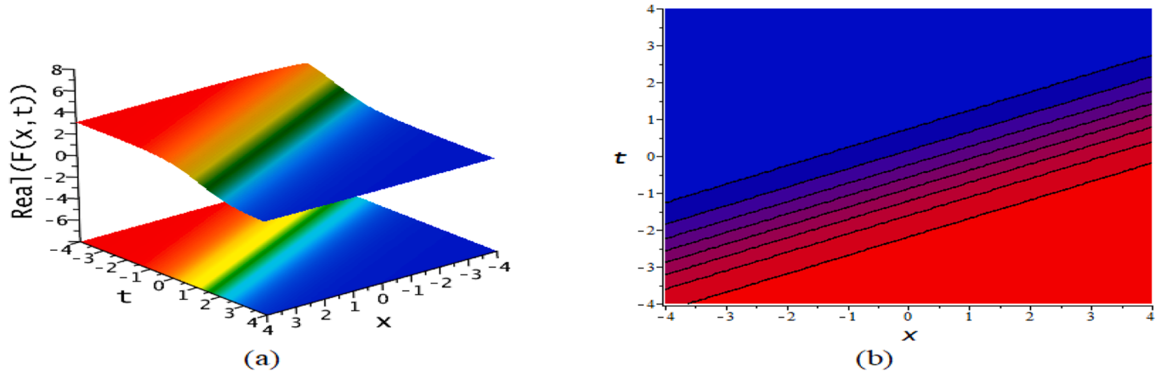

Fig. 11. 3D plot of kink shape soliton solutions of Eq. (33) for the parameters $a_1 = 0.2, \quad a_2 = 1, \quad \alpha = 0.2, \quad \beta = 0.5, m = -0.5, h = 1, B_1 = 0.1, \quad B_1 = 1, \quad g = -2t$.

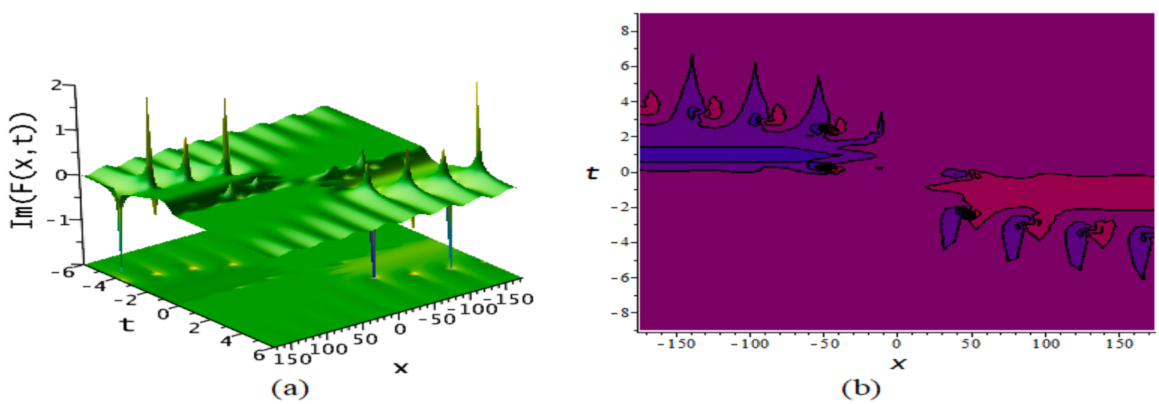

Fig. 12. 3D plot of kink shape with periodic wave soliton solutions of Eq. (33) for the parameters $a_1 = 0.2, \quad a_2 = 1, \quad \alpha = 0.2, \quad \beta = 0.5, m = -0.5, h = 1, B_1 = 0.1, B_1 = 1, \quad g = secht$.

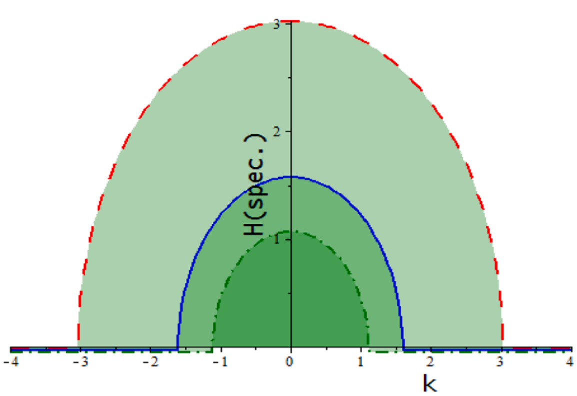

Fig. 13. Gain spectrum of MI for different values of $[a_1 = 1, a_2 = 1, \quad \alpha = 0.1, \beta = 0.1, \quad \phi = -0.1, \gamma = 1], \quad [\rho = 0.01, \quad 0.03, \quad 0.05]$.

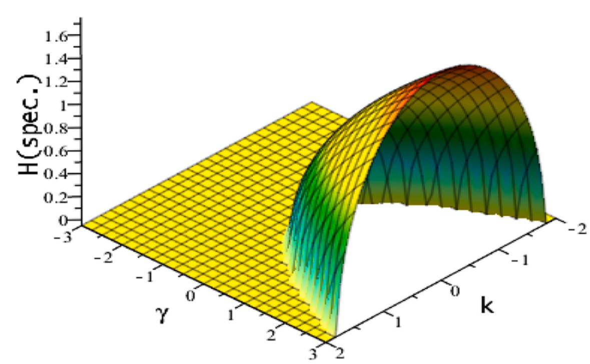

Fig. 14. 3D plot of Gain spectrum of MI for different values of $[a_1 = 1, a_2 = 1, \alpha = 0.1, \beta = 0.1, \quad \phi = -0.1, \rho = 0.01]$.

limitations in constructing exact soliton solutions if the nonlinear model has a higher order derivative term concerning time and also it is non-integral. To apply this technique, the mathematical knowledge must be high in mathematics calculation. Otherwise, it cannot be used. This technique must be calculated in hand. For diverse functions of variable coefficients, we get more exact solutions than other methods. By selecting various time-dependent functions in place of the variable coefficients, we obtained more phenomena such as kink, anti-kink shape, the interaction of kink, anti-kink and singularities, interaction of instanton and kink shape, instanton shape, kink and bell interaction, anti-kink and bell interaction, kink and singular solitons, anti-kink and singular solitons, kink and singular interaction, anti-kink and singular interaction solutions from ascertained from a single solution.

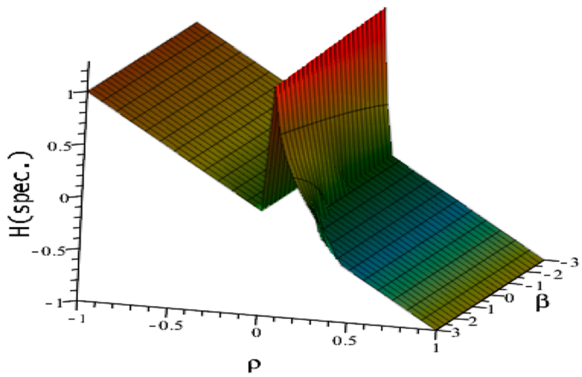

Fig. 15. 3D plot of Gain spectrum of MI for different values of [$a_1 = 1$, $a_2 = 1$, $\alpha = 0.1$, $\phi = -0.1$, $\gamma = 1$, $k = -0.1$].

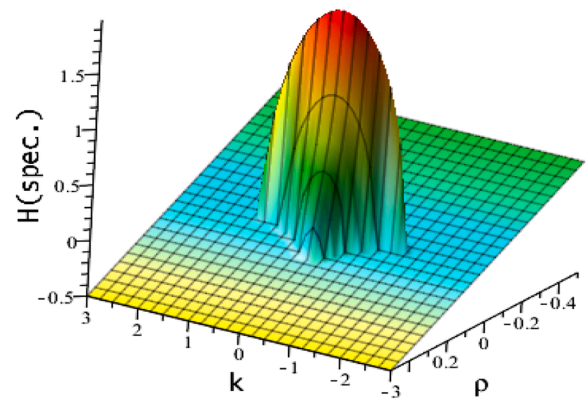

Fig. 16. 3D plot of Gain spectrum of MI for different values of [$a_1 = 1$, $a_2 = 2$, $\alpha = 0.1$, $\beta = 0.1$, $\phi = -0.1$, $\gamma = 1$].

Comparison of ESME and other methods

EMSE method	Other methods
In the EMSE method, we used the traveling wave variable in the form $\eta = h(t)x + g(t)$. Here $h(t)$ is wave length and $g(t)$is wave spread. The wavelength and spread are both differentiable functions of time. Since The wavelength and spread are both functions of time, so we can consider many differentiable functions. So, by applying the EMSE method, we get more exact solutions and can explain different novel phenomena.	The analytic methods except the EMSE method discussed in the introductory section, are used traveling the wave variable in the form $\eta = kx \pm \omega t$ to convert the nonlinear evolution equation into ordinary differential equations form. Here k is wave length and ω is wave spread are both arbitrary constants.

6.2. Comparison with the work of Hyunsoo Kima et al.[40]

Here, we compare the solutions attained in this study with the solutions attained by Hyunsoo Kima et al. in [40]. Variations of the required parameters and corresponding solutions are compared for several phenomena.

7. Conclusion

In this manuscript, we have successfully implemented the time-dependent variable coefficient solitary wave solution of the telegraph model by using the impressive EMSE method. To get the variable coefficient solution we applied the wave formation as $\eta = h(t)x + g(t)$. For

Comparison of the solutions attained by Hyunsoo Kima et al.[40] and this study

Solutions attained by Hyunsoo Kima et al.[40]	Solutions attained in this study
(i) if we set $\alpha = -1$, $\beta = 0.25$, $\omega = 0.5$, then Eq. (31) becomes, $u_2(x,t) = 2\dfrac{2+e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}{1+e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}$. It can be written as $u_2(x,t) = 2\dfrac{2+\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}{1+\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}$.	(i) if we set $\alpha = -1, \beta = 2,\ g = 0.5t,\ Q = \theta = B_0 = 1, B_1 = 4,\ h = \frac{\sqrt{11}}{2},\ a_1 = \frac{4}{11}$, then Eq. (33) becomes, $U(x,t) = 2\dfrac{2+e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}{1+e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}$. It can be written as $U(x,t) = 2\dfrac{2+\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}{1+\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}$.
(iii) if we set $\alpha = -1$, $\beta = 0.25$, $\omega = 0.5$, then Eq. (31) (negative sign) and Eq. (32) becomes, $u_2(x,t) = 2\dfrac{e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}{1+e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}$. It can be written as $u_2(x,t) = 2\dfrac{\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}{1+\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}$.	(iii) if we set $\alpha = -1, \beta = 2,\ g = 0.5t,\ Q = \theta = B_0 = 1, B_1 = 4,\ h = \frac{\sqrt{11}}{2},\ a_1 = \frac{4}{11}$, then Eq. (23) becomes, $U(x,t) = 2\dfrac{e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}{1+e^{\frac{\sqrt{11}}{2}x+\frac{1}{2}t}}$. It can be written as $U(x,t) = 2\dfrac{\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}{1+\cosh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)+\sinh\left(\frac{\sqrt{11}}{2}x+\frac{1}{2}t\right)}$.
(iii) if we set $\alpha = 0.2$, $\beta = -1.25$, $\omega = 0.5$, then Eq. (31) becomes, $u_2(x,t) = 2\dfrac{2+\cos\left(\frac{1}{25}x+\frac{i}{2}t\right)+i\sin\left(\frac{1}{25}x+\frac{i}{2}t\right)}{1+\cos\left(\frac{1}{25}x+\frac{i}{2}t\right)+i\sin\left(\frac{1}{25}x+\frac{i}{2}t\right)}$.	(iii) if we set $\alpha = -1, \beta = 2,\ g = 0.5t,\ Q = \theta = B_0 = 1, B_1 = 4,\ h = \frac{1}{25},\ a_1 = 625$, then Eq. (35) becomes, $U(x,t) = 2\dfrac{2+\cos\left(\frac{1}{25}x+\frac{i}{2}t\right)+i\sin\left(\frac{1}{25}x+\frac{i}{2}t\right)}{1+\cos\left(\frac{1}{25}x+\frac{i}{2}t\right)+i\sin\left(\frac{1}{25}x+\frac{i}{2}t\right)}$.

the diverse function of the variable coefficient, the obtained solitary waves of the classical nonlinear telegraph model, have broad applications across physics engineering, and biology. In nonlinear optics, they ensure robust information transmission in optical fibers, resisting dispersion and maintaining their shape and velocity. Furthermore, in biology, they model nerve impulses, offering insights into neuronal communication and neurological disorders. These solitary waves, governed by the intricate interplay of nonlinear effects like self-interaction and dispersion, provide a fundamental framework for understanding complex wave phenomena. The enhanced modified simple equation (EMSE) technique has several benefits, including the ability to bargain more time-varying solitary wave solutions directly, without any discretization, linearization, or perturbation. This method makes it possible to create flexible-constrained solitary wave solutions, which are essential for explaining intricate nonlinear phenomena such as instanton soliton, a kinky periodic wave, lump wave, kink wave, anti-kink double periodic wave, diverse types periodic wave, lump with bell wave periodic wave, periodic lump wave, etc. the obtained solution is presented and explained by illustrating the three-dimensional plot with density plot and corresponding contour plots. The movement role of the waves is investigated, and the stability analysis of the acquired solutions is discussed using the modulation instability analysis through the figurative approach, demonstrating the accuracy and stability of all produced solutions. This study's solutions are validated and truly helpful for nonlinear scientific computations. The presented equation's strength is its computational appeal, predictability, and reliability. We do not doubt that the approaches mentioned here will be extremely important in subsequent studies.

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CRediT authorship contribution statement

Anuz Kumar Chakrabarty: Conceptualization, Software, Visualization. **Sonia Akter:** Software, Writing – review & editing. **Mahtab Uddin:** Methodology, Visualization, Writing – review & editing. **Md. Mamunur Roshid:** Investigation, Visualization, Writing – original draft. **Alrazi Abdeljabbar:** . **Harun Or-Roshid:** Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The simulation data used to support the findings of this study are available from the corresponding author upon request.

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