

On Some Universal Constructions for Fuzzy Topological Spaces



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*Dedicated
To My Parents*

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At first, I am giving my unlimited thanks to the Almighty Allah for giving me strength, patience and capability to complete this course of study.

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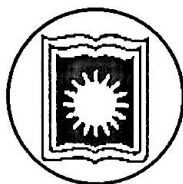
I feel pleased and privileged to fulfill my parent's ambition and I am greatly indebted to them for their unmeasured, continuing inspiration and bearing the inconvenience during my this work.

Finally, I acknowledge that I am alone responsible for the errors and shortcomings in this study if there be any and sorry for that.

(Mohammed Harunor Rashid)

Department of Mathematics

September-2005



No

Dated

Certificate

This is to certify that the M.Phil. thesis entitled "On Some Universal Constructions for Fuzzy Topological Spaces" which is being submitted by Mohammed Harunor Rashid in fulfillment of the requirement for the degree of M.Phil. in Mathematics, University of Rajshahi, Rajshahi, Bangladesh is a record of bonafide research work done by him under my supervision. I believe that the results embodied in the thesis are new and it has not been submitted elsewhere for any degree.

To the best of my knowledge Mohammed Harunor Rashid bears a good moral character and is mentally and physically fit to get the degree. I wish him a bright future and every success in his life.

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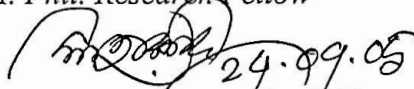
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STATEMENT OF ORIGINALITY

I declare that the contents in my M. Phil. thesis entitled “ On Some Universal Constructions for Fuzzy Topological Spaces” are original and accurate to the best of my Knowledge. I also certify that the materials contained in my research work have not been previously published or written by any person for a degree or diploma.

Rajshahi University
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ABSTRACT

In this work, many known universal objects such as products, equalizers, pullbacks and their duals, namely, coproducts, coequalizers and pushouts for fuzzy topological spaces are studied in detail. Another known concepts of inverse and direct limits on inverse and direct system of fuzzy topological spaces are also studied in more detail. Besides this, we explored some properties of the singular homology groups of fuzzy topological spaces by using the concepts of direct limits on fuzzy direct systems of fuzzy topological spaces.

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INTRODUCTION

In 1965, Zadeh L.A. first introduced the concept of fuzzy sets in his classic paper [127]. He interpreted a fuzzy set on a set X is a mapping from X into the unit interval $I=[0, 1]$ and generalizing the notion of characteristic function on a set X . In 1967, Goguen [42] defined fuzzy set by replacing the unit interval I by a completely distributive lattice L with an order reversing involution. This extended idea of fuzzy sets using the completely distributive lattice L became a suitable framework for the theory of L -fuzzy sets.

Fuzzy sets have become a concern and tool for persons working in a variety of fields. Particularly, Mathematicians working in many different areas such as logic, topology, algebra, probability theory, category theory etc. have shown keen interest and have done a lot of research in the area of fuzzy sets. In the same way, researchers are applying fuzzy sets in the areas of general systems, intelligent systems, artificial intelligence, decision theory and optimization, approximate reasoning, sociology and behavioral science etc. Also, the theory of fuzzy sets has already affected a wide variety of disciplines such as control theory, medicines, information processing, cluster analysis, genetics, electrical engineering, operations research and pattern recognition etc.

The notion of fuzzy sets as stated by Zadeh inspired many mathematicians to generalize the important abstract concepts and structures of the present mathematics into the framework of fuzzy sets. In this regards, we could like to mention that fuzzy topological spaces, fuzzy groups, fuzzy vector spaces and fuzzy topological vector spaces, fuzzy topological groups, a categorical accommodation of various notions of fuzzy topology and singular homology groups of fuzzy topological spaces were introduced by Cang C. L.[24], Rosenfeld A.[96], Katsara A. K. and Liu D. B.[55], Foster D. H.[36], Rodabaugh S. E.[97] and Liu Wang-jin and Zheng Chong-you [63] respectively.

In 1984, Katsara and Petalas [57] also introduced a unified theory of fuzzy topologies, fuzzy uniformities and fuzzy proximities. Considerable progress has also been made in successful development of fuzzy algebra, fuzzy calculus, fuzzy logic, fuzzy

operations research, fuzzy category theory and singular homology groups of fuzzy topological spaces etc.

In fact, in 1968 Chang C. L. [24] paved the way for subsequent growth of numerous topological concepts. It is worth to mention that in 1976, Lowen R. [66] pointed out that in Chang's definition of fuzzy topological spaces [24], constant functions between fuzzy topological spaces need not be fuzzy continuous. It was a major deviation from the corresponding result in general topology. To deal with this situation, Lowen R. [66] modified the definition of fuzzy topological spaces by including all constant fuzzy sets in a fuzzy topology. He also gave a new definition of fuzzy compactness which he proved to be a natural generalization of compactness in ordinary topology and which also safeguards the Tychonoff theorem. Furthermore, in 1977 Lowen R. [65] introduced the notions of initial and final fuzzy topologies, which are two appropriate concepts to generalize the topological ones from the categorical point of view.

A further development of fuzzy topology was made by Sarkar Mira [101] and Hutton B. [50], where they considered L-fuzzy sets and which was introduced by Goguen J. A. [42]. This developing fuzzy topology was considered as a L-fuzzy topology. The present state of ongoing research in fuzzy topology can be divided into two separate sections, one of which is exclusively using the unit interval I to describe fuzziness (Chang's fuzzy topologies) and the other using L-fuzzy topologies. In our investigation, we have preferred the concepts of fuzzy topology, which has developed by Chang C. L. [24].

In 1980, Pu Pao-Ming and Liu Ying-Ming [88] successfully gave a new definition of fuzzy point, which takes singleton or ordinary point as a special case. They also introduced the very important notions of quasi-coincidence and quasi-neighborhoods of fuzzy points. With this new concept of fuzzy points and quasi-neighborhoods, they successfully established the Moore-Smith convergence in fuzzy setting.

Two of the important topologies, which play very vital roles in general topology, are product and quotient topologies. Wong C. K [122] extended these notions to fuzzy setting and later extensively investigated by many authors including Lowen R. [65],

Hutton B. [50], Pu Pao-Ming and Liu Ying-Ming [89], Mashhour et al [73], Christoph F. T. [25] and Erceg M. A. [32] etc. We have made good use of these fuzzy spaces for investigation of universal objects such as product, equalizer, pullback and their duals, namely, coproduct, coequalizer, pushout and also in the study of direct and inverse limits on direct and inverse system of fuzzy topological spaces.

The present thesis entitled **“On Some Universal Constructions for Fuzzy Topological Spaces”** is devoted to the study of some important concepts of category theory for fuzzy topological spaces. There are many interesting universal objects such as products, equalizers, pullbacks and their duals, namely, coproducts, coequalizers, pushouts that are introduced by Mac Lane S. [69] in general topology. But only two objects, namely, products and coproducts have been studied in fuzzy setting by Hutton B. [50], Mashhour et al [73], Pu Pao-Ming and Liu Ying-Ming [88, 89] and Wong C. K. [122]. Behera A. [17] has introduced the concepts of fuzzy equalizers, fuzzy pullbacks and their duals, namely, fuzzy coequalizers, fuzzy pushouts for fuzzy topological spaces and explored the results concerning all such universal objects. Sahu J.K. [98] has introduced the concepts of fuzzy inverse and direct limits on inverse and direct system of fuzzy topological spaces and explored their properties and characterizations. We studied these universal objects in more detail and developed theories of the above-cited universal objects. Also, we explore some properties of singular homology groups of fuzzy topological spaces in terms of fuzzy direct limit on direct system of fuzzy topological spaces to give them an enriched flavor.

The materials of this thesis has been divided into six chapters and a brief discussion of this we mention below:

The first chapter is incorporated with some basic concepts, definitions and known results on fuzzy sets, fuzzy topological spaces and different mappings on fuzzy topological spaces, which are necessary for the subsequent chapters. Results are included without proof and can be seen in the papers referred to.

We have started our research work from the second chapter. In this chapter, we recall definitions of fuzzy products, fuzzy equalizers, fuzzy pullbacks and their duals,

namely, fuzzy coproducts, fuzzy coequalizers, fuzzy pushouts and results on such universal objects, which are introduced and proved by Behera A. [17]. Also, we have added some new results of these concepts.

In chapter three, we recall the definitions of fuzzy inverse system, fuzzy inverse limit, fuzzy direct system, fuzzy direct limit, which are introduced by Sahu J.K.[98]. We have added some new results and presented the explicit constructions of these limits.

In chapter four, We have studied the mappings between two inverse systems and the mappings between two direct systems of fuzzy topological spaces. Also, We have constructed the mappings between two inverse limits and the mappings between two direct limits of fuzzy topological spaces and explored some of their properties.

In chapter five, we have characterized fuzzy inverse limits in terms of fuzzy products, fuzzy equalizers, fuzzy pullbacks and fuzzy direct limits in terms of fuzzy coproducts, fuzzy coequalizers, and fuzzy pushouts.

In the last chapter of this thesis, we recall the entire construction of singular homology theory of fuzzy topological spaces, which are developed by Liu Wang-jin and Zheng Chong-you [63]. We have added some new results to fuzzy algebraic topology using the concepts of direct limits of fuzzy topological spaces.

CHAPTER-1

Pre-requisites

In this chapter we recall some definitions and known results on fuzzy sets, fuzzy topological spaces and fuzzy groups. Some more definitions and results are included in the relevant chapters. We consider this chapter as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when required.

1.1 FUZZY SETS:

In 1965, Zadeh L. A. introduced the following revolutionary concepts of fuzzy set.

1.1.1. DEFINITION ([127]):

Let X be a set of points $\{x : x \in X\}$. A fuzzy set λ in X is a function $\lambda : X \rightarrow I$, where I is the unit closed interval $[0,1]$. For each $x \in X$, $\lambda(x) \in I$ denotes a degree or the grade of membership of x . We denote the set of all fuzzy sets in X by I^X . Ordinary subsets of X (crisp sets) are those members of I^X which take the values 0 and 1 only. In particular, the crisp sets that always take the values 0 and 1 on X are denoted by 0 and 1 respectively.

1.1.2. DEFINITION:

Let λ be a fuzzy set in X . Then the set $\{x \in X : \lambda(x) > 0\}$ is called the support of λ and is denoted by λ_0 or $\text{supp}\lambda$. If $A \subseteq X$, then 1_A will denote the characteristic function of A . We also write 1_x for the characteristic function of $\{x\} \subseteq X$.

We will denote the fuzzy sets on a set X by the letters $\lambda, \mu, \alpha, \beta, \gamma, \eta, \dots$ and ordinary functions will denote by the letters $f, g, h, p, q, r, s, \dots$ etc.

1.1.3. DEFINITION ([127]):

Let α and β be fuzzy sets in X . Let α^c be the complement of the fuzzy set α in X . Then for all $x \in X$, we define

- (i) $\alpha = \beta$ if and only if $\alpha(x) = \beta(x)$.

- (ii) $\alpha \subseteq \beta$ if and only if $\alpha(x) \leq \beta(x)$.
- (iii) $\lambda = \alpha \cup \beta$ if and only if $\lambda(x) = \max[\alpha(x), \beta(x)]$.
- (iv) $\mu = \alpha \cap \beta$ if and only if $\mu(x) = \min[\alpha(x), \beta(x)]$.
- (v) $\eta = \alpha^c$ if and only if $\eta(x) = 1 - \alpha(x)$.

More generally, for a family of fuzzy sets $\{\alpha_i, i \in J\}$, their union $\bigcup_{i \in J} \alpha_i$ and

intersection $\bigcap_{i \in J} \alpha_i$ are also fuzzy sets which are defined by

- (i) $\theta = \bigcup_{i \in J} \alpha_i$ if and only if $\theta(x) = \sup_{i \in J} \{\alpha_i(x)\}$, for all $x \in X$ and
- (ii) $\varphi = \bigcap_{i \in J} \alpha_i$ if and only if $\varphi(x) = \inf_{i \in J} \{\alpha_i(x)\}$, for all $x \in X$ respectively.

1.1.4. REMARK:

We shall usually denote the union (supremum) and intersection (infimum) operations on fuzzy sets by $\vee$ and $\wedge$ respectively. For $\alpha, \beta \in I^X$ we have the following properties:

1. $\alpha \vee \alpha = \alpha, \alpha \wedge \alpha = \alpha$ (Idempotent law),
2. $\alpha \vee \beta = \beta \vee \alpha, \alpha \wedge \beta = \beta \wedge \alpha$ (Commutative law),
3. $(\alpha \vee \beta) \vee \gamma = \alpha \vee (\beta \vee \gamma)$ and $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ (Associative law),
4. $(\alpha \vee \beta) \wedge \alpha = \alpha$ and $(\alpha \wedge \beta) \vee \alpha = \alpha$ (Absorption law),
5. $(\alpha \vee \beta) \wedge (\alpha \vee \gamma) = \alpha \vee (\beta \wedge \gamma)$ and $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = \alpha \wedge (\beta \vee \gamma)$ (Distributive law),
6. $\alpha \vee 1 = 1, \alpha \wedge 1 = \alpha$ and $\alpha \vee 0 = \alpha, \alpha \wedge 0 = 0$ (Bipolarity),
7. $(\alpha^c)^c = 1 - \alpha^c = 1 - (1 - \alpha) = \alpha$ (Reflexivity),
8. $(\alpha \vee \beta)^c = \alpha^c \wedge \beta^c, (\alpha \wedge \beta)^c = \alpha^c \vee \beta^c$ (De-Morgan's law).

Complementary law of cantor set does not hold for a fuzzy set, such as if $\alpha \in I^X$, then $\alpha \vee \alpha^c \neq 1, \alpha \wedge \alpha^c \neq 0$.

For the proof of this, take for instance $\alpha(x) = \frac{1}{2}$,

$$\text{then } \alpha(x) \vee \alpha^c(x) = \frac{1}{2} \vee (1 - \frac{1}{2}) = \frac{1}{2} \vee \frac{1}{2} = \frac{1}{2} \neq 1$$

$$\text{and } \alpha(x) \wedge \alpha^c(x) = \frac{1}{2} \wedge (1 - \frac{1}{2}) = \frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2} \neq 0, \text{ for all } x \in X.$$

1.1.5. DEFINITION:

If $r \in I$ and λ is a fuzzy set in X defined by $\lambda(x) = r$, for all $x \in X$, then we called λ is a constant fuzzy set in X .

1.1.6. DEFINITION ([13]):

If X and Y are any two sets and $\alpha \in I^X, \beta \in I^Y$, then their Cartesian product $\alpha \times \beta$ is a fuzzy set in $X \times Y$ defined by $(\alpha \times \beta)(x, y) = \min\{\alpha(x), \beta(y)\}$, for $(x, y) \in X \times Y$.

1.2. FUZZY POINT AND FUZZY SINGLETON :

In 1980, Pu Pao-Ming and Liu Ying-Ming [88] presented the following definition of fuzzy point.

1.2.1. DEFINITION ([88]):

A fuzzy point x_r in X is a special type of fuzzy set in X with membership function $x_r(y) = r$, for $x = y$ and $x_r(y) = 0$, for $x \neq y$, where $0 < r < 1$ and $\forall x, y \in X$. The fuzzy point x_r is said to have support x and value r . We also write this as rl_x .

1.2.2. DEFINITION:

A fuzzy point x_r in X is said to belong to a fuzzy set α in X , denoted $x_r \in \alpha$ if and only if $x_r(x) < \alpha(x)$ and $x_r(y) < \alpha(y)$ if $y \neq x$ i.e. $x_r \in \alpha \Rightarrow r < \alpha(x)$.

1.2.3. DEFINITION:

A fuzzy singleton in X is a fuzzy set in X which is zero everywhere except at one point, say x , where it takes value, say r , with $r \in (0, 1]$ i.e. $0 < r \leq 1$. We denote it by x_r . Also, $x_r \in \alpha \Leftrightarrow r \leq \alpha(x)$.

1.2.4. THEOREM ([59]):

For all fuzzy points x_r in X , we have

- (i) $\alpha \subseteq \beta$ if and only if $x_r \in \alpha \Rightarrow x_r \in \beta$,
- (ii) $\alpha = \beta$ if and only if $x_r \in \alpha \Leftrightarrow x_r \in \beta$,
- (iii) $x_r \in \alpha \vee \beta$ if and only if $x_r \in \alpha$ or $x_r \in \beta$,
- (iv) $x_r \in \alpha \wedge \beta$ if and only if $x_r \in \alpha$ and $x_r \in \beta$.

More generally,

- (v) $x_r \in \bigvee_{i=1}^n \alpha_i$ if and only if $x_r \in \alpha_i$, for some α_i , $i=1,2,\dots,n$,
- (vi) $x_r \in \bigwedge_{i \in J} \alpha_i$ if and only if $x_r \in \alpha_i$, for all $i \in J$, where J is an arbitrary index set and n is a natural number.

1.2.5. THEOREM ([88]):

A fuzzy set α in X is the union of all its fuzzy points' i.e. $\alpha = \bigvee_{x_r \in \alpha} x_r$.

1.3. FUNCTIONS ON FUZZY SETS:

In 1968, Chang C. L. presented the following definitions relating to function on fuzzy sets.

1.3.1. DEFINITION ([24]):

Let $f : X \rightarrow Y$ be a mapping and α be a fuzzy set in X . Then the image $f(\alpha)$ is a fuzzy set in Y which is defined by $f(\alpha)(y) = \sup \{\alpha(x) : f(x) = y, f^{-1}(y) \neq \emptyset\}$, $f(\alpha)(y) = 0$ if $f^{-1}(y) = \emptyset$.

1.3.2. DEFINITION ([24]):

Let $f : X \rightarrow Y$ be a mapping and β be a fuzzy set in Y . Then the inverse image $f^{-1}(\beta)$ is a fuzzy set in X which is defined by $f^{-1}(\beta)(x) = \beta(f(x))$, $x \in X$.

1.3.3. THEOREM ([24]):

Let $f : X \rightarrow Y$ be a function. Then

- (i) $\alpha_1 \leq \alpha_2 \Rightarrow f(\alpha_1) \subseteq f(\alpha_2)$, for all $\alpha_1, \alpha_2 \in I^X$.

- (ii) $\beta_1 \leq \beta_2 \Rightarrow f^{-1}(\beta_1) \subseteq f^{-1}(\beta_2)$, for all $\beta_1, \beta_2 \in I^Y$.
- (iii) $\alpha \subseteq f^{-1}(f(\alpha))$, for any $\alpha \in I^X$.
- (iv) $\beta \supseteq f(f^{-1}(\beta))$, for any $\beta \in I^Y$.
- (v) $f^{-1}(\beta^c) = (f^{-1}(\beta))^c$, for any $\beta \in I^Y$.
- (vi) $(f(\alpha))^c \subseteq f(\alpha^c)$, for any $\alpha \in I^X$.
- (vii) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions and $g \circ f : X \rightarrow Z$ be the composition of f and g . Then $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$, for any $\mu \in I^Z$.

1.3.4. REMARK:

If $f : X \rightarrow Y$ is a function and if $\{\alpha_i : i \in K\}$ is a family of fuzzy sets in X and $\{\beta_j : j \in J\}$ is a family of fuzzy sets in Y , then

- (i) $f^{-1}\left(\bigvee_{j \in J} \beta_j\right) = \bigvee_{j \in J} f^{-1}(\beta_j)$;
- (ii) $f^{-1}\left(\bigwedge_{j \in J} \beta_j\right) = \bigwedge_{j \in J} f^{-1}(\beta_j)$;
- (iii) $f\left(\bigvee_{i \in K} \alpha_i\right) = \bigvee_{i \in K} f(\alpha_i)$;
- (iv) $f\left(\bigwedge_{i \in K} \alpha_i\right) = \bigwedge_{i \in K} f(\alpha_i)$.

1.3.5. THEOREM ([76]):

Let $f : X \rightarrow Y$ be a mapping from a set X into a set Y with $\alpha \in I^X$ and $\beta \in I^Y$.

Then

- (i) If x_r is a fuzzy point in X , then $f(x_r) = [f(x)]_r$ is a fuzzy point in Y ;
- (ii) If x_r is a fuzzy point in α , then $f(x_r)$ is a fuzzy point in $f(\alpha)$;
- (iii) The fuzzy point $f(x_r)$ in β if and only if x_r is a fuzzy point in $f^{-1}(\beta)$;
- (iv) If x_r is a fuzzy point in Y , then $f^{-1}(x_r)$ need not be a fuzzy point in X . However, if f is injective and $x_r \in f(X)$, then $f^{-1}(x_r)$ is a fuzzy point in X and is then defined as $f^{-1}(x_r) = [f^{-1}(x)]_r$.

1.3.6. DEFINITION ([13]):

Let $f : X \rightarrow Y$ be a function. The graph of f is a function $g : X \rightarrow X \times Y$ such that $g(x) = (x, f(x))$ for all $x \in X, y \in Y$ such that $f(x) = y$.

1.3.7. THEOREM ([13]):

If $g : X \rightarrow X \times Y$ is the graph of a function $f : X \rightarrow Y$, then for any fuzzy sets $\alpha \in I^X$ and $\beta \in I^Y$, the result $g^{-1}(\alpha \times \beta) = \alpha \wedge f^{-1}(\beta)$ holds.

1.4. FUZZY TOPOLOGICAL SPACES:

Chang C. L gives the following definition of a fuzzy topological space.

1.4.1. DEFINITION ([24]):

Let X be a set. A class t of fuzzy sets in X is called a fuzzy topology on X if t satisfies the following conditions:

- (i) $0, 1 \in t$;
- (ii) If $\alpha, \beta \in t$, then $\alpha \wedge \beta \in t$;
- (iii) If $\{\alpha_i : i \in K\}$ is a family of fuzzy sets in t , then $\bigvee_{i \in K} \alpha_i \in t$.

The pair (X, t) is called a fuzzy topological space, in short fts. The members of t are called t -open fuzzy sets and their complements are called t -closed fuzzy sets. When there is no risk of confusion, we will denote an fts (X, t) simply by X and t -open or t -closed fuzzy sets by open and closed fuzzy sets respectively.

1.4.2. DEFINITION ([66]):

Lowen R. modified the definition of a fuzzy topological space of Chang C. L. [24] by adding another condition. In the sense of Lowen R., the definition of fuzzy topological space is as follows:

Let X be a set and $t \subseteq I^X$. Then t is called a fuzzy topology on X if the following condition holds:

- (i) $0, 1 \in t$;
- (ii) If $\alpha, \beta \in t$, then $\alpha \wedge \beta \in t$;
- (iii) If $\{\alpha_i : i \in K\}$ is a family of fuzzy sets in t , then $\bigvee_{i \in K} \alpha_i \in t$ and

(iv) t contains all constant fuzzy sets in X .

The pair (X, t) is called a fuzzy topological space.

We shall use the concept of fuzzy topological space due to Chang C. L.[24] unless otherwise stated.

1.4.3. DEFINITION:

Let X be a set and d be the class of all fuzzy sets in X . Observe that d satisfies the axioms for a fuzzy topology t on X . This fuzzy topology d on X is called the discrete fuzzy topology and the pair (X, d) is called a discrete fuzzy topological space.

1.4.4. DEFINITION:

Let X be a set and i be a fuzzy topology on X consisting of fuzzy sets 0 and 1 alone. Then $i = \{0, 1\}$ is called the indiscrete fuzzy topology and the pair (x, i) is called the indiscrete fuzzy topological space.

1.4.5. DEFINITION:

Let (X, t) be a fuzzy topological space with fuzzy T_1 -space. Then $\{x\}$, for all $x \in X$ is a closed fuzzy set in (X, t) .

1.4.6. DEFINITION ([24]):

Let λ be a fuzzy set in an fts (X, t) . Then the fuzzy closure $\bar{\lambda}$ and fuzzy interior λ^0 of λ are the fuzzy sets in X defined by $\bar{\lambda} = \inf\{\mu : \lambda \subseteq \mu, \mu \in t^c\}$ and $\lambda^0 = \sup\{\mu : \mu \subseteq \lambda, \mu \in t\}$ respectively.

1.4.7. NOTE:

We observe that

- (i) $\lambda^0 = \left(\bar{\lambda}^c\right)^c$
- (ii) $\lambda \in I^X$ is a fuzzy open if and only if $\lambda = \lambda^0$.
- (iii) $\lambda \in I^X$ is a fuzzy closed if and only if $\lambda = \bar{\lambda}$.
- (iv) $\lambda^0 \subseteq \lambda \subseteq \bar{\lambda}$, for any $\lambda \in I^X$.

(v) If $\alpha \subseteq \beta$, then $\bar{\alpha} \subseteq \bar{\beta}$ and $\alpha^0 \subseteq \beta^0$. Also, $\overline{\bar{\alpha}} = \bar{\alpha}$ and $(\alpha^0)^0 = \alpha^0$.

1.4.8. DEFINITION ([41]):

Let (X, t) be an fts and $A \subseteq X$. We define the relative fuzzy topology for A by $t_A = \{A \wedge \eta : \eta \in t\}$. Then t_A is a fuzzy topology on A . This fuzzy topology on A is called the subspace fuzzy topology on A . The pair (A, t_A) is called the fuzzy subspace of (X, t) . A subspace (A, t_A) of the fts (X, t) is called fuzzy open (closed) subspace if and only if the set A is fuzzy open (closed) in X .

1.4.9. THEOREM ([41]):

Let (A, t_A) be a fuzzy subspace of an fts (X, t) and α be any fuzzy set in (A, t_A) .

Then

- (i) α is t_A -closed if and only if $\alpha = A \wedge \mu$, for some t -closed fuzzy set μ in X .
- (ii) The t_A -closure $cl_A(\alpha)$ and the t -closure $\bar{\alpha}$ of α are related by $cl_A(\alpha) = A \wedge \bar{\alpha}$.

1.4.10. DEFINITION ([36]):

Let (X, t) be an fts. Then a subfamily B of t is called a base for t if and only if each member of t can be expressed as a supremum (union) of members of B ; and a subfamily S of t is a subbase for t if and only if the family of all the finite infimum (intersection) of members of S is a base for t .

1.4.11. DEFINITION ([24]):

Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : (X, t) \rightarrow (Y, s)$ be any function. Then f is called

- (i) Fuzzy continuous if and only if $f^{-1}(u) \in t$, for each $u \in s$.
- (ii) Fuzzy open if and only if $f(\mu) \in s$, for each $\mu \in t$.
- (iii) Fuzzy closed if and only if $f(\lambda) \in s^c$, for each $\lambda \in t^c$.
- (iv) Fuzzy homeomorphism if f is bijective, fuzzy continuous and fuzzy open.
- (v) Fuzzy identification if f is fuzzy continuous, surjective and for each $\mu \in I^Y$, $f^{-1}(\mu) \in t$ implies $\mu \in s$.

1.4.12. DEFINITION ([122]):

Let $\{(X_i, t_i) : i \in K\}$ be a collection of fuzzy topological spaces. Let $X = \prod_{i \in K} X_i$ be their Cartesian product and $p_i : X \rightarrow X_i$ be the projection map from X on to X_i . Then the fuzzy topology t on X is called the product fuzzy topology for X if all the projections $p_i : X \rightarrow X_i$ are fuzzy continuous. The product fuzzy topology t is also called the coarsest fuzzy topology on X and the pair (X, t) is called the product fuzzy topological space.

1.4.13. DEFINITION ([122]):

Let (X, t) be an fts and R be an equivalence relation on X . Let X/R be the quotient set and $p : X \rightarrow X/R$ be the projection (quotient map). Let u be the family of fuzzy sets in X defined by $u = \{\mu : p^{-1}(\mu) \in t\}$. Then u is obviously a fuzzy topology. This topology u is called the quotient fuzzy topology for X/R . The pair $(X/R, u)$ is called the fuzzy quotient space of (X, t) . When no confusion arise, the fuzzy quotient space will be denoted by X/R .

1.5. SOME FUZZY CONCEPTS:

Several concepts of fuzzy compactness, fuzzy Hausdorffness, fuzzy t_1 -space, fuzzy regular space and fuzzy normal space are available in fuzzy literature. We recall some of these notions below which we have used in our subsequent works.

1.5.1. DEFINITION ([77]):

Let (X, t) be an fts and $\alpha \in I^X$. A collection $u \subseteq I^X$ is called a fuzzy cover of α if and only if $[\bigvee \{\mu : \mu \in u\}](x) = 1$, for all $x \in \alpha_0$ (support of α). If the members of u are fuzzy open, then u is called a fuzzy open cover of α .

A fuzzy set $\alpha \in I^X$ is fuzzy compact if and only if every fuzzy open cover of α has a finite sub-cover.

1.5.2.DEFINITION:

An fts (X, t) is called fuzzy T_1 -space if and only if for all pair of fuzzy singletons x_r, y_s ($x \neq y$) in X , there exists $u, v \in t$ such that $x_r \in u \subseteq \text{co}y_s$ and $y_s \in v \subseteq \text{co}x_r$.

1.5.3.THEOREM:

For an fts (X, t) the following are equivalent:

(i) For each pair of distinct fuzzy singletons x_r, y_s in X , there exists $u, v \in t$ such that

$$x_r \in u \subseteq \text{co}y_s \text{ and } y_s \in v \subseteq \text{co}x_r.$$

(ii) For all $x, y \in X$, $x \neq y$, there exists $u, v \in t$ such that $u(x) = 1, u(y) = 0$, and

$$v(x) = 0, v(y) = 1.$$

(iii) $\{x\}$, for all $x \in X$, is a closed fuzzy set in (X, t) .

(iv) $D(X)$, the diagonal of X is closed in $(X \times X, t \times d)$ where d is the discrete fuzzy topology on X .

1.5.4. DEFINITION ([108]):

An fts (X, t) is called t_2 -space or Hausdorff space if and only if it satisfies the following axiom:

Each pair of distinct fuzzy points x_r and y_s , ($x \neq y$) in X , there exists fuzzy open sets $u, v \in t$ such that $x_r \in u, y_s \in v$ and $u \wedge v = 0$.

1.5.5. PROPOSITION:

Let $\{(X_i, t_i) : i \in K\}$ be a collection of fuzzy Hausdorff spaces and let (X, t) be the product fuzzy topological spaces. Then (X, t) is also a Hausdorff space.

PROOF:

Let $x_r = \{\alpha_i : i \in K\}$ and $y_s = \{\beta_i : i \in K\}$ be distinct fuzzy points in X . Then x_r and y_s must differ in at least one co-ordinate space, say X_{j_0} i.e. $\alpha_{j_0} \neq \beta_{j_0}$. By the hypothesis, X_{j_0} is fuzzy Hausdorff space. Then there exists fuzzy open sets u and v of X_{j_0} such that $\alpha_{j_0} \in u$, $\beta_{j_0} \in v$ and $u \wedge v = 0$. By the definition of product fuzzy topological space, the projection $p_{j_0} : X \rightarrow X_{j_0}$ is fuzzy continuous. Accordingly $p_{j_0}^{-1}(u)$ and $p_{j_0}^{-1}(v)$ are fuzzy open disjoint sets of X containing x_r and y_s respectively. Hence (X, t) is also a fuzzy Hausdorff space.

1.5.6. DEFINITION ([88]):

Let (X, t) be an fts and α be a non-empty fuzzy set in X . Then α is called dense if $\bar{\alpha} = 1$. Note that a fuzzy set α in X is dense if each non-zero fuzzy open set intersects α . We need the following definition and result for our subsequent work.

1.5.7. DEFINITION ([98]):

Let (X, t) and (Y, s) be two fts with equivalence relation R and S defined on them respectively. A map $f : (X, t) \rightarrow (Y, s)$ is called relation-preserving if $xRx' \Rightarrow f(x)Sf(x')$, for each $x, x' \in X$.

1.5.8. THEOREM ([98]):

Let (X, t) and (Y, s) be two fuzzy topological spaces with equivalence relations R and S respectively defined on them and let $f : (X, t) \rightarrow (Y, s)$ be a relation-preserving fuzzy continuous map. Then there is a unique fuzzy continuous map $f_* : X/R \rightarrow Y/S$ such that $f_* \circ p_X = p_Y \circ f$ i.e. the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p_X \downarrow & & \downarrow p_Y \\ X/R & \xrightarrow{f_*} & Y/S \end{array} \quad (1)$$

is commutative. Furthermore if f is a fuzzy identification, then so also is f_* .

1.5.9. DEFINITION:

Let G be a set, which is closed under binary operations. Then a fuzzy set in G is a function $\mu : G \rightarrow I = [0,1]$ from G into $I=[0, 1]$.

1.5.10. DEFINITION:

Let G be a group. A fuzzy groupoid μ in G will be called a fuzzy group in G if $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$.

CHAPTER-2

Constructions of Some Universal Objects In Fuzzy Topological Spaces

2.1. INTRODUCTION:

In general topology, there are many interesting universal objects, such as products, equalizers, pullbacks and their duals, namely coproducts, coequalizers and pushouts. Only products and coproducts have been studied in fuzzy setting by Hutton [50], Mashhour et al [73], Pu and Liu [88, 89] and Wong [122]. Behera [17] has introduced the concepts of fuzzy equalizers, fuzzy pullbacks and their duals, namely, fuzzy coequalizers, fuzzy pushouts for fuzzy topological spaces and has tried to get the result about all such universal objects. In this paper, we study and prove some results on these universal objects. We use several definitions and results relevant to this article, without record, as they are standard by now.

2.2. FUZZY PRODUCTS:

In fact, the concepts of fuzzy products have already appeared in the research work of Hutton [49], Mashhour et al [73], Pu and Liu [88,89] and Wong [122]. The following definition of fuzzy product is given by Behera [17] in the form of a universal property:

2.2.1. DEFINITION [127]:

Let $\{X_i, i \in I\}$ be a family of fuzzy topological spaces. Then a fuzzy product (X, α_i) of the family $\{X_i : i \in I\}$ is a fuzzy topological space X together with a family of

fuzzy continuous (f-continuous) maps $\alpha_i : X \rightarrow X_i$ having the universal property: given any fts Y and a family $\{\varphi_i : Y \rightarrow X_i\}$ of f-continuous maps, there exists a unique f-continuous map $\theta : Y \rightarrow X$ such that

$$\alpha_i \circ \theta = \varphi_i, \text{ for each } i \in I.$$

In other words, the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha_i} & X_i \\ & \swarrow \theta & \uparrow \varphi_i \\ & Y & \end{array} \quad (2)$$

is commutative for each $i \in I$.

2.2.2. REMARKS:

- (i) By abuse of language, X is called a product of the family of objects $\{X_i : i \in I\}$.
- (ii) By abuse of notation, we often write a product of the family of objects $\{X_i : i \in I\}$ by $\prod X_i$. We observe that $\prod X_i$ is just a fuzzy topological space.

2.2.3. PROPOSITION [17]:

If (X, α_i) is a fuzzy product of the family $\{X_i : i \in I\}$ of fuzzy topological spaces and $\varphi, \psi : Y \rightarrow X$, Y being an fts, are f-continuous maps such that $\alpha_i \circ \varphi = \alpha_i \circ \psi$ for each $i \in I$, then $\varphi = \psi$.

PROOF:

Since (X, α_i) is a fuzzy product of the family $\{X_i : i \in I\}$ of fuzzy topological spaces, then $\alpha_i : X \rightarrow X_i$ is a family of f-continuous maps and having universal property: given any fts Y and a family $\{\varphi_i : Y \rightarrow X_i\}$ of f-continuous maps, there exists a unique f-continuous map $\varphi : Y \rightarrow X$ such that $\alpha_i \circ \varphi = \varphi_i$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha_i} & X_i \\
 \nearrow \varphi & & \uparrow \varphi_i \\
 & Y &
 \end{array} \quad (3)$$

is commutative. Also, since $\psi: Y \rightarrow X$ is an f -continuous map, then we have $\alpha_i \circ \psi = \varphi_i$.

Therefore, $\alpha_i \circ \varphi = \alpha_i \circ \psi$

$$\Rightarrow (\alpha_i \circ \varphi)(y) = (\alpha_i \circ \psi)(y)$$

$$\Rightarrow \alpha_i(\varphi(y)) = \alpha_i(\psi(y))$$

$$\Rightarrow \varphi(y) = \psi(y), \quad \text{by left cancellation law for fuzzy products.}$$

$$\Rightarrow \varphi = \psi, \quad \text{by right cancellation law for fuzzy products.}$$

$\therefore \varphi = \psi$, this completes the proof.

2.2.4. PROPOSITION:

Consider a fuzzy product (X, α_i) of the family $\{X_i: i=1,2\}$ of fuzzy topological spaces with injections u_1, u_2 and projections p_1, p_2 with zero. Then u_1 is the kernel of p_2 .

PROOF:

Let $X = X_1 \times X_2$. Then the injections and projections are written as $u_i: X_i \rightarrow X$ and $p_i: X \rightarrow X_i$, for $i=1,2$. Since (X, α_i) is a fuzzy product, then $\{\alpha_i: X \rightarrow X_i\}$ is a family of f -continuous maps and having universal property: given any f ts Y and a family $\{\varphi_i: Y \rightarrow X_i, i=1,2\}$ of f -continuous maps, there exists a unique f -continuous map $\alpha: Y \rightarrow X$ such that $p_i \circ \alpha = \varphi_i$ and $u_i \circ \varphi_i$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xleftarrow{p_i, \alpha_i} & X_i \\
 \nearrow \alpha & & \uparrow \varphi_i \\
 & Y &
 \end{array} \quad (4)$$

is commutative for $i=1,2$.

Also, since p_1 and p_2 are projections with zero, then $p_i \circ \alpha = 0$ i.e. $p_1 \circ \alpha = 0$ and $p_2 \circ \alpha = 0$.

Now, $u_i \circ \varphi_i = \alpha$

$$\Rightarrow u_1 \circ \varphi_1 = \alpha, \text{ for } i = 1$$

$$\Rightarrow p_2(u_1 \circ \varphi_1) = p_2 \circ \alpha = 0$$

$$\Rightarrow p_2 \circ u_1 = 0, \text{ as } u_1 \text{ is injection.}$$

$$\Rightarrow u_1 \in \text{Kernel of } p_2.$$

Therefore, u_1 is the kernel of p_2 . The proof is now complete.

2.3. FUZZY COPRODUCT:

The dual notion of fuzzy products of the family $\{X_i: i \in I\}$ of fuzzy topological spaces is called the fuzzy coproduct of the family $\{X_i: i \in I\}$. The concept of fuzzy coproduct has already been appeared in the research work of Wong [116] and Pu and Liu [88,89]. The following definition of fuzzy coproduct is given by Behera [17] in the form of a couniversal property:

2.3.1. DEFINITION [17]:

Let $\{X_i: i \in I\}$ be a family of fuzzy topological spaces. Then a fuzzy coproduct (X, α_i) of the family $\{X_i: i \in I\}$ is a fuzzy topological space X together with a family of f -continuous maps $\{\alpha_i: X_i \rightarrow X\}$ having couniversal property: given any fts Y and a family $\{\varphi_i: X_i \rightarrow Y, i \in I\}$ of f -continuous maps, there exists a unique f -continuous map $\theta: X \rightarrow Y$ such that $\theta \circ \alpha_i = \varphi_i$ for each $i \in I$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\theta} & Y \\
 \alpha_i \swarrow & & \nearrow \varphi_i \\
 & X_i &
 \end{array}
 \quad (5)$$

is commutative, for each $i \in I$.

2.3.2. REMARK:

(i) By abuse of language, X is called the fuzzy coproduct of the family of objects

$$\{X_i: i \in I\}.$$

(ii) By abuse of notation, we often write a fuzzy coproduct of the family of objects

$$\{X_i: i \in I\} \text{ by } \coprod X_i. \text{ We observe that } \coprod X_i \text{ is just a fuzzy topological space.}$$

2.3.3. PROPOSITION [17]:

If $\{X, \alpha_i\}$ is a fuzzy coproduct of the family $\{X_i: i \in I\}$ of fuzzy topological spaces and $\varphi, \psi: X \rightarrow Y$, Y being an fts, are f -continuous maps such that $\varphi \circ \alpha_i = \psi \circ \alpha_i$ for each $i \in I$, then $\varphi = \psi$.

PROOF:

Since (X, α_i) is a fuzzy coproduct of the family $\{X_i: i \in I\}$ of fuzzy topological spaces, then $\{\alpha_i: X_i \rightarrow X\}$ are family of f -continuous maps having couniversal property: given any fts Y and a family $\{\varphi_i: X_i \rightarrow Y\}$ of f -continuous maps, there exists a unique f -continuous map $\varphi: X \rightarrow Y$ such that $\varphi \circ \alpha_i = \varphi_i$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xleftarrow{\alpha_i} & X_i \\
 \searrow \varphi & & \downarrow \varphi_i \\
 & & Y
 \end{array}
 \quad (6)$$

is commutative.

Also, since $\psi: X \rightarrow Y$ is a f -continuous map, then $\psi \circ \alpha_i = \varphi_i$.

Now, we have

$$\varphi \circ \alpha_i = \psi \circ \alpha_i$$

$$\Rightarrow (\varphi \circ \alpha_i)(x_i) = (\psi \circ \alpha_i)(x_i), \text{ for each } x_i \in X_i.$$

$$\Rightarrow \varphi(\alpha_i(x_i)) = \psi(\alpha_i(x_i))$$

$$\Rightarrow \varphi = \psi, \text{ by left cancellation law for fuzzy co-products.}$$

This completes the proof.

2.4. FUZZY EQUALIZER:

We now recall another universal construction for fuzzy topological spaces that will be important to us, namely, the fuzzy equalizer from Behera[17].

2.4.1. DEFINITION [17]:

Let X and Y be two fuzzy topological spaces and $\alpha, \beta : X \rightarrow Y$ be a parallel pair of f -continuous maps between X and Y . An equalizer (E, λ) of α and β is an f ts together with an f -continuous map $\lambda : E \rightarrow X$ with the following properties:

$$(i) \quad \alpha \circ \lambda = \beta \circ \lambda$$

(ii) (Universal property) for any f ts Z and f -continuous map $\varphi : Z \rightarrow X$ satisfying

$\alpha \circ \varphi = \beta \circ \varphi$, there exists a unique f -continuous map $\mu : Z \rightarrow E$ such that $\varphi = \lambda \circ \mu$ i.e. the

following diagram

$$\begin{array}{ccccc} E & \xrightarrow{\lambda} & X & \xrightarrow{\alpha} & Y \\ & \nearrow \mu & \uparrow \varphi & & \\ & & Z & & \end{array} \quad (7)$$

is commutative.

2.4.2. PROPOSITION:

Let (E, λ) be an equalizer of the parallel pair of f -continuous maps $\alpha, \beta: X \rightarrow Y$. Then a f -continuous map $\lambda: E \rightarrow X$ is an injective map. Moreover, any two equalizers of α and β are isomorphic subobject of X .

PROOF:

Let X and Y be two fuzzy topological spaces and $\alpha, \beta: X \rightarrow Y$ be the parallel pair of f -continuous maps. Since (E, λ) is an equalizer together with a f -continuous map $\lambda: E \rightarrow X$, then we have, $\alpha \circ \lambda = \beta \circ \lambda$. Also, for any given f ts Z and f -continuous map $\varphi: Z \rightarrow X$ satisfying $\alpha \circ \varphi = \beta \circ \varphi$, there exists a unique f -continuous map $\mu: Z \rightarrow E$ such that $\varphi = \lambda \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 E & \xrightarrow{\lambda} & X \xrightarrow{\alpha, \beta} Y \\
 \swarrow \mu & & \uparrow \varphi \\
 & & Z
 \end{array} \tag{8}$$

is commutative .

Suppose $\mu_1, \mu_2: Z \rightarrow E$ are f -continuous maps with $\lambda \circ \mu_1 = \lambda \circ \mu_2 = \varphi$. Then $\alpha \circ \varphi = \alpha \circ \lambda \circ \mu_1 = \beta \circ \lambda \circ \mu_2 = \beta \circ \varphi \Rightarrow \alpha \circ \varphi = \beta \circ \varphi$, so there is a unique f -continuous map $\mu: Z \rightarrow E$ with $\varphi = \lambda \circ \mu$. But both μ_1 and μ_2 are such f -continuous maps, so $\mu_1 = \mu_2$. This implies that λ is an injective map.

2nd part:

We know, if (E, λ) and (E', λ') are both equalizers of α and β , then there is a unique isomorphism $\mu: E \rightarrow E'$ for which $\lambda' \circ \mu = \lambda$.

Now, suppose (E', λ') is also an equalizer of α and β . Then we have an f -continuous map

$\mu': E' \rightarrow E$ such that $\lambda \circ \mu' = \lambda'$.

$$\begin{aligned}
 \text{Now,} \quad \lambda \circ \mu \circ \mu' &= (\lambda' \circ \mu) \circ \mu \circ \mu' \\
 &= (\lambda' \circ \mu) \circ I_{E'} \\
 &= \lambda' \circ \mu \\
 &= (\lambda \circ \mu') \circ \mu \\
 &= \lambda \circ (\mu' \circ \mu) \\
 &= \lambda \circ I_E \\
 &= \lambda
 \end{aligned}$$

$\Rightarrow \mu \circ \mu' = I_{E'}$, by left cancellation law.

Similarly, we can show that $\mu' \circ \mu = I_E$ and μ is an isomorphism with inverse μ' and hence any two equalizers of α and β are isomorphic. This completes the proof.

2.5. FUZZY COEQUALIZER:

Already we introduced the concepts of fuzzy equalizer. The dual notion of fuzzy equalizer is called the fuzzy coequalizer.

2.5.1. DEFINITION [17]:

Let X and Y be two fuzzy topological spaces and $\alpha, \beta : X \rightarrow Y$ be a parallel pair of f -continuous maps between X and Y . Then a fuzzy coequalizer (Q, δ) of α and β is an f ts together with an f -continuous map $\delta: Y \rightarrow Q$ with the following properties:

(i) $\delta \circ \alpha = \delta \circ \beta$

(ii) (Couniversal property) for any fts Z and an f -continuous map $\varphi: Y \rightarrow Z$ satisfying $\varphi \circ \alpha = \varphi \circ \beta$, there exists a unique f -continuous map $\mu: Q \rightarrow Z$ such that $\varphi = \mu \circ \delta$ i.e. the following diagram

$$\begin{array}{ccccc}
 Q & \xleftarrow{\delta} & Y & \xleftarrow{\alpha, \beta} & X \\
 \searrow \mu & & \downarrow \varphi & & \\
 & & Z & &
 \end{array} \quad (8)$$

is commutative.

2.5.2. THEOREM:

Show that any fuzzy coequalizer is surjective.

Proof:

Let X and Y be two fuzzy topological spaces and $\alpha, \beta : X \rightarrow Y$ be a parallel pair of f -continuous maps between X and Y . Also, let (Q, δ) be a fuzzy coequalizer of α and β together with a f -continuous map $\delta: Y \rightarrow Q$ such that $\delta \circ \alpha = \delta \circ \beta$. For any fts Z and an f -continuous map $\varphi: Y \rightarrow Z$ satisfying $\varphi \circ \alpha = \varphi \circ \beta$, there exists a unique f -continuous map $\mu: Q \rightarrow Z$ such that $\varphi = \mu \circ \delta$. Let $q \in Q$ and $z \in Z$ such that $\mu(q) = z$.

Also, for every $y \in Y$, $\varphi(y) = z$ and then $\varphi(y) = \mu(q)$.

Thus we have $\varphi = \mu \circ \delta$

$$\Rightarrow \varphi(y) = (\mu \circ \delta)(y), \text{ for each } y \in Y.$$

$$\Rightarrow \varphi(y) = \mu(\delta(y))$$

$$\Rightarrow \mu(q) = \mu(\delta(y))$$

$\Rightarrow q = \delta(y)$, by left cancellation law. This implies that δ is surjective and hence a fuzzy coequalizer is surjective. This completes the proof.

2.6. FUZZY PULLBACK:

We shall introduce another universal construction for fuzzy topological spaces in this section, namely, the fuzzy pullback.

2.6.1. DEFINITION [17]:

Let $\alpha: X \rightarrow Z$ and $\beta: Y \rightarrow Z$ be a pair of f -continuous maps. Then the triplet (P, λ, μ) is called the fuzzy pullback of α and β if

(i) P is an fts and $\lambda: P \rightarrow X, \mu: P \rightarrow Y$ are the f -continuous maps such that $\alpha \circ \lambda = \beta \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\lambda} & X \\
 \mu \downarrow & & \downarrow \alpha \\
 Y & \xrightarrow{\beta} & Z
 \end{array} \quad (9)$$

is commutative.

(ii) (Universal property) for any fts W and f -continuous maps $\phi: W \rightarrow X, \psi: W \rightarrow Y$ satisfying $\alpha \circ \phi = \beta \circ \psi$, there exists a unique f -continuous map

$\theta: W \rightarrow P$ such that $\phi = \lambda \circ \theta, \psi = \mu \circ \theta$ i.e. the following diagram

$$\begin{array}{c}
 W \\
 \begin{array}{ccc}
 & \nearrow \varphi & \\
 \theta \searrow & & \\
 \psi \searrow & P \xrightarrow{\lambda} X & \\
 & \downarrow \mu \quad \downarrow \alpha & \\
 & Y \xrightarrow{\beta} Z &
 \end{array}
 \end{array}
 \quad (10)$$

is commutative.

2.6.2. THEOREM:

In the diagram (9), if α is surjective, then so is μ (i.e. one can say that a fuzzy pullback of a surjective is a surjective).

PROOF:

Let $\alpha: X \rightarrow Z$ and $\beta: Y \rightarrow Z$ be a pair of f-continuous maps and the triplet (P, λ, μ) be a fuzzy pullback of α and β . Then P is an fts and $\lambda: P \rightarrow X, \mu: P \rightarrow Y$ are the f-continuous maps such that $\alpha \circ \lambda = \beta \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\lambda} & X \\
 \downarrow \mu & & \downarrow \alpha \\
 Y & \xrightarrow{\beta} & Z
 \end{array}
 \quad (11)$$

is commutative.

According to our theorem, α is a surjective. Then for every $x \in X$, $z \in Z$, we have $\alpha(x) = z$. Let $y \in Y$. Then for every $z \in Z$, $\beta(y) = z$. Thus $\alpha(x) = \beta(y)$.

Now, $\alpha \circ \lambda = \beta \circ \mu$

$$\Rightarrow (\alpha \circ \lambda)(p) = (\beta \circ \mu)(p) \text{ , for each } p \in P.$$

$$\Rightarrow \alpha(\lambda(p)) = \beta(\mu(p))$$

$$\Rightarrow \alpha(x) = \beta(\mu(p)) \text{ , for each } p \in P, x = \lambda(p).$$

$$\Rightarrow \beta(y) = \beta(\mu(p))$$

$$\Rightarrow y = \mu(p) \text{ , by left cancellation law. This implies that } \mu \text{ is surjective}$$

and we can say that a fuzzy pullback of a surjective is surjective. Hence the proof is complete.

2.6.3. EXAMPLE

If $\alpha: X \rightarrow Z$ and $\beta: Y \rightarrow Z$ is a pair of f -continuous maps, then the fuzzy pullback

$$\begin{array}{ccc} P & \xrightarrow{\lambda} & X \\ \mu \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\beta} & Z \end{array} \quad (12)$$

is constructed by setting

$P = \{ (x,y) : \alpha(x) = \beta(y) \}$, with $\lambda(x,y) = x$ and $\mu(x,y) = y$. This example is a characteristic of fuzzy pullback with fuzzy products and P is a subobject of $X \times Y$.

2.6.4. PROPOSITION:

If the diagram (1.12) is a fuzzy pullback with fuzzy products, then

$\langle \lambda, \mu \rangle : P \rightarrow X \times Y$ is monic.

PROOF:

Let $\alpha: X \rightarrow Z$ and $\beta: Y \rightarrow Z$ be a pair of f -continuous maps. Since the above diagram (1.12) is a fuzzy pullback with fuzzy products, then the fuzzy pullback can be constructed by setting $P = \{ (x,y) : \alpha(x) = \beta(y) \}$, with $\lambda(x,y) = x$ and $\mu(x,y) = y$.

Now, $\langle \lambda(x,y), \mu(x,y) \rangle = (x,y)$

$\Rightarrow \langle \lambda, \mu \rangle(x,y) = (x,y)$, this relation means that $\langle \lambda, \mu \rangle : P \rightarrow X \times Y$ is monic. This completes the proof.

2.7. FUZZY PUSHOUT:

The dual notion of fuzzy pullback is that of fuzzy pushout and here we use the couniversal property.

2.7.1. DEFINITION [17]:

Let $\alpha: X \rightarrow Y$ and $\beta: X \rightarrow Z$ be a pair of f-continuous maps. Then the triplet (Q, λ, μ) is called the fuzzy pushout of α and β if

(i) Q is an fts and $\lambda: Y \rightarrow Q$, $\mu: Z \rightarrow Q$ are f-continuous maps such that $\lambda \circ \alpha = \mu \circ \beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & Y \\
 \beta \downarrow & & \downarrow \lambda \\
 Z & \xrightarrow{\mu} & Q
 \end{array} \quad (13)$$

is commutative .

(ii) (Couniversal property) for any fts W and f-continuous maps $\varphi: Y \rightarrow W$,

$\psi: Z \rightarrow W$ satisfying $\varphi \circ \alpha = \psi \circ \beta$, there exists a unique f-continuous map $\theta: Q \rightarrow W$ such that $\varphi = \theta \circ \lambda$, $\psi = \theta \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & Y \\
 \beta \downarrow & & \downarrow \lambda \\
 Z & \xrightarrow{\mu} & Q
 \end{array}
 \begin{array}{c}
 \searrow \varphi \\
 \downarrow \theta \\
 \swarrow \psi
 \end{array}
 \begin{array}{c}
 \\
 \\
 \end{array}
 W \quad (14)$$

is commutative.

2.8. SOME IMPLICATIONS:

We now show that fuzzy pullbacks and fuzzy equalizers are related and fuzzy pushouts and fuzzy coequalizers are also related.

2.8.1. THEOREM [17]:

The following are equivalent:

- (a) The fuzzy equalizer of any two f -continuous maps $\alpha, \beta : X \rightarrow Y$ exist.
- (b) The fuzzy pullback of any two f -continuous maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are exist.

PROOF:

Firstly, suppose that (a) is true. Then there is an fts E and a f -continuous map $\delta : E \rightarrow X$ such that $\alpha \circ \delta = \beta \circ \delta : E \rightarrow Y$. Consider any two f -continuous maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ and an fts E together with $\delta : E \rightarrow X$ and

$\lambda = \alpha \circ \delta = \beta \circ \delta : E \rightarrow Y$. Then we have $f \circ \delta = g \circ \lambda$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Z \\
 \delta \uparrow & \alpha & \beta \downarrow \\
 E & \xrightarrow{\lambda} & Y
 \end{array}$$

(15)

is commutative. This implies that the triplet (E, δ, λ) is a fuzzy pullback and hence $(a) \Rightarrow (b)$.

Next, let (b) is true. Then for any fts P and f -continuous maps $\lambda: P \rightarrow X$ and $\mu: P \rightarrow Y$ such that $f \circ \lambda = g \circ \mu: P \rightarrow Z$.

Consider any two f -continuous maps $\alpha, \beta: X \rightarrow Y$ and an fts P . Then there exists a f -continuous map $\lambda: P \rightarrow X$ such that $\alpha \circ \lambda = \beta \circ \lambda$. This implies that (P, λ) is a fuzzy equalizer. Thus $(b) \Rightarrow (a)$. This completes the proof.

The above theorem can be dualized as follows:

2.8.2. THEOREM [17]:

The following are equivalent:

- (a) The fuzzy coequalizer of any two f -continuous maps $\alpha, \beta: X \rightarrow Y$ exist.
- (b) The fuzzy pushout of any two f -continuous maps $f: X \rightarrow Y$ and $g: X \rightarrow Z$ exist.

PROOF:

Firstly, suppose that (a) is true. Then for any fts Q and a f -continuous map

$\delta: Y \rightarrow Q$ such that $\delta \circ \alpha = \delta \circ \beta: X \rightarrow Q$. Let $f: X \rightarrow Y$ and $g: X \rightarrow Z$ be two f -continuous maps. For an fts Q and f -continuous maps $\delta: Y \rightarrow Q$ and $\lambda: Z \rightarrow Q$ such that $\delta \circ f = \lambda \circ g: X \rightarrow Q$. This implies that (Q, δ, λ) is a fuzzy pushout. Thus $(a) \Rightarrow (b)$.

Next, let (b) is true. Then for any fts Q and f -continuous maps $\delta: Y \rightarrow Q$ and $\lambda: Z \rightarrow Q$ such that $\delta \circ f = \lambda \circ g: X \rightarrow Q$.

Let $\alpha, \beta: X \rightarrow Y$ be two f -continuous maps and Q be an fts. Then there exists an f -continuous map $\delta: Y \rightarrow Q$ such that $\delta \circ \alpha = \delta \circ \beta$. This implies that (Q, δ) is a fuzzy coequalizer. Therefore (b) $\Rightarrow$ (a). This completes the proof.

CHAPTER-3

Inverse and Direct Limits with Fuzzy Topological Spaces

3.1. INTRODUCTION:

In this section, we introduce the concepts of inverse system and direct system for fuzzy topological spaces and study inverse limits and direct limits on such spaces. We also introduce the explicit constructions of these limits. We shall use many important concepts of fuzzy topology such as product fuzzy topology, quotient fuzzy topology, fuzzy continuity etc. in the study of inverse limits and direct limits for fuzzy topological spaces.

3.2. INVERSE LIMITS FOR FUZZY TOPOLOGICAL SPACES:

In this section, we introduce the concepts of an inverse system and the inverse limits of this system for fuzzy topological spaces.

3.2.1. DEFINITION ([17]):

A directed set L is a partially ordered set $(L, \leq)$ such that for each $\alpha, \beta \in L$, there exists $\gamma \in L$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

3.2.2. DEFINITION ([98]):

An inverse system $\{X_\alpha, f_{\alpha\beta}: \alpha, \beta \in L\}$ of fuzzy topological spaces consists of

- (i) a collection of fuzzy topological spaces $\{X_\alpha : \alpha \in L\}$ indexed by the directed set L .

(ii) a collection of fuzzy continuous functions $\{f_{\alpha\beta}: X_\beta \rightarrow X_\alpha\}$ one for each pair (α, β) of indices satisfying $\alpha \leq \beta$ such that

(a) $f_{\alpha\alpha} = \text{Id}_{X_\alpha} : X_\alpha \rightarrow X_\alpha$ for each $\alpha \in L$.

(b) for $\alpha \leq \beta \leq \gamma$, $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$ i.e. the following diagram .

$$\begin{array}{ccc}
 X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \\
 & \nwarrow f_{\beta\gamma} & \uparrow f_{\alpha\gamma} \\
 & & X_\gamma
 \end{array} \quad (16)$$

is commutative.

3.2.3. DEFINITION ([98]):

An inverse limit of an inverse system $\{X_\alpha, f_{\alpha\beta}: \alpha, \beta \in L\}$ of fuzzy topological spaces is an fts X together with f -continuous maps $\varphi_\alpha: X \rightarrow X_\alpha, \alpha \in L$ such that

(a) for $\alpha \leq \beta$, $\varphi_\alpha = f_{\alpha\beta} \circ \varphi_\beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \\
 & \nwarrow \varphi_\beta & \uparrow \varphi_\alpha \\
 & & X
 \end{array} \quad (17)$$

is commutative .

(b) (Universal property) given any fts Y and fuzzy continuous maps $\psi_\alpha: Y \rightarrow X_\alpha$,

$\alpha \in L$ with $\psi_\alpha = f_{\alpha\beta} \circ \psi_\beta$ for $\alpha \leq \beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \\
 & \nwarrow \psi_\beta & \uparrow \psi_\alpha \\
 & & Y
 \end{array} \quad (18)$$

is commutative , then there exists a unique f-continuous map $\mu : Y \rightarrow X$ such that $\psi_\alpha = \varphi_\alpha \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi_\alpha} & X_\alpha \\
 \swarrow \mu & & \uparrow \psi_\alpha \\
 & Y &
 \end{array} \quad (19)$$

is commutative .

The inverse limit of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces is denoted by $\varprojlim X_\alpha$ and is defined by $\{X, \varphi_\alpha : \alpha \in L\}$.

3.2.4. LEMMA:

If in an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces, all the f-continuous functions $f_{\alpha\beta} : X_\beta \rightarrow X_\alpha$ are one-one, then all the f-continuous maps $\varphi_\alpha : X \rightarrow X_\alpha$ are also one-one.

PROOF:

Let $x, y \in X$, where $x = (x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \dots)$, $y = (y_{\alpha_1}, y_{\alpha_2}, y_{\alpha_3}, \dots)$ and $\varphi_\alpha(x) = \varphi_\alpha(y)$. Now, let $\beta \in L$, we want to prove that $\varphi_\beta(x) = \varphi_\beta(y)$. Let $\alpha, \beta \leq \gamma$. Then $\varphi_\gamma f_{\alpha\gamma}(x_\gamma) = \varphi_\gamma f_{\alpha\gamma}(y_\gamma)$, so $\varphi_\gamma(x) = \varphi_\gamma(y)$. Thus, $\varphi_\beta(x) = \varphi_\gamma f_{\beta\gamma}(y) = \varphi_\beta(y)$. Therefore φ_α is one-one. This completes the proof of the lemma.

3.2.5. PROPOSITION ([98]):

The inverse limits $\{X, \varphi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces is unique up to fuzzy homeomorphism .

PROOF:

Let $\{X, \varphi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ be two inverse limits of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces. Since $\{X, \varphi_\alpha : \alpha \in L\}$ is an inverse limit of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces, then there exists a f -continuous map $\mu : Y \rightarrow X$ such that $\psi_\alpha = \varphi_\alpha \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi_\alpha} & X_\alpha \\
 \swarrow \mu & & \uparrow \psi_\alpha \\
 & Y &
 \end{array} \quad (20)$$

is commutative.

Similarly, since $\{Y, \psi_\alpha : \alpha \in L\}$ is an inverse limit of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces, then there exists a f -continuous map $\eta : X \rightarrow Y$ such that $\psi_\alpha \circ \eta = \varphi_\alpha$ i.e. the following diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{\psi_\alpha} & X_\alpha \\
 \swarrow \mu & & \uparrow \varphi_\alpha \\
 & X &
 \end{array} \quad (21)$$

is commutative.

Now, consider the diagrams

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi_\alpha} & X_\alpha \\
 \swarrow \mu \circ \eta & & \uparrow \varphi_\alpha \\
 & X &
 \end{array} \quad (22)$$

and

$$\begin{array}{ccc}
 Y & \xrightarrow{\psi_\alpha} & X_\alpha \\
 \eta \circ \mu \swarrow & & \uparrow \psi_\alpha \\
 & Y &
 \end{array}
 \quad (23)$$

By the uniqueness condition from the definition of inverse limits of an inverse system, we conclude that $\mu \circ \eta = \text{Id}_X$ and $\eta \circ \mu = \text{Id}_Y$. Hence the inverse limits $\{X, \varphi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces are fuzzy homeomorphic and is therefore unique. This completes the proof.

3.2.6. INVERSE LIMIT SYSTEM FOR THE SEQUENCE OF FUZZY TOPOLOGICAL SPACES:

The concepts of inverse limit system in this section are of recent importance in fuzzy topology. We begin with a special case of an inverse limit system; we study that special case and then indicate how the general concept is defined.

3.2.7. DEFINITION:

Let $X_0, X_1, X_2, \dots$ be a countable collection of fuzzy topological spaces and suppose that for each $n > 0$, there is a f -continuous map $f_n : X_n \rightarrow X_{n-1}$. The sequence of fuzzy topological spaces and mappings $\{X_n, f_n\}$ is called an inverse limit sequence and may be represented by means of the diagram

$$\dots \rightarrow X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0.$$

Clearly, if $n > m$, there is a f -continuous mapping $f_{n,m} : X_n \rightarrow X_m$ given by the

$$\text{composition } f_{n,m} = f_{m+1} \circ f_{m+2} \circ \dots \circ f_{n-1} \circ f_n.$$

Consider a sequence of points $(x_0, x_1, \dots, x_n, \dots)$ such that $x_n = f_{n+1}(x_{n+1})$ for all $n \geq 0$.

Consider a map $\varphi : n \rightarrow \prod_{n=0}^{\infty} X_n$ by $\varphi(n) = x_n$, where n is a non-negative integer.

Then $(x_0, x_1, \dots, x_n, \dots)$ is a point of the fuzzy product $\prod_{n=0}^{\infty} X_n$. Therefore, the set of all

such sequences is in this way a subset of $\prod_{n=0}^{\infty} X_n$ and has a fuzzy topology as a

subspace. This fuzzy topological space is the inverse limit space of the sequence

$\{X_n, f_n\}$. We shall denote it by X_{∞} .

3.2.8. LEMMA:

If $\{X_n, f_n\}$ is an inverse limit sequence, if each f_n is an onto mapping and if $\{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$ is a set of points with x_{n_i} in X_{n_i} for $i=1, 2, 3, \dots$ and if $i < j$ such that $f_{n_j, n_i}(x_{n_j}) = x_{n_i}$ then there is a point in X_{∞} whose co-ordinate in X_{n_i} is x_{n_i} , $i=1, 2, 3, \dots$.

PROOF:

We prove the above lemma (3.2.8) for two cases. The first in which the set $\{x_{n_i}\}$ is infinite and in the second case, the set is finite.

The first case is easy. For arbitrary n , there is a least integer n_j with $n_j \geq n$, if $n_j = n$, set $x_n = x_{n_j}$; if $n_j > n$, define $x_n = f_{n_j, n}(x_{n_j})$. Clearly, the sequence $(x_0, x_1, \dots, x_n, \dots)$ so defined is a point of X_{∞} .

In the second case, there is a greatest n_i , say n_k . For $n < n_k$, we may define $x_n = f_{n_k, n}(x_{n_k})$. Now, suppose that for some $m \geq n_k$, we have already defined x_m . Since

f_{n+1} is onto, there is at least one point x in X_{m+1} such that $f_{m+1}(x) = x_m$. Choose any such point as x_{m+1} . Now, by induction the existence of our desired sequence follows.

3.2.9. REMARK:

In fact, if the mappings f_n are not onto in the above lemma (3.2.8), then the inverse limit spaces X_∞ may be empty.

PROOF:

Consider a sequence of countable discrete fuzzy topological space

$$X_n = \bigcup_{m=1}^{\infty} x_{n,m}, n=0, 1, 2, \dots$$

Define a f -continuous map $f_n : X_n \rightarrow X_{n-1}$ by $f_n(x_{n,m}) = x_{n-1,m+1}$. Then we have an inverse limit sequence. But if we begin with a point $x_{0,j}$ of X_0 and attempt to form a point of X_∞ , we can only construct the first j co-ordinates and then are forced to stop and X_∞ has no points in it at all. Therefore X_∞ is empty, if the mappings f_n are not onto.

3.2.10. EXPLICIT CONSTRUCTION OF THE FUZZY INVERSE LIMITS:

There is no guarantee of the existence of an inverse limit for an inverse system of fuzzy topological spaces in the definition of inverse limit of an inverse system which are given above. For this, we introduce the explicit construction in the following way:

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ be an inverse system of fuzzy topological spaces. Consider the fuzzy product $\prod_{\alpha \in L} X_\alpha = \{f : L \rightarrow \bigvee_{\alpha \in L} X_\alpha / f(\alpha) \in L\}$ and for each $\alpha \in L$.

Let π'_α be its projection on to the factor X_α i.e. $\pi'_\alpha : \prod_{\alpha \in L} X_\alpha \rightarrow X_\alpha$ be a map

such that $\pi'_\alpha(f) = f(\alpha)$. Now, for $f_{\alpha\beta} : X_\beta \rightarrow X_\alpha$ and

$\prod_{\alpha \in L} X_\alpha = \{f : L \rightarrow \bigvee_{\alpha \in L} X_\alpha / f(\alpha) \in L\}$, whenever $\alpha \leq \beta$ in L , we define

$X = \{f \in \bigvee_{\alpha \in L} X_\alpha : f_{\alpha\beta}(f(\beta)) = f(\alpha)\}$, whenever $\alpha \leq \beta$ in L .

Let $\pi_\alpha : X \rightarrow X_\alpha$ be the restriction of the projections $\pi'_\alpha : \prod_{\alpha \in L} X_\alpha \rightarrow X_\alpha$.

3.2.11. LEMMA:

If $\{X, \varphi_\alpha : \alpha \in L\}$ is the inverse limit of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces, then each projections

$\pi'_\alpha : \prod_{\alpha \in L} X_\alpha \rightarrow X_\alpha$ is continuous.

PROOF:

Let $\{X, \varphi_\alpha : \alpha \in L\}$ be an inverse limit of an inverse system

$\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces and $X = \prod_{\alpha \in L} X_\alpha$. Also, we have for an

inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces, $\{f_{\alpha\beta} : X_\beta \rightarrow X_\alpha\}$ is the collection of fuzzy continuous functions one for each pair (α, β) of indices

satisfying $\alpha \leq \beta$. Define $\pi'_\alpha : \prod_{\alpha \in L} X_\alpha \rightarrow X_\alpha$ by $\pi'_\alpha(f) = f(\alpha)$.

Now, consider a map $\pi'_\alpha : X \rightarrow X_\alpha$ such that $\pi'_\alpha = f_{\alpha\beta} \circ \pi'_\beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \\
 \pi'_\beta \swarrow & & \uparrow \pi'_\alpha \\
 & X &
 \end{array}
 \quad (24)$$

is commutative .

Given any fts Y and f -continuous maps $\psi_\alpha : Y \rightarrow X_\alpha$, $\alpha \in L$ with $\psi_\alpha = f_{\alpha\beta} \circ \psi_\beta$ for $\alpha \leq \beta$, there exists a unique f -continuous map $\mu : Y \rightarrow X$ such that $\psi_\alpha = \pi'_\alpha \circ \mu$.

Since $\{X, \varphi_\alpha : \alpha \in L\}$ is the inverse limit of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ and $\pi'_\alpha = \varphi_\alpha$, then from the above discussion we can say that $\{X, \pi'_\alpha : \alpha \in L\}$ is

a fuzzy product. By the definition of fuzzy product, we have $\{\pi'_\alpha : X \rightarrow X_\alpha\}$ is

a family of f -continuous maps. Therefore, each projection $\pi'_\alpha : \prod_{\alpha \in L} X_\alpha \rightarrow X_\alpha$ are

continuous. This completes the proof of the lemma.

3.2.12. LEMMA:

The inverse limit $\{X, \varphi_\alpha : \alpha \in L\}$ of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ is a closed subspace of the fuzzy product space $\prod_{\alpha \in L} X_\alpha$

PROOF:

Let $\{X_\alpha, \alpha \in L\}$ be a family of fuzzy topological spaces and $\{X, \varphi_\alpha : \alpha \in L\}$ is the inverse limit of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$. Then $\{f_{\alpha\beta} : X_\beta \rightarrow X_\alpha\}$ are the collection of fuzzy continuous functions with $\alpha \leq \beta$ in a directed set L such that $f_{\alpha\beta}(x_\beta) = x_\alpha$, for each $x_\beta \in X_\beta$, $x_\alpha \in X_\alpha$. Let $\{x_\alpha\}$ be a fuzzy topological space consisting of one point x_α from each fuzzy topological spaces X_α and satisfying the condition $f_{\alpha\beta}(x_\beta) = x_\alpha$ for $\alpha \leq \beta$ in a directed set L .

Let $X = \prod_{\alpha \in L} X_\alpha = \{f : L \rightarrow \bigvee_{\alpha \in L} X_\alpha \mid f(\alpha) \in X_\alpha\}$. Then such an fts $\{x_\alpha\}$ identified with the

point f in the fuzzy product spaces $\prod_{\alpha \in L} X_\alpha$ having coordinates $f(\alpha) = x_\alpha$. Hence the

collection of all such fts $\{x_\alpha\}$ constitutes a subspaces of the fuzzy product spaces

$\prod_{\alpha \in L} X_\alpha$ and this subspaces is the inverse limit of the inverse system

$\{X_\alpha, f_{\alpha\beta}: \alpha, \beta \in L\}$. Next, we have to show that this inverse limit is a fuzzy closed

subspace of the fuzzy product spaces $\prod_{\alpha \in L} X_\alpha$.

We know, every fuzzy singleton set is closed. Therefore, the fuzzy singleton set $\{x_\alpha\}$ is closed and hence the collections of all such fts are closed. Hence the inverse limit

$\{X, \varphi_\alpha: \alpha \in L\}$ of an inverse system $\{X_\alpha, f_{\alpha\beta}: \alpha, \beta \in L\}$ is a closed subspace of the fuzzy product space $\prod_{\alpha \in L} X_\alpha$. This completes the proof of the lemma.

[Remark: This theorem may not be true be T_0 - space]

3.2.13. THEOREM ([98]):

Show that $\{X, \pi_\alpha: \alpha \in L\} = \lim_{\leftarrow} \{X_\alpha, f_{\alpha\beta}: \alpha, \beta \in L\}$.

PROOF:

Consider a fuzzy product $\prod_{\alpha \in L} X_\alpha = \{f: L \rightarrow \bigvee_{\alpha \in L} X_\alpha \mid f(\alpha) \in X_\alpha\}$ of fuzzy

topological spaces $\{X_\alpha, \alpha \in L\}$ and

$X = \left\{ f \in \prod_{\alpha \in L} X_\alpha : f_{\alpha\beta}(f(\beta)) = f(\alpha), \text{ whenever } \alpha \leq \beta \text{ in } L \right\}$. Then for any $\alpha \leq \beta$ and

any $f \in X$, we have $f_{\alpha\beta}(\pi_\alpha(f)) = f_{\alpha\beta}(f(\beta)) = f(\alpha) = \pi_\alpha(f)$. Thus we have the

following commutative diagram:

$$\begin{array}{ccc}
 X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \\
 \pi_\beta \swarrow & & \nearrow \pi_\alpha \\
 & X &
 \end{array}
 \quad (25)$$

Next, consider an fts Y and a family of fuzzy continuous maps $\psi_\alpha : Y \rightarrow X_\alpha, \alpha \in L$

with $\psi_\alpha = f_{\alpha\beta} \circ \psi_\beta$ for $\alpha \leq \beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha \\
 \psi_\beta \swarrow & & \nearrow \psi_\alpha \\
 & Y &
 \end{array}
 \quad (26)$$

is commutative. Define $\mu : Y \rightarrow \prod_{\alpha \in L} X_\alpha$ by $(\mu(y_t))(\alpha) = \psi_\alpha(y_t)$ for any $y_t \in Y$ and

any $\alpha \in L$. Suppose that $\alpha \leq \beta$, then $f_{\alpha\beta}(\mu(y_t))(\beta) = f_{\alpha\beta}(\psi_\beta(y_t))$

$= \psi_\alpha(y_t) = (\mu(y_t))(\alpha)$ and this shows that $\mu(y_t) \in X$. Thus, we have a fuzzy

continuous map $\mu : Y \rightarrow X$ and μ is uniquely defined. Hence for any $y_t \in Y$,

$\pi_\alpha(\mu(y_t)) = (\mu(y_t))(\alpha) = \psi_\alpha(y_t), \alpha \in L$ and we have the following commutative

diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{\pi_\alpha} & X_\alpha \\
 \mu \swarrow & & \nearrow \psi_\alpha \\
 & Y &
 \end{array}
 \quad (27)$$

This completes the proof of the theorem.

3.2.14. EXAMPLE:

Let L be a partially ordered set and X be any fts. If each $X_\alpha = X$ and each $f_{\alpha\beta}$ is the identity map, then $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ is called the trivial inverse system. Then obviously the inverse limit of this system is X itself.

3.2.15. EXAMPLE:

Let L be a partially ordered set. Then the inverse limit $\{X, \varphi_\alpha : \alpha \in L\}$ of an inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ is the fuzzy product of the fuzzy topological spaces $\{X_\alpha : \alpha \in L\}$.

3.3. DIRECT LIMITS FOR FUZZY TOPOLOGICAL SPACES:

Now, we discuss the dual notion of the inverse limit for fuzzy topological spaces. The dual notion of the inverse limit is called the direct limit. In this section, we define direct limit on direct system of fuzzy topological spaces.

3.3.1. DEFINITION ([98]):

A direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces consists of

(a) a collection of fuzzy topological spaces $\{X_\alpha : \alpha \in L\}$ indexed by the directed set L .

(b) a collection of fuzzy continuous functions $\{f^{\alpha\beta} : X_\alpha \rightarrow X_\beta\}$ one for each pair (α, β)

of indices satisfying $\alpha \leq \beta$ such that

(i) $f^{\alpha\alpha} = Id_{X_\alpha} : X_\alpha \rightarrow X_\alpha$ for each $\alpha \in L$.

(ii) for $\alpha \leq \beta \leq \gamma$, $f^{\beta\gamma} \circ f^{\alpha\beta} = f^{\alpha\gamma}$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{f^{\alpha\beta}} & X_\beta \\
 & \searrow f^{\alpha\gamma} & \downarrow f^{\beta\gamma} \\
 & & X_\gamma
 \end{array}$$

(28)

is commutative.

3.3.2. DEFINITION ([98]):

A direct limit of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces, is a fuzzy topological space X together with f -continuous maps $\varphi_\alpha : X_\alpha \rightarrow X, \alpha \in L$ such that

(a) for $\alpha \leq \beta$, $\varphi_\alpha = \varphi_\beta \circ f^{\alpha\beta}$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{f^{\alpha\beta}} & X_\beta \\
 \searrow \varphi_\alpha & & \downarrow \varphi_\beta \\
 & & X
 \end{array} \quad (29)$$

is commutative .

(b) (Co-universal property) for any fts Y and f -continuous maps

$\psi_\alpha : X_\alpha \rightarrow Y, \alpha \in L$ with $\psi_\alpha = \psi_\beta \circ f^{\alpha\beta}$ for $\alpha \leq \beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{f^{\alpha\beta}} & X_\beta \\
 \searrow \psi_\alpha & & \downarrow \psi_\beta \\
 & & Y
 \end{array} \quad (30)$$

is commutative , then there exists a unique f -continuous map $\mu : X \rightarrow Y$ such that

$\psi_\alpha = \mu \circ \varphi_\alpha, \alpha \in L$ i.e. the following diagram

$$\begin{array}{ccc}
 & X & \xrightarrow{\mu} & Y \\
 \uparrow \varphi_\alpha & \nearrow \psi_\alpha & & \\
 X_\alpha & & &
 \end{array} \quad (31)$$

is commutative .

The direct limit of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces is denoted by $\lim_{\rightarrow} X_\alpha$ and is defined by $\{X, \varphi_\alpha : \alpha \in L\}$.

3.3.3. LEMMA:

If in a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces, all the f-continuous functions $f^{\alpha\beta} : X_\alpha \rightarrow X_\beta$ are one-one, then all the f-continuous maps $\varphi_\alpha : X_\alpha \rightarrow X$ are one-one.

PROOF:

Let $x_\alpha, y_\alpha \in X_\alpha$ and suppose that $\varphi_\alpha(x_\alpha) = \varphi_\alpha(y_\alpha)$ that is $[x_\alpha] = [y_\alpha]$. Then there exists $\alpha \leq \beta$ such that $f^{\alpha\beta}(x_\alpha) = f^{\alpha\beta}(y_\alpha)$, since $f^{\alpha\beta}$ are one-one. Thus we infer that $x_\alpha = y_\alpha$. Thus we see that φ_α is one-one. This implies the proof of the lemma.

3.3.4. PROPOSITION ([98]):

The direct limits $\{X, \varphi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces are unique up to fuzzy homeomorphism.

PROOF:

Let $\{X, \varphi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ are two direct limits of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces. Since $\{X, \varphi_\alpha : \alpha \in L\}$ is a direct limit, then by definition of a direct limit of a direct system, there exists a unique f-continuous map $\mu : X \rightarrow Y$ such that $\psi_\alpha = \mu \circ \varphi_\alpha, \alpha \in L$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\mu} & Y \\
 \uparrow \varphi_\alpha & \nearrow \psi_\alpha & \\
 X_\alpha & &
 \end{array} \quad (32)$$

is commutative .

Similarly, since $\{Y, \psi_\alpha : \alpha \in L\}$ is a direct limit, then there exists a unique f-

continuous map $\lambda : Y \rightarrow X$ such that $\lambda \circ \psi_\alpha = \varphi_\alpha$ i.e. the following diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{\lambda} & X \\
 \nwarrow \psi_\alpha & & \uparrow \varphi_\alpha \\
 & X_\alpha &
 \end{array} \quad (33)$$

is commutative .

Now, consider the diagrams

$$\begin{array}{ccc}
 X & \xleftarrow{\varphi_\alpha} & X_\alpha \\
 \nwarrow \lambda \circ \mu & & \downarrow \varphi_\alpha \\
 & X &
 \end{array} \quad (34)$$

and

$$\begin{array}{ccc}
 Y & \xleftarrow{\psi_\alpha} & X_\alpha \\
 \nwarrow \mu \circ \gamma & & \downarrow \psi_\alpha \\
 & Y &
 \end{array} \quad (35)$$

By the uniqueness condition in the definition of a direct limit of a direct system, we have $\lambda \circ \mu = \text{Id}_X$ and $\mu \circ \gamma = \text{Id}_Y$. Hence the direct limits $\{X, \varphi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ are fuzzy homeomorphic and therefore they are fuzzy homeomorphism.

This completes the proof of the proposition.

3.3.5. LEMMA:

If $\alpha \leq \beta$, $\alpha, \beta \in L$ and $\pi_\alpha : X_\alpha \rightarrow X$ then show that $\pi_\beta \circ f^{\alpha\beta} = \pi_\alpha$.

PROOF:

Let $\{X, \pi_\alpha : \alpha \in L\}$ be a direct limit of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces. Since $\{X, \pi_\alpha : \alpha \in L\}$ is a direct limit of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$, then we have $\pi_\beta \circ f^{\alpha\beta} = \pi_\alpha$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{f^{\alpha\beta}} & X_\beta \\
 \searrow \pi_\alpha & & \downarrow \pi_\beta \\
 & & X
 \end{array} \tag{36}$$

is commutative. This completes the proof of the lemma.

3.3.6. LEMMA:

If x is any element of X , then there is an α in L and an x_α in X_α such that

$$\pi_\alpha(x_\alpha) = x.$$

PROOF:

Let $\{X, \pi_\alpha : \alpha \in L\}$ be a direct limit of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces. Since $\{X, \pi_\alpha : \alpha \in L\}$ is a direct limit of a direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$, then we have $\pi_\alpha : X_\alpha \rightarrow X$ is a fuzzy continuous map. Let $x \in X$ and x_α be the any element of X_α . Then for each $\alpha \in L$, we have $\pi_\alpha(x_\alpha) = x$. This completes the proof.

3.4. EXPLICIT CONSTRUCTION OF THE FUZZY DIRECT LIMIT:

There is no guarantee of the existence of a direct limit for a direct system of fuzzy topological spaces in the definition of direct limit, which are given above. Now, we give an idea about explicit construction of the fuzzy direct limit as follows:

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ be a direct system of fuzzy topological spaces. Consider the disjoint union $\bigvee_{\alpha \in L} X_\alpha$ of fuzzy topological spaces. Define a relation $\sim$ on $\bigvee_{\alpha \in L} X_\alpha$ as follows: For $x_\alpha \in X_\alpha, x_\beta \in X_\beta$, we have $x_\alpha \sim x_\beta$ iff there exists $\gamma \in L$ such that $\alpha, \beta \leq \gamma$, since L is directed and $f^{\alpha\gamma}(x_\alpha) = f^{\beta\gamma}(x_\beta)$.

3.4.1. LEMMA ([98]):

Show that $\sim$ is an equivalence relation on $\bigvee_{\alpha \in L} X_\alpha$.

PROOF:

We prove $\sim$ is an equivalence relation on $\bigvee_{\alpha \in L} X_\alpha$ and we need to show that $\sim$ is

(a) Reflexive relation

(b) Symmetric relation and

(c) Transitive relation.

Now, for any $x_\alpha \in X_\alpha$, $x_\alpha \sim x_\alpha$, because $f^{\alpha\alpha} = \text{Id}_{X_\alpha}$ and $\alpha \leq \alpha$. Hence $\sim$ is a reflexive relation.

For $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$, let $x_\alpha \sim x_\beta$. Then there exists $\gamma \in L$ such that $\alpha, \beta \leq \gamma$ and $f^{\alpha\gamma}(x_\alpha) = f^{\beta\gamma}(x_\beta) \Rightarrow f^{\beta\gamma}(x_\beta) = f^{\alpha\gamma}(x_\alpha) \Rightarrow$ there exists $\gamma \in L$ such that $\beta, \alpha \leq \gamma \Rightarrow x_\beta \sim x_\alpha$.

This implies that $\sim$ is a symmetric relation.

Finally, let $x_\alpha \sim x_\beta$. Then there exists $\varepsilon \in L$ such that $\alpha, \beta \leq \varepsilon$ and

$f^{\alpha\varepsilon}(x_\alpha) = f^{\beta\varepsilon}(x_\beta)$. Again, let $x_\beta \sim x_\gamma$. Then there exists $\delta \in L$ and

$f^{\beta\delta}(x_\beta) = f^{\gamma\delta}(x_\gamma)$. Choose η such that $\varepsilon, \delta \leq \eta$. Then we have

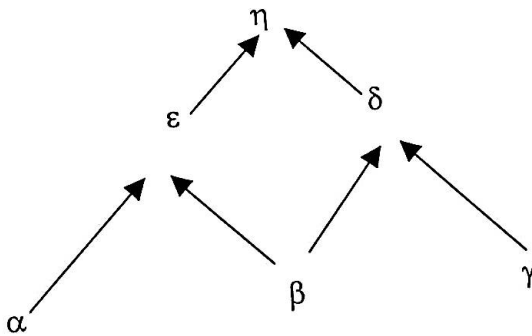
$$\begin{aligned} f^{\alpha\eta}(x_\alpha) &= (f^{\varepsilon\eta} \circ f^{\alpha\varepsilon})(x_\alpha) = f^{\varepsilon\eta}(f^{\alpha\varepsilon}(x_\alpha)) = f^{\varepsilon\eta}(f^{\beta\varepsilon}(x_\beta)) = f^{\beta\eta}(x_\beta) = (f^{\delta\eta} \circ f^{\beta\delta})(x_\beta) \\ &\Rightarrow f^{\alpha\eta}(x_\alpha) = f^{\delta\eta}(f^{\beta\delta}(x_\beta)) = f^{\delta\eta}(f^{\gamma\delta}(x_\gamma)) = (f^{\delta\eta} \circ f^{\gamma\delta})(x_\gamma) = f^{\gamma\eta}(x_\gamma) \end{aligned}$$

$\Rightarrow x_\alpha \sim x_\gamma$. This implies that $\sim$ is a transitive relation. Therefore $\sim$ is an equivalence relation on $\bigvee_{\alpha \in L} X_\alpha$. This completes the proof.

3.4.2. NOTE:

The transitivity of an equivalence relation can also be proved by a lattice diagram.

Consider the lattice diagram



where the arrow from α to ε indicates that $\alpha \leq \varepsilon$ and that there exists a fuzzy continuous map

$f^{\alpha\epsilon} : X_\alpha \rightarrow X_\epsilon$. Similarly, another f-continuous map $f^{\epsilon\eta} : X_\epsilon \rightarrow X_\eta$. Thus

$$\begin{aligned} f^{\alpha\eta}(x_\alpha) &= (f^{\epsilon\eta} \circ f^{\alpha\epsilon})(x_\alpha) = f^{\epsilon\eta}(f^{\alpha\epsilon}(x_\alpha)) = f^{\epsilon\eta}(f^{\beta\epsilon}(x_\beta)) = f^{\beta\eta}(x_\beta) = (f^{\delta\eta} \circ f^{\beta\delta})(x_\beta) \\ \Rightarrow f^{\alpha\eta}(x_\alpha) &= f^{\delta\eta}(f^{\beta\delta}(x_\beta)) = f^{\delta\eta}(f^{\gamma\delta}(x_\gamma)) = (f^{\delta\eta} \circ f^{\gamma\delta})(x_\gamma) = f^{\gamma\eta}(x_\gamma) \end{aligned}$$

$\Rightarrow x_\alpha \sim x_\gamma$. This implies that $\sim$ is a transitive relation.

3.4.3. REMARK:

Let $[x_\alpha]$ denote the equivalence class of x_α and let $X = \frac{(\bigvee_{\alpha \in L} X_\alpha)}{t}$ be the quotient fuzzy topological spaces. Let $\varphi_\alpha : X_\alpha \rightarrow X, \alpha \in L$ be the usual quotient map such that $\varphi_\alpha(x_\alpha) = [x_\alpha]$. Note that each φ_α is f-continuous and $0 \subset X$ is fuzzy open (closed) iff $\varphi_\alpha^{-1}(0)$ is fuzzy open (closed) in X_α , for each $\alpha \in L$. Clearly, $X = \bigvee_{\alpha \in L} \{\varphi_\alpha(X_\alpha) : \alpha \in L\}$, so that X is non-empty, whenever at least one X_α is non-empty. Now, we show that $\{X, \varphi_\alpha : \alpha \in L\}$ is the fuzzy direct limit of the direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces.

3.4.4. THEOREM ([98]):

Show that $\{X, \varphi_\alpha : \alpha \in L\} = \lim_{\rightarrow} \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$.

PROOF:

Define $\varphi_\alpha : X_\alpha \rightarrow X, \alpha \in L$ by $\varphi_\alpha(x_\alpha) = [x_\alpha]$ for any $x_\alpha \in X_\alpha$. Also,

let $f^{\alpha\beta} : X_\alpha \rightarrow X_\beta$ be a f-continuous map. Then

$$(\varphi_\beta \circ f^{\alpha\beta})(x_\alpha) = \varphi_\beta(f^{\alpha\beta}(x_\alpha)) = [f^{\alpha\beta}(x_\alpha)]. \text{ Thus } \varphi_\alpha(x_\alpha) = (\varphi_\beta \circ f^{\alpha\beta})(x_\alpha), \text{ for } \alpha \leq \beta$$

$\Rightarrow \varphi_\alpha = \varphi_\beta \circ f^{\alpha\beta}$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{\varphi_\alpha} & X \\
 f^{\alpha\beta} \downarrow & \nearrow \varphi_\beta & \\
 X_\beta & &
 \end{array} \quad (38)$$

is commutative .

Next, we established the universal property of $\{X, \varphi_\alpha : \alpha \in L\}$. Let $\{Y, \psi_\alpha : \alpha \in L\}$ with

$\psi_\alpha : X_\alpha \rightarrow Y$ be such that $\psi_\alpha = \psi_\beta \circ f^{\alpha\beta}$ for $\alpha \leq \beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{\varphi_\alpha} & Y \\
 f^{\alpha\beta} \downarrow & \nearrow \psi_\beta & \\
 X_\beta & &
 \end{array} \quad (39)$$

is commutative . Define a map $\mu : X \rightarrow Y$ by $\mu([x_\alpha]) = \psi_\alpha(x_\alpha)$. Then μ is well defined.

For $x_\alpha \sim x_\beta$, there exists $\gamma \in L$ such that $\alpha, \beta \leq \gamma$, then $f^{\alpha\gamma}(x_\alpha) = f^{\beta\gamma}(x_\beta)$. Thus

$$\psi_\alpha(x_\alpha) = (\psi_\gamma \circ f^{\alpha\gamma})(x_\alpha) = (\psi_\gamma \circ f^{\beta\gamma})(x_\beta) = \psi_\beta(x_\beta) \Rightarrow \mu([x_\alpha]) = \mu([x_\beta]) \Rightarrow \mu \text{ is unique.}$$

Clearly, μ is f -continuous. Thus $(\mu \circ \varphi_\alpha)(x_\alpha) = \psi_\alpha(x_\alpha) = \mu([x_\alpha])$ i.e. the following

diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\mu} & Y \\
 \uparrow \varphi_\alpha & \nearrow \psi_\alpha & \\
 X_\alpha & &
 \end{array} \quad (40)$$

is commutative .

From the above discussion we conclude that $\{X, \varphi_\alpha : \alpha \in L\}$ is the direct limit of the direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces.

Therefore, $\{X, \varphi_\alpha : \alpha \in L\} = \varinjlim \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$.

3.4.5. EXAMPLE:

Let L be any directed set and X be any fts. The direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ in which each $X_\alpha = X$ and $f^{\alpha\beta} = \text{Identity map}$ is called the trivial direct system and the direct limit of this system is X .

Chapter-4

Mappings on Inverse Systems and Direct Systems of Fuzzy Topological Spaces

4.1. INTRODUCTION:

In this section, we introduce the mappings between two inverse systems of fuzzy topological spaces such that this map gives us a map between the two inverse limits of the inverse systems of fuzzy topological spaces. Also, in the same way we introduce the mappings between two direct systems of fuzzy topological spaces and obtain a map between the direct limits of the respective direct systems of fuzzy topological spaces and study the various properties of both the mappings.

4.2. MAPPINGS OF INVERSE SYSTEMS:

In this section, we introduce the concepts of the mappings between two inverse systems of fuzzy topological spaces and examine some of their properties.

4.2.1. DEFINITION ([98]):

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ and $\{Y_\alpha, g_{\alpha\beta} : \alpha, \beta \in L\}$ be two inverse systems of fuzzy topological spaces over the same partially ordered set L . Then the map $\varphi : \{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Y_\alpha, g_{\alpha\beta} : \alpha, \beta \in L\}$ between two inverse systems of fuzzy topological spaces is called a mapping of inverse system if φ is a collection $\{\varphi_n\}$ of fuzzy

continuous maps $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ one for each $\alpha \in L$ such that $g_{\alpha\beta} \circ \varphi_\beta = \varphi_\alpha \circ f_{\alpha\beta}$ whenever $\alpha \leq \beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{\varphi_\alpha} & Y_\alpha \\
 \uparrow f_{\alpha\beta} & & \uparrow g_{\alpha\beta} \\
 X_\beta & \xrightarrow{\varphi_\beta} & Y_\beta
 \end{array} \quad (41)$$

is commutative.

4.2.2. DEFINITION:

Let $\varphi : \{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Y_\alpha, g_{\alpha\beta} : \alpha, \beta \in L\}$ be a continuous map of inverse systems. Then the unique fuzzy continuous map $\phi : X \rightarrow Y$, where $\{X, \phi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ are the inverse limits of the inverse systems $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ and $\{Y_\alpha, g_{\alpha\beta} : \alpha, \beta \in L\}$ respectively is called the mapping between two inverse limits if for each $\alpha \in L$, $\psi_\alpha \circ \phi = \varphi_\alpha \circ \phi_\alpha$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\phi} & Y \\
 \downarrow \phi_\alpha & & \downarrow \psi_\alpha \\
 X_\alpha & \xrightarrow{\varphi_\alpha} & Y_\alpha
 \end{array} \quad (42)$$

is commutative $\square$.

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ be an inverse system of fuzzy topological spaces with directed set L and let $J \subseteq L$ such that J is also a directed set. Let $\{X_{\alpha'}, f_{\alpha'\beta'} : \alpha', \beta' \in J\}$ be another inverse system of fuzzy topological spaces with directed set J .

4.2.3. THEOREM:

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ be an inverse system of fuzzy topological spaces with directed set L and let $J \subseteq L$ such that J is a directed set co-final with L (i.e. for every $\alpha \in L$ there exists a $\alpha' \in J$ such that $\alpha \leq \alpha'$). Then $\lim_{\leftarrow} \{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\} \cong \lim_{\leftarrow} \{X_{\alpha'}, f_{\alpha'\beta'} : \alpha', \beta' \in J\}$.

PROOF:

Let $\lim_{\leftarrow} \{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\} = (X, \varphi_\alpha : \alpha \in L)$ and

$\lim_{\leftarrow} \{X_{\alpha'}, f_{\alpha'\beta'} : \alpha', \beta' \in J\} = (X', \varphi_{\alpha'} : \alpha' \in J)$. Consider the mapping $\psi : X \rightarrow X'$ by

$\psi(x_\alpha) = x_{\alpha'}$, where $x_\alpha \in \prod_{\alpha \in L} X_\alpha$ and $x_{\alpha'} \in \prod_{\alpha' \in J} X_{\alpha'}$. Then ψ is a homomorphism. Now,

we show that ψ is one-one. If $\psi(x_\alpha) = x_{\alpha'}$ and $\alpha \in L$, then for any $\alpha' \in J$ with $\alpha \leq \alpha'$,

we have $\varphi_\alpha = f_{\alpha\alpha'} \varphi_{\alpha'}$ i.e. $x_{\alpha'}$ determines x_α and so ψ is one-one.

To prove that ψ is onto. For any given $x_{\alpha'} \in X'$, define a $x_\alpha \in \prod_{\alpha \in L} X_\alpha$ as follows: For

given $\alpha \in L$ choose a $\alpha' \in J$ with $\alpha \leq \alpha'$ and define $\varphi_\alpha = f_{\alpha\alpha'} \varphi_{\alpha'}$. Note that x_α is well

defined because if $\alpha \leq \beta' \in J$, then we can choose an upper bound γ' of α' and β' in J .

Then $f_{\alpha\alpha'} \varphi_{\alpha'} = f_{\alpha\alpha'} f_{\alpha'\gamma'} \varphi_{\gamma'} = f_{\alpha\gamma'} \varphi_{\gamma'} = f_{\alpha\beta'} f_{\beta'\gamma'} \varphi_{\gamma'} = f_{\alpha\beta'} \varphi_{\beta'}$ and thus φ_α does not depend on the choice of α' .

To prove $x_\alpha \in X$, choose $\alpha \leq \beta, \alpha, \beta \in L$. Then there exists an $\gamma' \in J$ with $\beta \leq \gamma'$ since J is co-final with L . Let us compute $f_{\alpha\beta} \varphi_\beta = f_{\alpha\beta} f_{\beta\gamma'} \varphi_{\gamma'} = f_{\alpha\gamma'} \varphi_{\gamma'} = \varphi_\alpha$ which was to be proved. Since $\psi(x_\alpha) = x_{\alpha'}$ is trivial, we conclude that ψ is an isomorphism, completing the proof of the theorem.

4.2.4. DEFINITION:

Let $\{X_n, f_n\}$ and $\{Y_n, g_n\}$ be two inverse limit sequences of fuzzy topological spaces. Then the mapping $\varphi : \{X_n, f_n\} \rightarrow \{Y_n, g_n\}$ is called a mapping of inverse limit system for the sequence of fuzzy topological spaces if φ is a collection $\{\varphi_n\}$ of fuzzy continuous mapping $\varphi_n : X_n \rightarrow Y_n$ satisfying the condition $g_n \circ \varphi_n = \varphi_{n-1} \circ f_n$, $n \geq 1$ i.e. the following diagram

$$\begin{array}{ccccccc}
 \dots & \rightarrow & X_n & \xrightarrow{f_n} & X_{n-1} & \xrightarrow{f_{n-1}} & X_{n-2} \rightarrow \dots \rightarrow X_1 \xrightarrow{f_1} X_0 \\
 & & \downarrow \varphi_n & & \downarrow \varphi_{n-1} & & \downarrow \varphi_{n-2} \dots \dots \downarrow \varphi_1 & \downarrow \varphi_0 \\
 \dots & \rightarrow & Y_n & \xrightarrow{g_n} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_{n-2} \rightarrow \dots \rightarrow Y_1 \xrightarrow{g_1} Y_0
 \end{array} \quad (43)$$

is commutative.

4.2.5. THEOREM:

The mapping $\psi : X_\infty \rightarrow Y_\infty$ induced by the mapping $\varphi : \{X_n, f_n\} \rightarrow \{Y_n, g_n\}$ is f-continuous.

PROOF:

Since Y_∞ is contained in $\prod_{n=0}^{\infty} Y_n$, then the mapping $\psi : X_\infty \rightarrow Y_\infty$ may be regarded as

a mapping of X_∞ into $\prod_{n=0}^{\infty} Y_n$. Let $x = (x_0, x_1, x_2, \dots)$ be a point of X_∞ . Then

$\psi(x) = (\psi(x_0), \psi(x_1), \dots)$ is defined by a f-continuous mapping $\varphi_n : X_n \rightarrow Y_n$ and is

indeed a point of Y_∞ follows from the equations

$g_n \circ (\varphi_n(x_n)) = \varphi_{n-1} \circ (f_n(x_n)) = \varphi_{n-1}(x_{n-1})$. Since the mappings $\varphi_n : X_n \rightarrow Y_n$ are f-

continuous and φ is the collection $\{\varphi_n\}$ of fuzzy continuous mappings $\varphi_n : X_n \rightarrow Y_n$, then the mappings $\varphi : \{X_n, f_n\} \rightarrow \{Y_n, g_n\}$ induces a f-continuous map $\psi : X_\infty \rightarrow Y_\infty$. This completes the proof of the theorem.

4.2.6. LEMMA:

If X is an fts and if for each α in an index set L , $X_\alpha = X$, then the diagonal in

$\prod_{\alpha \in L} X_\alpha$ consisting of all constant functions $\psi : L \rightarrow \bigvee_{\alpha \in L} X_\alpha$ is homeomorphic to X .

(Its proof is easy)

4.2.7. LEMMA:

Let $X_0, X_1, X_2, \dots$ each be the same fts X and let $f_n : X_n \rightarrow X_{n-1}$, $n > 0$, be the identity mapping. Then the inverse limit space X_∞ of the sequence $\{X_n, f_n\}$ is homeomorphic to X .

PROOF:

The points of X_∞ are sequences $(x, x, x, \dots)$, x in X , and there is an obvious f-continuous map $h : X_\infty \rightarrow X$ defined by $h(x, x, x, \dots) = x$. The only question that remains is whether the fuzzy topologies of X and X_∞ are equivalent. But X_∞ is the diagonal in

$\prod_{n=0}^{\infty} X_n$. Therefore by the lemma (4.2.6) we can say that the inverse limit space X_∞ of the

sequence $\{X_n, f_n\}$ is homeomorphic to X . This completes the proof of the lemma.

4.2.8. THEOREM:

Let $\{X_n, f_n\}$ and $\{Y_n, g_n\}$ be two inverse limit sequences of compact T_2 -spaces and let $\varphi = \{\varphi_n\}$ be a mapping of $\{X_n, f_n\}$ into $\{Y_n, g_n\}$ such that each $\varphi_n : X_n \rightarrow Y_n$ is onto. Then the induced mapping $\psi : X_\infty \rightarrow Y_\infty$ is also onto.

PROOF:

Let $(y_0, y_1, y_2, \dots)$ be a point in Y_∞ . For each n , let e the set $\varphi_n^{-1}(y_n)$. The subsets X_n' exists, since φ_n is onto. Define the mappings $f_n' = f_n / X_n'$ (f_n restricted to X_n'). Then $\{X_n', f_n'\}$ is an inverse limit sequence of compact spaces, for if x_n is in X_n' , and then $f_n'(x_n)$ is in $\varphi_{n-1}^{-1}g_n(y_n) = \varphi_{n-1}^{-1}(y_{n-1}) = X_{n-1}'$. Hence X_∞' exists and any point in X_∞' is mapped by ψ onto $(y_0, y_1, y_2, \dots)$.

4.2.9. REMARKS:

The concepts of an inverse limit sequence of fuzzy topological spaces are closely related to an inverse limit sequence of fuzzy groups. Let $\{G_0, G_1, G_2, \dots\}$ be a countable collection of fuzzy groups and $\varphi_n : G_n \rightarrow G_{n-1}, n \geq 1$ be a f -continuous homomorphism. The inverse limit group G_∞ of such a sequence is the collection of all sequences $(g_0, g_1, g_2, \dots)$ with $g_i \in G_i$ and such that $g_i = \varphi_{i+1}(g_{i+1})$ for all i . Clearly G_∞ is indeed a fuzzy group. Note that G_∞ always contains at least one element, namely, $(e_0, e_1, e_2, \dots)$, where e_i denotes the identity element of G_i .

4.2.10. DEFINITION:

Let $\{G_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ and $\{G'_{\alpha'}, f'_{\alpha'\beta'} : \alpha', \beta' \in L'\}$ be two inverse systems of fuzzy groups over directed sets L and L' respectively. We define an f-homomorphism $\varphi : \{G_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{G'_{\alpha'}, f'_{\alpha'\beta'} : \alpha', \beta' \in L'\}$ with an order-preserving mapping $\phi : L' \rightarrow L$ and for each $\alpha' \in L'$, a f-homomorphism $\varphi_{\alpha'} : G_{\phi(\alpha')} \rightarrow G'_{\alpha'}$ such that $\varphi_{\alpha'} \circ f_{\phi(\alpha')\phi(\beta')} = f'_{\alpha'\beta'} \circ \varphi_{\beta'}$ i.e. the following diagram

$$\begin{array}{ccc}
 G_{\phi(\alpha')} & \xrightarrow{\varphi_{\alpha'}} & G'_{\alpha'} \\
 \uparrow f_{\phi(\alpha')\phi(\beta')} & & \uparrow f'_{\alpha'\beta'} \\
 G_{\phi(\beta')} & \xrightarrow{\varphi_{\beta'}} & G'_{\beta'}
 \end{array} \tag{45}$$

is commutative.

4.2.11. THEOREM:

The f-homomorphism $\varphi : \{G_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{G'_{\alpha'}, f'_{\alpha'\beta'} : \alpha', \beta' \in L'\}$ induces a f-homomorphism $\varphi_\infty : G_\infty \rightarrow G'_\infty$.

PROOF:

The inverse limit group G_∞ of the inverse system $\{G_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy group over the directed set L is that fuzzy subgroup of the direct sum $\sum_{\alpha \in L} G_\alpha$ consisting of all sets $\{x_\alpha\}$, one element from each fuzzy group G_α , for which $f_{\alpha\beta}(x_\beta) = x_\alpha$ whenever $\alpha \leq \beta$ in L . Thus let $\{x_\alpha\}$ be an element of G_∞ . The f-homomorphism $\varphi : \{G_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{G'_{\alpha'}, f'_{\alpha'\beta'} : \alpha', \beta' \in L'\}$ consists of an order-preserving

map $\phi : L' \rightarrow L$ and for each $\alpha' \in L'$, a f-homomorphism $\varphi_{\alpha'} : G_{\phi(\alpha')} \rightarrow G'_{\alpha'}$. Thus we have $x'_{\alpha'} = \varphi_{\alpha'}(x_{\phi(\alpha')})$. Also, if $\alpha' \leq \beta'$, then we have $\varphi_{\alpha'} \circ f_{\phi(\alpha')\phi(\beta')} = f'_{\alpha'\beta'} \circ \varphi_{\beta'}$ i.e. the following diagram

$$\begin{array}{ccc}
 G_{\phi(\alpha')} & \xrightarrow{\varphi_{\alpha'}} & G'_{\alpha'} \\
 \uparrow f_{\phi(\alpha')\phi(\beta')} & & \uparrow f'_{\alpha'\beta'} \\
 G_{\phi(\beta')} & \xrightarrow{\varphi_{\beta'}} & G'_{\beta'}
 \end{array} \quad (46)$$

is commutative. The above commutative relation tells us that $f'_{\alpha'\beta'}(x'_{\beta'}) = x'_{\alpha'}$. Thus $\{x'_{\alpha'}\}$ is an element of G'_{∞} . Now, we define $\varphi_{\infty} : G_{\infty} \rightarrow G'_{\infty}$ by $\varphi_{\infty}(\{x_{\alpha}\}) = \{x'_{\alpha'}\}$. Then φ_{∞} is the induced f-homomorphism φ . This completes the proof of the theorem.

4.3. MAPPING OF DIRECT SYSTEMS:

In this section, we introduce the mappings between two direct systems of fuzzy topological spaces and study their several features.

4.3.1. DEFINITION ([98]):

Let $\{X_{\alpha}, f^{\alpha\beta} : \alpha, \beta \in L\}$ and $\{Y_{\alpha}, g^{\alpha\beta} : \alpha, \beta \in L\}$ be two direct systems of fuzzy topological spaces over the same directed set L . Then the map $\varphi : \{X_{\alpha}, f^{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Y_{\alpha}, g^{\alpha\beta} : \alpha, \beta \in L\}$ between two direct systems of fuzzy topological spaces is called a mapping of direct system if φ is a collection $\{\varphi_{\alpha}\}$ of fuzzy continuous maps $\varphi_{\alpha} : X_{\alpha} \rightarrow Y_{\alpha}$ for each $\alpha \in L$ such that $g^{\alpha\beta} \circ \varphi_{\alpha} = \varphi_{\beta} \circ f^{\alpha\beta}$ whenever $\alpha \leq \beta$ i.e. the diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{\varphi_\alpha} & Y_\alpha \\
 \downarrow f^{\alpha\beta} & & \downarrow g^{\alpha\beta} \\
 X_\beta & \xrightarrow{\varphi_\beta} & Y_\beta
 \end{array} \tag{47}$$

is commutative.

4.3.2. DEFINITION:

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ and $\{Y_\alpha, g^{\alpha\beta} : \alpha, \beta \in L\}$ be two direct systems of fuzzy topological spaces over the same directed set L with direct limits $\{X, \phi_\alpha : \alpha \in L\}$ and $\{Y, \psi_\alpha : \alpha \in L\}$ respectively and let $\varphi : \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Y_\alpha, g^{\alpha\beta} : \alpha, \beta \in L\}$ be a map between two direct systems of fuzzy topological spaces. Then the unique f -continuous map $\phi : X \rightarrow Y$ is called a mapping between two direct limits if $\psi_\alpha \circ \phi_\alpha = \phi \circ \phi_\alpha$ for each $\alpha \in L$ i.e. the following diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{\varphi_\alpha} & Y_\alpha \\
 \downarrow \phi_\alpha & & \downarrow \psi_\alpha \\
 X & \xrightarrow{\phi} & Y
 \end{array} \tag{48}$$

is commutative.

4.3.3. DEFINITION:

The map ϕ as constructed in the above definition (4.3.2) is called a direct limit of φ and written as $\phi = \lim_{\rightarrow} \varphi$.

4.3.4. PROPOSITION:

If $\varphi : \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Y_\alpha, g^{\alpha\beta} : \alpha, \beta \in L\}$ and $\eta : \{Y_\alpha, g^{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Z_\alpha, h^{\alpha\beta} : \alpha, \beta \in L\}$ are fuzzy continuous maps between direct system of fuzzy topological spaces over the same directed set L , then $\eta \circ \varphi : \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Z_\alpha, h^{\alpha\beta} : \alpha, \beta \in L\}$ is also a fuzzy continuous map between direct system of fuzzy topological spaces and $\lim_{\rightarrow}(\eta \circ \varphi) = \lim_{\rightarrow} \eta \circ \lim_{\rightarrow} \varphi$.

PROOF:

Let $\varphi : \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Y_\alpha, g^{\alpha\beta} : \alpha, \beta \in L\}$ and $\eta : \{Y_\alpha, g^{\alpha\beta} : \alpha, \beta \in L\} \rightarrow \{Z_\alpha, h^{\alpha\beta} : \alpha, \beta \in L\}$ be mappings of direct systems. Then φ is the collection $\{\varphi_\alpha\}$ of f-continuous map $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ and η is the collection $\{\eta_\alpha\}$ of f-continuous map $\eta_\alpha : Y_\alpha \rightarrow Z_\alpha$. Also, we have the following commutative diagrams

$$\begin{array}{ccccc}
 X_\alpha & \xrightarrow{\varphi_\alpha} & Y_\alpha & \xrightarrow{\eta_\alpha} & Z_\alpha \\
 f^{\alpha\beta} \downarrow & & g^{\alpha\beta} \downarrow & & h^{\alpha\beta} \downarrow \\
 X_\beta & \xrightarrow{\varphi_\beta} & Y_\beta & \xrightarrow{\eta_\beta} & Z_\beta
 \end{array} \quad \text{and} \quad (49)$$

$$\begin{array}{ccccc}
 X_\alpha & \xrightarrow{\varphi_\alpha} & Y_\alpha & \xrightarrow{\eta_\alpha} & Z_\alpha \\
 \phi_\alpha \downarrow & & \psi_\alpha \downarrow & & \theta_\alpha \downarrow \\
 X & \xrightarrow{\phi} & Y & \xrightarrow{\theta} & Z
 \end{array} \quad (50)$$

Then we have the relations, $g^{\alpha\beta} \circ \varphi_\alpha = \varphi_\beta \circ f^{\alpha\beta}$, $h^{\alpha\beta} \circ \eta_\alpha = \eta_\beta \circ g^{\alpha\beta}$

and $\psi_\alpha \circ \varphi_\alpha = \phi \circ \phi_\alpha$, $\theta_\alpha \circ \eta_\alpha = \theta \circ \psi_\alpha$, where $\{X, \phi_\alpha : \alpha \in L\} = \lim_{\rightarrow} \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$,

$$\{Y, \psi_\alpha : \alpha \in L\} = \varinjlim \{Y_\alpha, g^{\alpha\beta} : \alpha, \beta \in L\}, \{Z, \theta_\alpha : \alpha \in L\} = \varinjlim \{Z_\alpha, h^{\alpha\beta} : \alpha, \beta \in L\},$$

$\phi = \varinjlim \varphi, \theta = \varinjlim \eta$. Hence $\theta \circ \phi = \varinjlim \eta \circ \varinjlim \varphi$. But $\theta \circ \phi$ is the direct limit of $\eta \circ \varphi$

i.e. $\theta \circ \phi = \varinjlim (\eta \circ \varphi)$. Therefore $\varinjlim (\eta \circ \varphi) = \varinjlim \eta \circ \varinjlim \varphi$ and this completes the proof of

the proposition $\square$.

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ be a direct system of fuzzy topological spaces with directed set L and let $I \subseteq L$ such that I is also a directed set. Let $\{X_{\alpha'}, f^{\alpha'\beta'} : \alpha', \beta' \in I\}$ be another direct system of fuzzy topological spaces with directed set I .

4.3.5. THEOREM:

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ be a direct system of fuzzy topological spaces with directed set L and let $I \subseteq L$ such that I is a directed set co-final with L (i.e. for every $\alpha \in L$ there exists a $\alpha' \in I$ such that $\alpha \leq \alpha'$). Then

$$\varinjlim \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\} \cong \varinjlim \{X_{\alpha'}, f^{\alpha'\beta'} : \alpha', \beta' \in I\}.$$

PROOF:

$$\text{Let } \varinjlim \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\} = (X, \varphi_\alpha : \alpha \in L)$$

and $\varinjlim \{X_{\alpha'}, f^{\alpha'\beta'} : \alpha', \beta' \in I\} = (X', \varphi_{\alpha'} : \alpha' \in I)$. An element of X is of the form $[x_L]$,

where the index L indicates that we take its closure in the system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$.

Consider the mapping $\psi : [x_L] \rightarrow [x_I]$, where $x_{\alpha'} \in X_{\alpha'}$, $\alpha' \in I$ and the subscript I denotes the closure in the system $\{X_{\alpha'}, f^{\alpha'\beta'} : \alpha', \beta' \in I\}$. The domain of ψ is X ,

because $[x_L] \in X, x_{\alpha'} \in X_{\alpha'}, \alpha \in L$, then there exists a $\alpha' \in I$ such that $\alpha \leq \alpha'$.

Since $f^{\alpha\alpha'}(x_{\alpha'}) = x_{\alpha'}$, then we get $[x_L] = [f^{\alpha\alpha'}(x_L)]$ and $f^{\alpha\alpha'}(x_{\alpha'}) \in X_{\alpha'}, \alpha' \in I$. It is obvious that ψ maps X onto X' and ψ is a homomorphism.

Next, we show that ψ is one-one. Assume that $[x_I] = [y_I]$ and $x_{\alpha'} \in X_{\alpha'}, y_{\beta'} \in X_{\beta'}, \alpha', \beta' \in I$. Then there exists a $\gamma' \in I$ such that $\alpha' \leq \gamma', \beta' \leq \gamma'$ and $f^{\alpha'\gamma'}(x_{\alpha'}) = f^{\beta'\gamma'}(y_{\beta'})$. This implies that $[x_L] = [y_L]$, so ψ is one-one. This completes the proof of the theorem.

Chapter-5

Some Properties of Inverse and Direct Limits for Fuzzy Topological Spaces

5.1. INTRODUCTION:

In this section, we recall some definitions such as fuzzy products, fuzzy equalizers, fuzzy pullbacks and their duals, namely, fuzzy coproducts, fuzzy coequalizers and fuzzy pushouts. Also, we show that fuzzy products, fuzzy equalizers, and fuzzy pullbacks are the substitute limits of the fuzzy inverse limits and dually fuzzy coproducts, fuzzy coequalizers and fuzzy pushouts are the substitute limits of the fuzzy direct limits.

5.2. FUZZY PRODUCT AS A FUZZY INVERSE LIMIT:

We can obtain fuzzy product as an inverse limit of certain inverse system of fuzzy topological spaces.

5.2.1. THEOREM ([98]):

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ be an inverse system of fuzzy topological spaces, where all the $f_{\alpha\beta}$'s are identity fuzzy continuous maps. Then $\{X, \varphi_\alpha : \alpha \in L\} = \varprojlim \{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ if and only if (X, φ_α) is the fuzzy product of the fuzzy topological spaces $\{X_\alpha : \alpha \in L\}$.

5.3. FUZZY COPRODUCT AS A FUZZY DIRECT LIMIT:

We can obtain fuzzy coproduct as a direct limit of certain direct system of fuzzy topological spaces.

5.3.1. THEOREM:

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ be a direct system of fuzzy topological spaces. Then $\{X, \varphi_\alpha : \alpha \in L\} = \varinjlim \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ if and only if (X, φ_α) is the fuzzy coproduct of the fuzzy topological spaces $\{X_\alpha : \alpha \in L\}$.

PROOF:

Let $\{X, \varphi_\alpha : \alpha \in L\} = \varinjlim \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$. Then we have fuzzy continuous maps $\varphi_\alpha : X_\alpha \rightarrow X$, for each $\alpha \in L$. Consider a fuzzy topological space Y and a family of fuzzy continuous maps $\psi_\alpha : X_\alpha \rightarrow Y$, for each $\alpha \in L$. Then by the co-universal property of fuzzy direct limit, there exists a fuzzy continuous map $\mu : X \rightarrow Y$ such that $\psi_\alpha = \mu \circ \varphi_\alpha, \alpha \in L$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\mu} & Y \\
 \varphi_\alpha \swarrow & & \uparrow \psi_\alpha \\
 & X_\alpha &
 \end{array} \tag{51}$$

is commutative. Therefore (X, φ_α) is the fuzzy coproduct of the family $\{X_\alpha : \alpha \in L\}$ of fuzzy topological spaces.

Conversely, suppose that (X, φ_α) is the fuzzy coproduct of the family $\{X_\alpha : \alpha \in L\}$ of fuzzy topological spaces. Thus we have fuzzy inclusion maps $\varphi_\alpha : X_\alpha \rightarrow X$ for each $\alpha \in L$. If Y is any fuzzy topological space and $\psi_\alpha : X_\alpha \rightarrow Y$ are fuzzy continuous maps, then by the definition of fuzzy coproduct, there exists a unique fuzzy continuous map $\mu : X \rightarrow Y$ such that $\psi_\alpha = \mu \circ \varphi_\alpha, \alpha \in L$, showing that $\{X, \varphi_\alpha : \alpha \in L\} = \varinjlim \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$. This completes the proof of the theorem.

5.4. FUZZY EQUALIZER AS FUZZY INVERSE LIMIT:

We can obtain fuzzy equalizer as an inverse limit of certain inverse system of fuzzy topological spaces.

5.4.1. THEOREM ([98]):

Consider the fuzzy inverse system $\{X_1, X_2, f, g, Id_{X_1}, Id_{X_2} / f, g : X_1 \rightarrow X_2\}$. Then the fuzzy inverse limit of $\{X_1, X_2, f, g, Id_{X_1}, Id_{X_2}\}$ can be obtained as the fuzzy equalizer of $f, g : X_1 \rightarrow X_2$ and vice-versa.

5.5. FUZZY COEQUALIZER AS A FUZZY DIRECT LIMIT:

We can obtain fuzzy coequalizer as a direct limit of certain direct system of fuzzy topological spaces.

5.5.1. THEOREM:

Consider the direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ with $\alpha \leq \beta$ and $\alpha, \beta = 1, 2$. Then the fuzzy direct limit of $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ can be obtained as the fuzzy coequalizer of $f^{\alpha\beta}, g^{\alpha\beta} : X_\alpha \rightarrow X_\beta$, for $\alpha \leq \beta$ and $\alpha, \beta = 1, 2$.

PROOF:

First suppose that $\{X, \varphi_\alpha, \alpha \in L\}$ is the direct limit of the direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ with $\alpha \leq \beta$ and $\alpha, \beta = 1, 2$. Then for $\alpha < \beta$, we have the following diagram

$$\begin{array}{ccc}
 X_1 & \xrightarrow{f^{12}} & X_2 \\
 \searrow \varphi_1 & & \downarrow \varphi_2 \\
 & & X
 \end{array} \quad (52)$$

with $\varphi_1 = \varphi_2 \circ f^{12} = \varphi_2 \circ g^{12}$. Now, for any fuzzy topological space Y and fuzzy continuous maps $\psi_\alpha : X_\alpha \rightarrow Y$, $\alpha = 1, 2$; we have $\psi_1 = \psi_2 \circ f^{12} = \psi_2 \circ g^{12}$. By the couniversal property of the fuzzy direct limit, there exists a unique f -continuous map $\mu : X \rightarrow Y$ such that $\psi_2 = \mu \circ \varphi_2, \alpha \in L$ i.e. the following diagram

$$\begin{array}{ccccc}
 X_1 & \xrightarrow{f^{12}, g^{12}} & X_2 & \xrightarrow{\varphi_2} & X \\
 & \searrow \psi_2 & & \downarrow \mu & \\
 & & & & Y
 \end{array} \quad (53)$$

is commutative. Thus (X, φ_2) is the fuzzy coequalizer of f^{12} and g^{12} .

Conversely, let $f^{12}, g^{12} : X_1 \rightarrow X_2$ be the f-continuous maps having fuzzy coequalizer (Q, q) . Then we have a f-continuous map $q : X_2 \rightarrow Q$ such that $q \circ f^{12} = q \circ g^{12}$ and therefore a diagram

$$\begin{array}{ccc}
 X_1 & \xrightarrow{f^{12}, g^{12}} & X_2 \\
 & \searrow & \downarrow q \\
 & & Q
 \end{array}
 \quad (54)$$

$q \circ f^{12} = q \circ g^{12}$

with $q \circ f^{12} = q \circ g^{12}$. Hence $(Q, q \circ f^{12}, q)$ is the fuzzy direct limit of the direct system

$\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ with $\alpha \leq \beta$ and $\alpha = 1, 2$. This completes the proof of the theorem.

5.6. FUZZY PULLBACK AS A FUZZY INVERSE LIMIT:

We can obtain fuzzy pullback as an inverse limit of certain inverse system of fuzzy topological spaces.

5.6.1. THEOREM ([98]):

Consider the inverse system $\{X, Y, Z, f, g / f : X \rightarrow Z, g : Y \rightarrow Z\}$. Then the fuzzy inverse limit of $\{X, Y, Z, f, g\}$ can be obtained as the fuzzy pullback of f and g .

5.7. FUZZY PUSHOUT AS FUZZY DIRECT LIMIT:

We can obtain fuzzy pushout as a direct limit of certain direct system of fuzzy topological spaces.

5.7.1. THEOREM:

Consider the fuzzy direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ with $\alpha \leq \beta$ and $\alpha, \beta = 1, 2, 3$.

Then the fuzzy direct limit of $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ can be obtained as the fuzzy pushout of $f^{12} : X_1 \rightarrow X_2$ and $f^{13} : X_1 \rightarrow X_3$.

PROOF:

First suppose that $\{X, \varphi_\alpha : \alpha \in L\}$ is the fuzzy direct limit of the fuzzy direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ with $\alpha \leq \beta$ and $\alpha, \beta = 1, 2, 3$. We consider the diagram

$$\begin{array}{ccccc}
 X_3 & \xleftarrow{f^{13}} & X_1 & \xrightarrow{f^{12}} & X_2 \\
 & \searrow \varphi_3 & \downarrow \varphi_1 & \nearrow \varphi_2 & \\
 & & X & &
 \end{array} \tag{55}$$

and then we have $\varphi_1 = \varphi_3 \circ f^{13} = \varphi_2 \circ f^{12}$. For any fuzzy topological space Y and a family of f -continuous maps $\psi_\alpha : X_\alpha \rightarrow Y, \alpha = 1, 2, 3$ satisfying $\psi_2 \circ f^{12} = \psi_3 \circ f^{13}$,

we have $\psi_1 = \psi_2 \circ f^{12} : X_1 \rightarrow Y$ as shown in the following diagram

$$\begin{array}{ccccc}
 X_3 & \xleftarrow{f^{13}} & X_1 & \xrightarrow{f^{12}} & X_2 \\
 & \searrow \psi_3 & \downarrow \psi_1 & \nearrow \psi_2 & \\
 & & Y & &
 \end{array}
 \quad (56)$$

By the couniversal property of the fuzzy direct limit, there exists a unique fuzzy continuous map $\mu : X \rightarrow Y$ such that $\psi_\alpha = \mu \circ \varphi_\alpha, \alpha = 2, 3$ i.e. the following diagram

$$\begin{array}{ccccc}
 & & & & Y \\
 & & & \nearrow \mu & \\
 & & X & \xleftarrow{\varphi_2} & X_2 \\
 & \nearrow \psi_3 & \uparrow \varphi_3 & \nwarrow \varphi_1 & \uparrow f^{12} \\
 & & X_3 & \xleftarrow{f^{13}} & X_1
 \end{array}
 \quad (57)$$

is commutative. Thus $\{X, \varphi_\alpha : \alpha \in L, \alpha = 2, 3\}$ is the fuzzy pushout of f^{12} and f^{13} .

Conversely, suppose that $\{X, \varphi_\alpha : \alpha \in L, \alpha = 2, 3\}$ is the fuzzy pushout of $f^{12} : X_1 \rightarrow X_2$ and $f^{13} : X_1 \rightarrow X_3$. Then we have $\varphi_1 = \varphi_2 \circ f^{12} = \varphi_3 \circ f^{13} : X_1 \rightarrow X$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xleftarrow{\varphi_2} & X_2 \\
 \uparrow \varphi_3 & \nearrow \varphi_1 & \uparrow f^{12} \\
 X_3 & \xleftarrow{f^{13}} & X_1
 \end{array} \quad (58)$$

is commutative and showing that $\{X, \varphi_\alpha : \alpha \in L, \alpha = 1, 2, 3\} = \varinjlim \{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ with $\alpha \leq \beta$ and $\alpha, \beta = 1, 2, 3$. This completes the proof of the theorem.

5.8. Fuzzy inverse and direct limits in a special way:

In this section, we mark the character of fuzzy inverse and fuzzy direct limits in a special way. This special way is that in the presence of fuzzy product, fuzzy inverse limits can be expressed in terms of fuzzy equalizer and fuzzy pullbacks individually. Also, dually in the presence of fuzzy coproduct, fuzzy direct limits can be expressed in terms of fuzzy coequalizer and fuzzy pushout differently.

5.8.1. THEOREM:

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ be inverse systems of fuzzy topological spaces. Then in the presence of fuzzy product, fuzzy equalizer is a substitute fuzzy inverse limit of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces.

PROOF:

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ be inverse systems of fuzzy topological spaces and let $d(f_{\alpha\beta})$ denote the domain of $f_{\alpha\beta}$ and $c(f_{\alpha\beta})$ the codomain of $f_{\alpha\beta}$. Let U be the class of

all fuzzy continuous maps $\{f_{\alpha\beta} : \alpha, \beta \in L\}$ of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$. Let X and Y denote the following fuzzy products:

$X = \prod_{\alpha \in L} X_\alpha$, $Y = \prod_{f_{\alpha\beta} \in U} c(f_{\alpha\beta})$. Then we have the projections $\pi_\alpha : X \rightarrow X_\alpha, \alpha \in L$ and

$p_{c(f_{\alpha\beta})} : Y \rightarrow c(f_{\alpha\beta}), f_{\alpha\beta} \in U$ respectively. Also, we have two different fuzzy continuous

maps $\pi_{c(f_{\alpha\beta})}, f_{\alpha\beta} \circ \pi_{d(f_{\alpha\beta})} : X \rightarrow c(f_{\alpha\beta}), f_{\alpha\beta} \in U$, which are shown in the following

diagram:

$$\begin{array}{ccc}
 & X & \\
 \pi_{d(f_{\alpha\beta})} \swarrow & & \searrow \pi_{c(f_{\alpha\beta})} \\
 d(f_{\alpha\beta}) & \xrightarrow{f_{\alpha\beta}} & c(f_{\alpha\beta})
 \end{array} \quad (59)$$

Take a fts X and a f -continuous map $\pi_{c(f_{\alpha\beta})}$. Then by the universal property of fuzzy

product, there is a unique f -continuous map $f : X \rightarrow Y$ such that $\pi_{c(f_{\alpha\beta})} = p_{c(f_{\alpha\beta})} \circ f$,

$f_{\alpha\beta} \in U$ i.e. the following diagram

$$\begin{array}{ccc}
 & X & \\
 f \downarrow & & \searrow \pi_{c(f_{\alpha\beta})} \\
 Y & \xrightarrow{p_{c(f_{\alpha\beta})}} & c(f_{\alpha\beta})
 \end{array} \quad (60)$$

is commutative. Similarly, Take a fts X and a f -continuous map $f_{\alpha\beta} \circ \pi_{d(f_{\alpha\beta})}$. Then by the universal property of fuzzy product, there is a unique f -continuous map $g : X \rightarrow Y$ such that $f_{\alpha\beta} \circ \pi_{d(f_{\alpha\beta})} = p_{c(f_{\alpha\beta})} \circ g$, $f_{\alpha\beta} \in U$ i.e. the following diagram

$$\begin{array}{ccc}
 X & & \\
 \downarrow g & \searrow f_{\alpha\beta} \circ \pi_{d(f_{\alpha\beta})} & \\
 Y & \xrightarrow{p_{c(f_{\alpha\beta})}} & c(f_{\alpha\beta})
 \end{array} \quad (61)$$

is commutative.

Let $e : E \rightarrow X$ be the fuzzy equalizer of f and g , where $f, g : X \rightarrow Y$ are the f -continuous maps with $f \circ e = g \circ e$. For each $f_{\alpha\beta} \in U$, we have $X_\beta = d(f_{\alpha\beta})$ and $X_\alpha = c(f_{\alpha\beta})$. Now,

$$\begin{aligned}
 f_{\alpha\beta} \circ (\pi_\beta \circ e) &= f_{\alpha\beta} \circ (\pi_{d(f_{\alpha\beta})} \circ e) = (f_{\alpha\beta} \circ \pi_{d(f_{\alpha\beta})}) \circ e \\
 &= (p_{c(f_{\alpha\beta})} \circ g) \circ e = p_{c(f_{\alpha\beta})} \circ (g \circ e) = p_{c(f_{\alpha\beta})} \circ (f \circ e) \\
 &= (p_{c(f_{\alpha\beta})} \circ f) \circ e = \pi_{c(f_{\alpha\beta})} \circ e = \pi_\alpha \circ e
 \end{aligned}$$

i.e. the diagram

$$\begin{array}{ccc}
 E & & \\
 \downarrow \pi_\beta \circ e & \searrow \pi_\alpha \circ e & \\
 X_\beta & \xrightarrow{f_{\alpha\beta}} & X_\alpha
 \end{array} \quad (62)$$

commutes. This implies that $\{E, \pi_\alpha \circ e : \alpha \in L\}$ is a fuzzy inverse limit of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$.

Next, we check the universal property for fuzzy inverse limits. Let $q_\alpha : Q \rightarrow X_\alpha$, $\alpha \in L$ be f-continuous maps with, $q_{c(f_{\alpha\beta})} = f_{\alpha\beta} \circ q_{d(f_{\alpha\beta})}$ i.e. the following diagram

$$\begin{array}{ccc}
 Q & & \\
 \downarrow q_{d(f_{\alpha\beta})} & \searrow q_{c(f_{\alpha\beta})} & \\
 d(f_{\alpha\beta}) & \xrightarrow{f_{\alpha\beta}} & c(f_{\alpha\beta})
 \end{array} \quad (63)$$

commutes. Then by the universal property of the fuzzy product, there is a unique f-continuous map $\varphi : Q \rightarrow X$ such that $\pi_\alpha \circ \varphi = q_\alpha$ for each $\alpha \in L$ i.e. the following diagram

$$\begin{array}{ccc}
 Q & & \\
 \downarrow \varphi & \searrow q_\alpha & \\
 X & \xrightarrow{\pi_\alpha} & X_\alpha
 \end{array} \quad (64)$$

is commutative. Thus we have $\pi_{d(f_{\alpha\beta})} \circ \varphi = q_{d(f_{\alpha\beta})}$ and $\pi_{c(f_{\alpha\beta})} \circ \varphi = q_{c(f_{\alpha\beta})}$. Observe that

$$\begin{aligned}
 p_{c(f_{\alpha\beta})} \circ (f \circ \varphi) &= (p_{c(f_{\alpha\beta})} \circ f) \circ \varphi = \pi_{c(f_{\alpha\beta})} \circ \varphi = q_{c(f_{\alpha\beta})} = f_{\alpha\beta} \circ q_{d(f_{\alpha\beta})} \\
 &= f_{\alpha\beta} \circ (\pi_{d(f_{\alpha\beta})} \circ \varphi) = (f_{\alpha\beta} \circ \pi_{d(f_{\alpha\beta})}) \circ \varphi = (\varphi_{c(f_{\alpha\beta})} \circ g) \circ \varphi = p_{c(f_{\alpha\beta})} \circ (g \circ \varphi) \\
 &\Rightarrow f \circ \varphi = g \circ \varphi, \text{ by cancellation law. Then by universal property of fuzzy equalizer,}
 \end{aligned}$$

there exists a unique f-continuous map $\mu : Q \rightarrow E$ such that $\varphi = e \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 E & \xrightarrow{e} & X \xrightarrow{f,g} Y \\
 \uparrow \mu & \nearrow \varphi & \\
 Q & &
 \end{array} \quad (65)$$

commutes. Thus $q_\alpha = \pi_\alpha \circ \varphi = \pi_\alpha \circ (e \circ \mu) = (\pi_\alpha \circ e) \circ \mu$, for each $\alpha \in L$ i.e. the diagram

$$\begin{array}{ccc}
 Q & & \\
 \downarrow \mu & \searrow q_\alpha & \\
 E & \xrightarrow{\pi_\alpha \circ e} & X_\alpha
 \end{array} \quad (66)$$

commutes and is showing that $\{E, \pi_\alpha \circ e : \alpha \in L\}$ is a fuzzy inverse limit of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$. This completes the proof of the theorem.

5.8.2. THEOREM:

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ be direct systems of fuzzy topological spaces. Then in the presence of fuzzy coproduct, fuzzy coequalizer is a substitute fuzzy direct limit of the direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces.

PROOF:

Follows from the dual version of theorem (5.8.1).

5.8.3. THEOREM:

Let $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ be inverse systems of fuzzy topological spaces. Then in the presence of fuzzy product, fuzzy pullback is a substitute fuzzy inverse limit of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces.

PROOF:

Firstly, we prove that fuzzy pullback is a fuzzy equalizer. Let the triplet (P, λ, μ) be the fuzzy pullback of α and β , where $\alpha: X \rightarrow Z$ and $\beta: Y \rightarrow Z$ are the pair of f-continuous maps. Then we have P is an fts and $\lambda: P \rightarrow X, \mu: P \rightarrow Y$ are the fuzzy continuous maps such that $\alpha \circ \lambda = \beta \circ \mu$ i.e. the following diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\lambda} & X \\
 \mu \downarrow & & \downarrow \alpha \\
 Y & \xrightarrow{\beta} & Z
 \end{array} \quad (67)$$

is commutative. Consider a pair of f-continuous maps $f, g: X \rightarrow Y$. Then for the f-continuous map $\lambda: P \rightarrow X$ we have $f \circ \lambda = g \circ \lambda$. This implies that (p, λ) is a fuzzy equalizer of f and g .

To check the universal property of fuzzy equalizer. Let W be an fts and $\delta: W \rightarrow X, \theta: W \rightarrow Y$ be two f-continuous maps satisfying $f \circ \delta = g \circ \theta$. Then by the universal property of fuzzy pullback, there exists a unique f-continuous map $h: W \rightarrow P$ such that $\delta = \lambda \circ h$ and $\theta = \mu \circ h$. Thus for the fts W and the f-continuous map $\delta: W \rightarrow X$ satisfying $f \circ \delta = g \circ \delta$, there exists a unique f-continuous map $h: W \rightarrow P$ such that $\delta = \lambda \circ h$ i.e. the following diagram

$$\begin{array}{ccccc}
 P & \xrightarrow{\lambda} & X & \xrightarrow{f} & Y \\
 & & \nwarrow h & & \nearrow \delta \\
 & & W & &
 \end{array}
 \quad (68)$$

is commutative. This implies that (p, λ) is a fuzzy equalizer of f and g together with an f -continuous map $\lambda : P \rightarrow X$. Therefore, fuzzy pullback is a fuzzy equalizer.

Now, by the theorem (5.8.1), we can easily show that fuzzy equalizer is a substitute fuzzy inverse limit of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces. This completes the proof of the theorem.

5.8.4. THEOREM:

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ be direct systems of fuzzy topological spaces. Then in the presence of fuzzy coproduct, fuzzy pushout is a substitute fuzzy direct limit of the direct system $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces.

PROOF:

Firstly, we prove that fuzzy pushout is a fuzzy coequalizer. Let the triplet (Q, λ, μ) be the fuzzy pushout of α and β , where $\alpha : X \rightarrow Y$ and $\beta : X \rightarrow Z$ are the pair of f -continuous maps. Then we have Q is an fts and $\lambda : Y \rightarrow Q, \mu : Z \rightarrow Q$ are the fuzzy continuous maps such that $\lambda \circ \alpha = \mu \circ \beta$ i.e. the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & Y \\
 \beta \downarrow & & \downarrow \lambda \\
 Z & \xrightarrow{\mu} & Q
 \end{array}
 \quad (69)$$

is commutative. Consider a pair of f -continuous maps $f, g : X \rightarrow Y$. Then for the f -continuous map $\lambda : Y \rightarrow Q$ we have $\lambda \circ f = \lambda \circ g$. This implies that (Q, λ) is a fuzzy coequalizer of f and g .

To check the couniversal property of fuzzy coequalizer. Let W be an fts and $\delta : Y \rightarrow W, \theta : Z \rightarrow W$ be two f -continuous maps satisfying $\delta \circ \alpha = \theta \circ \beta$. Then by the couniversal property of fuzzy pushout, there exists a unique f -continuous map $h : Q \rightarrow W$ such that $\delta = h \circ \lambda$ and $\theta = h \circ \mu$. Thus for the fts W and the f -continuous map $\delta : Y \rightarrow W$ satisfying $\delta \circ f = \delta \circ g$, there exists a unique f -continuous map $h : Q \rightarrow W$ such that $\delta = h \circ \lambda$ i.e. the following diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{f, g} & Y & \xrightarrow{\lambda} & Q \\
 & & \searrow \delta & & \downarrow h \\
 & & & & W
 \end{array} \tag{70}$$

is commutative. This implies that (Q, λ) is a fuzzy coequalizer of f and g together with an f -continuous map $\lambda : Y \rightarrow Q$. Therefore, fuzzy pushout is a fuzzy coequalizer.

Now, by the theorem (5.8.2), we can easily show that fuzzy coequalizer is a substitute fuzzy inverse limit of the inverse system $\{X_\alpha, f_{\alpha\beta} : \alpha, \beta \in L\}$ of fuzzy topological spaces. This completes the proof of the theorem.

Chapter-6

On Singular Homology Groups of Fuzzy Topological Spaces

6.1. INTRODUCTION:

The fundamental concepts of fuzzy algebraic topology, in particular, the singular homology groups of L-fuzzy topological spaces are introduced by Liu and Zheng [63]. They have defined $L = (L; \leq, \wedge, \vee)$ being a completely distributive lattice with order reversing involution, in such a way that they took the (usual) cubical singular homological groups of topological spaces as a special case. They have proved that singular homology groups are L-fuzzy homeomorphic invariants. In this section, we shall add some new results to fuzzy algebraic topology by using the concepts of direct limits of fuzzy topological spaces.

6.2. SINGULAR HOMOLOGY GROUPS OF FUZZY TOPOLOGICAL SPACES:

From the completeness point of view of the thesis developed by Liu and Zheng [63], we recall the entire construction of singular homology theory of fuzzy topological spaces. Let $L = (L; \leq, \wedge, \vee)$ be a completely distributive lattice which has least and greatest elements, say 0 and 1 respectively. Let X be a non-empty (usual) set. An L-fuzzy set in X is a map $\lambda : X \rightarrow L$ and the families of all L-fuzzy sets in X are denoted by L^X . It is clear that $L = (L; \leq, \wedge, \vee)$ is completely distributive lattice, which has least and greatest elements $\bar{0}$ and $\bar{1}$ respectively in a natural manner, where $\bar{0}(x) = 0$ and $\bar{1}(x) = 1$, for any $x \in X$.

6.2.1. DEFINITION ([63]):

Let $\tau \subseteq L^X$. Then τ is called an L-fuzzy topology on X if it satisfies the following conditions:

- (i) $\bar{0}, \bar{1} \in \tau$
- (ii) if $\lambda, \mu \in \tau$, then $\lambda \wedge \mu \in \tau$
- (iii) if $\lambda_i \in \tau, i \in \Delta$, then $\bigvee_{i \in \Delta} \lambda_i \in \tau$, where Δ is an index set.

The pair (X, τ) is called an L-fuzzy topological space (in short L-fts).

6.2.2. DEFINITION ([63]):

Let (X, τ) and (Y, σ) be two L-fuzzy topological spaces. A map $f: X \rightarrow Y$ is called L-fuzzy continuous if and only if for every $\mu \in \sigma$, $f^{-1}(\mu) \in \tau$, where $f^{-1}(\mu)(x) = \mu(f(x))$, for $x \in X$. The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is called L-fuzzy homeomorphism if and only if $f: X \rightarrow Y$ is bijective and both f and f^{-1} are L-fuzzy continuous. Also, the L-fuzzy topological spaces (X, τ) and (Y, σ) are said to be L-fuzzy homeomorphic if there exists an L-fuzzy homeomorphism $f: (X, \tau) \rightarrow (Y, \sigma)$.

6.2.3. REMARKS ([63]):

The properties of L-fuzzy continuous maps have been discussed in [11] and I (or I_i) always denotes (usual unit interval throughout this paper and $I^n = I \times I \times I \times \dots \times I$ (n factors, $n > 0$) unit n -cube. R is the set of real numbers.

6.2.4. DEFINITION ([63]):

Let $\tilde{I}$ be the set of all monotonic decreasing maps $\lambda: R \rightarrow L$ satisfying

- (i) $\lambda(t) = 1$, for $t < 0$,

(ii) $\lambda(t) = 0$, for $t > 0$.

For $\lambda, \mu \in \tilde{I}$, we define that $\lambda \approx \mu$ if and only if for all $t \in R$, $\lambda((t^-)) = \mu((t^-))$ and

$\lambda((t^+)) = \mu((t^+))$, where $\lambda((t^-)) = \inf_{s < t} \lambda(s)$, $\lambda((t^+)) = \sup_{s > t} \lambda(s)$. Then $\approx$ is called an

equivalence relation on $\tilde{I}$. The quotient set $\tilde{I}/\approx$ is called the L-fuzzy interval (in short

$I(L)$) and $[\lambda]$ will be used to denote the equivalence class of $\lambda \in \tilde{I}$.

We define an L-fuzzy topology T on $I(L)$ by taking $\{L_t, R_t : t \in R\}$ as a subbase, where $L_t([\lambda]) = \lambda((t^-)) \in L$ and $R_t([\lambda]) = \lambda((t^+)) \in L$. Then T is called the standard topology on $\tilde{I}$.

6.2.5. DEFINITION ([63]):

Let $I^n(L)$ be the product spaces of L-fuzzy intervals $I_1(L), I_2(L), \dots, I_n(L)$ with standard topology $T_1, T_2, \dots, T_n$ respectively, for $n \geq 1$. Then $I^n(L)$ is called the L-fuzzy basic n-cube and T^n denotes the product topology.

6.2.6. NOTE ([63]):

Let $p_i : I^n(L) \rightarrow I_i(L)$ be the natural projection, $i=1, 2, 3, \dots, n$, then $\{p_i^{-1}(L_t^{(i)}), p_i^{-1}(R_t^{(i)}) : t \in R, i=1, 2, 3, \dots, n\}$ is a subbase of T^n , where $\{L_t^{(i)}, R_t^{(i)} : t \in R\}$ is the subbase of standard topology T_i on $I_i(L)$.

6.2.7. DEFINITION ([63]):

Let $\nu_o \in \tilde{I}$, $\nu_o(t) = 1$, for $t < 0$ and $\nu_o(t) = 0$, for $t > 0$. Then $I^o(L) = [\nu_o] \in I(L)$ is called L-fuzzy basic 0-cube and $I^o(L)$ has the relative topology T^o .

6.2.8. DEFINITION ([63]):

Let (X, τ) be an L-fuzzy topological space. Then an L-fuzzy singular n-cube in (X, τ) is an L-fuzzy continuous map $\xi^n : (I^n(L), T^n) \rightarrow (X, \tau)$, $n \geq 0$. Let $S_n(X, \tau)$ denote the set of all L-fuzzy singular n-cube in (X, τ) .

6.2.9. DEFINITION ([63]):

Let ξ^n be an L-fuzzy singular n-cube, $n \geq 1$. Then $\xi^n \in S_n(X, \tau)$ is called degenerate if and only if there exists $i, 1 \leq i \leq n$ such that for any $[\lambda^i], [\mu^i] \in I_i(L)$, $\xi^n(\dots, [\lambda^i], \dots) = \xi^n(\dots, [\mu^i], \dots)$ i.e. $\xi^n([\lambda^1], \dots, [\lambda^i], \dots, [\lambda^n])$ does not depend on $[\lambda^i]$.

6.2.10. PROPOSITION ([63]):

Let (X, τ) be an L-fuzzy topological space. Then

- (i) $\xi^n \in S_n(X, \tau)$ if and only if $\xi^n : I^n(L) \rightarrow X$ is an L-fuzzy singular n-cube in X , $n \geq 0$;
- (ii) $\xi^n \in S_n(X, \tau)$ is degenerate if and only if $\xi^n : I^n(L) \rightarrow X$ is a degenerate singular n-cube in X , $n \geq 1$.

PROOF:

Let (X, τ) be an L-fuzzy topological space and $I^n(L)$ be the product spaces of L-fuzzy intervals $I_1(L), I_2(L), \dots, I_n(L)$ with standard topology $T_1, T_2, \dots, T_n$ respectively, where T^n denotes the product topology.

- (i) Let $\xi^n \in S_n(X, \tau)$. Since $S_n(X, \tau)$ is the set of all L-fuzzy singular n-cubes in (X, τ) , then $\xi^n : I^n(L) \rightarrow X$ is an L-fuzzy continuous map, $n \geq 0$ and hence $\xi^n : I^n(L) \rightarrow X$ is an L-fuzzy singular n-cube in X .

Conversely, let $\xi^n : I^n(L) \rightarrow X$ be an L-fuzzy singular n-cube in X , $n \geq 0$ and $S_n(X, \tau)$ be the set of all L-fuzzy singular n-cubes in (X, τ) . Then obviously, $\xi^n \in S_n(X, \tau)$.

(ii) Suppose that $\xi^n \in S_n(X, \tau)$, $n \geq 1$, is degenerate. Then there exists $i, 1 \leq i \leq n$ such that for any $[\lambda^i], [\mu^i] \in I_i(L)$, $\xi^n(\dots, [\lambda^i], \dots) = \xi^n(\dots, [\mu^i], \dots)$. Thus $\xi^n : I^n(L) \rightarrow X$ is a degenerate singular n-cube in (X, τ) .

Conversely, let $\xi^n : I^n(L) \rightarrow X$ is a degenerate singular n-cube in (X, τ) , $n \geq 1$. Then $\xi^n : I^n(L) \rightarrow X$ is an L-fuzzy singular n-cube in X and hence $\xi^n \in S_n(X, \tau)$, $n \geq 1$, is degenerate. This completes the proof of our proposition.

6.2.11. REMARKS:

Let $[\lambda^1], [\lambda^2], \dots, [\lambda^n]$ be the vertices of $I^n(L)$ and we define these vertices are the face of $I^n(L)$. For $i = 1, 2, \dots, n$, let $\varepsilon_n^i : (I^{n-1}(L), T^{n-1}) \rightarrow (I^n(L), T^n)$ be a f-continuous map such that $\varepsilon_n^i([\lambda^1], \dots, [\lambda^{n-1}]) = ([\lambda^1], \dots, [\nu_o], [\lambda^i], \dots, [\lambda^{n-1}])$, where $[\nu_o] \in I(L)$ and $\nu_o(t) = 1$, for $t < 0$ and $\nu_o(t) = 0$, for $t > 0$.

6.2.12. DEFINITION:

Let $\xi^n \in S_n(X, \tau)$, $n \geq 1$. For $i = 1, 2, \dots, n$, any L-fuzzy singular n-cube $\xi^n : I^n(L) \rightarrow X$ have n faces $d_i \xi^n$ defined by $d_i \xi^n = \xi^n \circ \varepsilon_n^i : I^{n-1}(L) \rightarrow X$. In other words, $d_i \xi^n$ is the map obtained by restricting ξ^n to the i th face of $I^n(L)$ and regarding this restriction as a map defined on $I^{n-1}(L)$ and $d_i \xi^n \in S_{n-1}(X, \tau)$.

6.2.13. PROPOSITION:

Let $\xi^n \in S_n(X, \tau)$ and $i < j$. Then $d_i d_j \xi^n = d_{j-1} d_i \xi^n$.

PROOF:

Let ξ^n be a L-fuzzy singular n-cube in $S_n(X, \tau)$. Then the map $d_i \xi^n$ defined by $d_i \xi^n([\lambda^1], [\lambda^2], \dots, [\lambda^{n-1}]) = \xi^n([\lambda^1], \dots, [\lambda^{i-1}], [\nu_o], [\lambda^i], \dots, [\lambda^{n-1}])$, where $\varepsilon_n^i : (I^{n-1}(L), T^{n-1}) \rightarrow (I^n(L), T^n)$ is a f-continuous map such that $\varepsilon_n^i([\lambda^1], \dots, [\lambda^{n-1}]) = ([\lambda^1], \dots, [\nu_o], [\lambda^i], \dots, [\lambda^{n-1}])$, where $[\nu_o] \in I(L)$ and $\nu_o(t) = 1$, for $t < 0$ and $\nu_o(t) = 0$, for $t > 0$.

First setting $[\lambda^j] = 0$ and then $[\lambda^i] = 0$ for $i < j$ in $\xi^n([\lambda^1], [\lambda^2], \dots, [\lambda^n])$ of $d_i d_j \xi^n$ amounts to the same as first setting $[\lambda^j] = 0$ and then setting equal to 0 the variable with the new number j-1 of $d_{j-1} d_i \xi^n$. Therefore, $d_i d_j \xi^n = d_{j-1} d_i \xi^n$. This completes the proof of our proposition.

6.2.14. DEFINITION ([63]):

Let (X, τ) be an L-fs and Z be the (usual) additive group of integers. For $n > 0$, $Q_n((X, \tau), Z)$ denotes the free abelian group with generators of all L-fuzzy singular n-cube ξ^n in (X, τ) i.e. every $\hat{c}_n \in Q_n((X, \tau), Z)$ has a unique representation as finite linear combination of L-fuzzy singular n-cubes, that is, $\hat{c}_n = \sum_{\alpha} a_{\alpha} \xi^n$, $a_{\alpha} \in Z$, $\xi^n \in S_n(X, \tau)$ and otherwise zero. We shall not distinguish between an L-fuzzy singular n-cube ξ^n and $1\xi^n \in Q_n((X, \tau), Z)$.

6.2.15. DEFINITION ([63]):

Let (X, τ) be an L-fs, $n \geq 1$. We define a homomorphism

$\partial_n : \mathcal{Q}_n((X, \tau), Z) \rightarrow \mathcal{Q}_{n-1}((X, \tau), Z)$ by $\partial_n(\hat{c}_n) = \sum_{\alpha} a_{\alpha} \partial_n(\xi^n)$, for that

$$\partial_n \xi^n = \sum_{i=1}^n (-1)^i d_i \xi^n.$$

6.2.16. THEOREM:

For $n > 1$, the composite $\partial_{n-1} \circ \partial_n : \mathcal{Q}_n((X, \tau), Z) \rightarrow \mathcal{Q}_{n-1}((X, \tau), Z)$ is the null homomorphism i.e. $(\partial_{n-1} \circ \partial_n) \xi^n = 0$.

PROOF:

From the proposition (2.13), we have

$$d_i d_j \xi^n = d_{j-1} d_i \xi^n, \text{ for } i < j. \quad \dots (1)$$

Let $\partial_n : \mathcal{Q}_n((X, \tau), Z) \rightarrow \mathcal{Q}_{n-1}((X, \tau), Z)$ be a homomorphism such that

$$\partial_n \xi^n = \sum_{i=1}^n (-1)^i d_i \xi^n. \text{ Then } (\partial_{n-1} \circ \partial_n) \xi^n = \sum_{i < j} (-1)^{i+j} d_i d_j \xi^n + \sum_{i \geq j} (-1)^{i+j} d_i d_j \xi^n. \text{ Now,}$$

using the relation (1) and switching the labels i and j in the second sum, then we

$$\text{have } (\partial_{n-1} \circ \partial_n) \xi^n = \sum_{j-1 \geq i} (-1)^{i+j} d_{j-1} d_i \xi^n + \sum_{k \geq i} (-1)^{i+k} d_k d_i \xi^n. \text{ These two sums are equal but}$$

opposite in sign and hence equal to zero i.e. $(\partial_{n-1} \circ \partial_n) \xi^n = 0$.

Also, $(\partial_{n-1} \circ \partial_n) \xi^n = 0$.

$$\Rightarrow (\partial_{n-1} \circ \partial_n)(\xi^n) = 0 \cdot \xi^n.$$

$$\Rightarrow (\partial_{n-1} \circ \partial_n) = 0, \text{ by L-fuzzy cancellation law.}$$

$\Rightarrow (\partial_{n-1} \circ \partial_n)$ is a null homomorphism. This completes the proof of our theorem.

6.2.17. DEFINITION ([63]):

Let $\partial_n : Q_n((X, \tau), Z) \rightarrow Q_{n-1}((X, \tau), Z)$ be a homomorphism. Then by theorem (2.16), we have $\text{Im } \partial_{n+1} \subseteq \text{Ker } \partial_n$. Then $C_n((X, \tau), Z) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}$ is called the homology group. The elements of $\text{Ker } \partial_n$ are called the cycles, the elements of $\text{Im } \partial_{n+1}$ are called the boundaries and the elements of $C_n((X, \tau), Z)$ are called the homology classes. Now, we define a homomorphism $\partial_n^* : C_n((X, \tau), Z) \rightarrow C_{n-1}((X, \tau), Z)$ by $\partial_n^* \left(\{ \hat{c}_n \} \right) = \left\{ \partial_n(\hat{c}_n) \right\}$, for $\hat{c}_n \in \text{Ker } \partial_n$. Then ∂_n^* is an induced homomorphism of ∂_n .

6.2.18. DEFINITION ([63]):

Let (X, τ) be an L-fuzzy topological space. We define

$$Z_n((X, \tau), Z) = \text{Ker } \partial_n^* = \left\{ c_n \in C_n((X, \tau), Z) : \partial_n^*(c_n) = 0, n \geq 1 \right\} \text{ and}$$

$$B_n((X, \tau), Z) = \text{Im } \partial_{n+1}^* = \partial_{n+1}^*(C_{n+1}((X, \tau), Z)), n \geq 0. \text{ Then } Z_n((X, \tau), Z) \text{ and } B_n((X, \tau), Z)$$

are called the group of singular n-cycles and group of n- boundaries in

(X, τ) respectively. Particularly, let $Z_0((X, \tau), Z) = C_0((X, \tau), Z)$. From the theorem

(2.16) we have $B_n((X, \tau), Z) \subseteq Z_n((X, \tau), Z), n \geq 0$. Hence we define

$$H_n((X, \tau), Z) = Z_n((X, \tau), Z) / B_n((X, \tau), Z), n \geq 0 \text{ and called it singular nth homology}$$

group of L-fits (X, τ) (with integral coefficient).

6.2.19. THEOREM ([63]):

Let $L = \{0,1\}$ and L -fts (X, τ) be the (usual) topological space.

Then $H_n((X, \tau), Z) = H_n(X, Z), n \geq 0$, where $H_n(X, Z)$ is the n -dimensional cubical singular homology group of X .

6.2.20. DEFINITION ([63]):

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a L -fuzzy continuous map. We define a homomorphism

$f_n : Q_n((X, \tau), Z) \rightarrow Q_n((Y, \sigma), Z), n \geq 0$ such that $f_n(\xi^n) = f \circ \xi^n$ for any $\xi^n \in S_n(X, \tau)$.

6.2.21. PROPOSITION ([63]):

For $n \geq 0$, the homomorphism $f_n : Q_n((X, \tau), Z) \rightarrow Q_n((Y, \sigma), Z)$ induces a

homomorphism $f_n^* : C_n((X, \tau), Z) \rightarrow C_n((Y, \sigma), Z), n \geq 0$.

6.2.22. PROPOSITION ([63]):

Let $(X, \tau), (Y, \sigma)$ and (U, ω) be the three L -fuzzy topological spaces. Then we have

(i) the homomorphism $Id_\# : C_n((X, \tau), Z) \rightarrow C_n((X, \tau), Z), n \geq 0$ induced by the identity map $Id : (X, \tau) \rightarrow (X, \tau)$, is the identity homomorphism.

(ii) if $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (U, \omega)$ are the L -fuzzy continuous maps, then

$(g_n \circ f_n)^*$ and $g_n^* \circ f_n^*$ are the same homomorphisms from $C_n((X, \tau), Z)$ to

$C_n((U, \omega), Z), n \geq 0$.

6.2.23. PROPOSITION ([63]):

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an L -fuzzy continuous map, $n \geq 1$. Then

(i) $f_{n-1} \circ \partial_n^X = \partial_n^Y \circ f_n : Q_n((X, \tau), Z) \rightarrow Q_{n-1}((Y, \sigma), Z);$

$$(ii) f_{n-1}^* \circ \partial_n^{X*} = \partial_n^{Y*} \circ f_n^* : C_n((X, \tau), Z) \rightarrow C_{n-1}((Y, \sigma), Z).$$

As results of proposition (2.23), for $n \geq 0$, we have

$f_n^*(Z_n((X, \tau), Z)) \subset Z_n((Y, \sigma), Z)$ and $f_n^*(B_n((X, \tau), Z)) \subset B_n((Y, \sigma), Z)$. Thus f_n^* induces a homomorphism $f^\# : H_n((X, \tau), Z) \rightarrow H_n((Y, \sigma), Z), n \geq 0$.

6.2.24. PROPOSITION ([63]):

Let $(X, \tau), (Y, \sigma)$ and (U, ω) be the three L-fuzzy topological spaces. Then we have

(i) the homomorphism $Id_* : H_n((X, \tau), Z) \rightarrow H_n((X, \tau), Z), n \geq 0$ induced by the identity map $Id : (X, \tau) \rightarrow (X, \tau)$, is the identity isomorphism, $n \geq 0$.

(ii) if $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (U, \omega)$ are the L-fuzzy continuous maps, then

$(g_n \circ f_n)^\#$ and $g_n^\# \circ f_n^\#$ are the same homomorphisms from $H_n((X, \tau), Z)$ to $H_n((U, \omega), Z), n \geq 0$.

6.2.25. PROPOSITION ([63]):

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an L-fuzzy homomorphism. Then

$f^\# : H_n((X, \tau), Z) \rightarrow H_n((Y, \sigma), Z), n \geq 0$ is a group isomorphism.

PROOF:

From the proposition (2.23), we have $f^\# : H_n((X, \tau), Z) \rightarrow H_n((Y, \sigma), Z), n \geq 0$ is

a group homomorphism. Let $c_n \in \text{Ker } \partial_n^{X*}$ such that $\partial_n^{X*}(c_n) = 0$ and $c_n' \in \text{Ker } \partial_n^{Y*}$ such that

$\partial_n^{Y*}(c_n') = 0$. Then we have $f^\#(\{c_n\}) = \{c_n'\}$.

Now, $f^\#(\{c_{n_1}\}\{c_{n_2}\}) = f^\#(\{c_{n_1}c_{n_2}\}) = \{c_{n_1}'c_{n_2}'\} = \{c_{n_1}'\}\{c_{n_2}'\} = f^\#(\{c_{n_1}\})f^\#(\{c_{n_2}\})$. Hence $f^\#$ is a group isomorphism, $n \geq 0$.

6.3. SOME CHARACTERS OF SINGULAR HOMOLOGY GROUPS OF AN L-FTS IN TERMS OF FUZZY DIRECT LIMITS:

In this section, we express the homology groups of an L-fts (X, τ) in terms of the homology groups of its fuzzy compact subspaces by using the concepts of fuzzy direct limits. Let $\{(X_\alpha, \tau_\alpha) : \alpha \in L\}$ be the collection of all fuzzy compact subspaces of L-fuzzy topological spaces (X, τ) , where L is a directed set.

6.3.1. THEOREM:

The singular homology group of an L-fuzzy topological space commutes with fuzzy direct limits.

PROOF:

Let $\{X_\alpha, f^{\alpha\beta} : \alpha, \beta \in L\}$ be a direct system of fuzzy topological spaces over the directed set L and let $\{X, \varphi_\alpha : \alpha \in L\}$ be the direct limit of this system i.e. $\varphi_\alpha : X_\alpha \rightarrow X$ and if $\alpha \leq \beta$, then $f^{\alpha\beta} : X_\alpha \rightarrow X_\beta$ and hence $\varphi_\alpha = \varphi_\beta \circ f^{\alpha\beta}$.

Let $\{(X_\alpha, \tau_\alpha) : \alpha \in L\}$ be the collection of all fuzzy compact subspaces of L-fuzzy topological spaces (X, τ) , where L is a directed set. Fix an integer $n \geq 0$ and let $M_\alpha = H_n((X_\alpha, \tau_\alpha), Z)$. For $\alpha \leq \beta$, the fuzzy inclusion map $f^{\alpha\beta} : X_\alpha \rightarrow X_\beta$ induces the group homomorphism $\theta^{\alpha\beta} = f_*^{\alpha\beta} : M_\alpha \rightarrow M_\beta$. Since $f^{\beta\gamma} \circ f^{\alpha\beta} = f^{\alpha\gamma}$ and $f^{\alpha\alpha} = Id_{X_\alpha}$, for $\alpha \leq \beta \leq \gamma$, then by proposition (2.24) we have $\theta^{\beta\gamma} \circ \theta^{\alpha\beta} = \theta^{\alpha\gamma}$ and $\theta^{\alpha\alpha} = Id_{M_\alpha}$. Thus

$\{M_\alpha, \theta^{\alpha\beta} : \alpha, \beta \in L\}$ is therefore a direct system of fuzzy abelian groups. Now, we are to show that $\{M, \varphi_{\alpha*} : \alpha \in L\}$ is the direct limit of this system.

Now, for each $\alpha \in L$, we have a fuzzy inclusion map $\varphi_\alpha : X_\alpha \rightarrow X$, so we have a homomorphism $\varphi_{\alpha*} : M_\alpha \rightarrow M$. Moreover, for $\alpha \leq \beta$ the commutative diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f^{\alpha\beta}} & X_\beta \\ & \searrow \varphi_\alpha & \downarrow \varphi_\beta \\ & & X \end{array} \quad (71)$$

i.e. $\varphi_\alpha = \varphi_\beta \circ f^{\alpha\beta}$ and by the proposition (2.24), we have a commutative diagram

$$\begin{array}{ccc} M_\alpha & \xrightarrow{\theta^{\alpha\beta}} & M_\beta \\ & \searrow \varphi_{\alpha*} & \downarrow \varphi_{\beta*} \\ & & M \end{array} \quad (72)$$

i.e. $\varphi_{\alpha*} = \varphi_{\beta*} \circ \theta^{\alpha\beta}$. For any abelian homology group $H_n((X, \tau), Z)$ and a homomorphism

$\psi_{\alpha*} : M_\alpha \rightarrow H_n((X, \tau), Z)$, $\alpha \in L$ with $\psi_{\alpha*} = \psi_{\beta*} \circ \theta^{\alpha\beta}$, for $\alpha \leq \beta$, there exists a unique

homomorphism $\psi : M \rightarrow H_n((X, \tau), Z)$ such that $\psi \circ \varphi_{\alpha*} = \psi_{\alpha*}$ i.e. the following

diagram

$$\begin{array}{ccc} M_\alpha & \xrightarrow{\psi_{\alpha*}} & H_n((X, \tau), Z) \\ & \searrow \varphi_{\alpha*} & \uparrow \psi \\ & & M \end{array} \quad (73)$$

is commutative. Therefore $\{M, \varphi_{\alpha*} : \alpha \in L\}$ is the direct limit of the direct system

$\{M_\alpha, \theta^{\alpha\beta} : \alpha, \beta \in L\}$ and is a singular homology group of an L-fuzzy topological space M

together with homomorphism $\varphi_{\alpha^*} : M_{\alpha} \rightarrow M$, $\alpha \in L$. Also, the direct limits of this system are denoted by $\varinjlim M_{\alpha}$. This completes the proof of the theorem.

6.3.2. NOTE:

The homology groups of an L-fuzzy topological space do not commute with inverse limits.

PROOF:

For the proof of the theorem (6.3.2) we present an example. For any integer $n \geq 1$ let (X_n, τ_n) be the chain complexes of L-fuzzy topological spaces with $(X_n)_q = 0$ if $q \neq 0$ or 1 and $(X_n)_1 \xrightarrow{(\partial_n)_1} (X_n)_0$ equal to $Z \rightarrow Z$, where the homomorphism is multiplication by 2. For each n let $f_n : X_n \rightarrow X_{n-1}$ be the chain map, which is multiplication by 3 on each chain groups, and for $n \leq m$ define $f_{n,m} : X_m \rightarrow X_n$ to be the composite $f_{n,m} = f_{n+1} \circ f_{n+2} \circ \dots \circ f_{n-1} \circ f_m$. Then $\{X_n, f_{n,m}\}$ is an inverse system whose inverse limit is the trivial chain complex of L-fuzzy topological spaces. Therefore $H_q(\varprojlim \{X_n, f_{n,m}\}) = 0$ for all q . However $H_0(X_n) \approx Z_2$ for all n and $f_{n,m}^* : H_0(X_m) \approx H_0(X_n)$ for all $n \leq m$. Therefore $\varprojlim \{H_0(X_n), f_{n,m}^*\} \approx Z_2$.

6.3.3. THEOREM:

The homomorphism $\psi : M \rightarrow H_n((X, \tau), Z)$ is an isomorphism.

PROOF:

For its proof (cf. [98]).

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