A FINITE DIFFERENCE SCHEME FOR MIXED BOUNDARY VALUES OF TWO DIMENSIONAL ELASTIC PROBLEMS WITH INTERNAL HOLE

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RECOMMENDATION OF THE BOARD OF EXAMINERS

The board of examiners hereby recommends to the Department of Mechanical Engineering, BUET, Dhaka, acceptance of the thesis, "**A FINITE DIFFERENCE SCHEME FOR MIXED BOUNDARY VALUES OF TWO DIMENSIONAL ELASTIC PROBLEMS WITH INTERNAL HOLE**", submitted by Md. Arifur Rahman, Roll – 040310024, in partial fulfillment of the requirement of the degree of Masters of Science in Mechanical Engineering.

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ABSTRACT

This thesis work presents the application of finite difference technique for the solution of an elastic problem of a rectangular plate having internal hole. This study also focuses on the stress concentration developed at the critical zones around the hole.

Elastic problems are formulated either in terms of deformation parameters or stress parameters. But, at the boundary, all the problems are invariably subjected to the mixture of both known deformations and stresses. Neither of the two formulations would however allow us to account fully of both of these two types of boundary conditions with equal sophistication in the region of transition of boundary conditions from one type to another.

A computer program has been developed to solve two dimensional plane strain and plane stress problems using the formulations of displacement potential function. Elastic body having internal hole or flaw poses a serious problem as it suffers stress concentration at the hole boundary. In finite difference technique, stress analysis of such a body becomes more complex due to management of boundary conditions at the external as well as at the internal boundary. A new methodology is proposed to manage the internal boundary conditions of the hole. A model problem has been solved and the results are analyzed and compared with the solution available in the literature to justify the validity of the solution scheme.

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DECLARATION

The author hereby declares that no portion of the work in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or institution of learning.

Md. Arifur Rahman

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NOMENCLATURE

Symbol	Meaning	Unit
x, y	Rectangular co-ordinates	(mm)
E	Elastic modulus of the material	(Pa)
μ	Poisson's ratio	(mm/mm)
u, v	Displacement component in the x and y-direction	(mm)
σ_x , σ_y, σ_{xy}	Stress component in the x-direction, y-direction and xy-plane	(Pa)
ψ	Potential function defined in terms of displacements	-
u_n, u_t	Normal and tangential displacement component on the physical boundary	(mm)
σ_n , σ_t	Normal and tangential stress component on the physical boundary	(Pa)
l, m	Direction cosines of the normal at any point on the boundary	-
h, k	Mesh lengths in the x and y-directions	(mm)

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CHAPTER – 1
INTRODUCTION

1.1 BACKGROUND

Analysis of stress and strain of any irregular-shaped two dimensional elastic body with internal hole due to the action of external load in the case of plane stress and plane strain state is the main work of this thesis. Elastic body with internal hole suffers stress concentration in the vicinity of the hole. Some critical sections can be found which are more vulnerable to loading. The critical sections may be different for different loading conditions. A numerical solution scheme has been developed to solve elastic problem of such type. Finite difference method has been used. Using this solution scheme, the state of stress and displacement components of a Rectangular Plate with internal circular hole have been obtained and compared with the known solution for verify the validity of the scheme. Designing of machine element needs such type of plate with different shape and size of internal hole. Theoretical and analytical solutions so far developed are limited to the solution of stress analysis of rectangular plate with internal hole with specific geometry and loading conditions. But in case of designing of machine element the geometry of the body and loading conditions may be arbitrary. So, stress analysis of such type of elastic problem is required. The solution scheme developed in this thesis work is capable of handling this type of problem.

Sometimes analytically it is not possible to solve a single or a system of partial differential equations for a certain boundary conditions, which is possible by numerical methods. For example, when the geometry and loading conditions of an elastic problem are relatively simple, analytical solutions of these types of boundary value problems can be obtained without too much difficulty. But, most of the engineering problems are dealing with structures and machine elements having complicated geometry and loading conditions. The analytical solution of these types of boundary value problems are formidable task. In some cases, where the analytical solution is obtained in terms of integrals of infinite series, involving non-elementary functions, the computation of numerical values for stresses and displacements is still a major task. To overcome these difficulties, several numerical techniques are applied to the boundary value problems of structural mechanics. The essential feature of these numerical techniques is that the original boundary value problems are replaced by a finite set of simultaneous algebraic equations. The solution of this set of simultaneous equations provides us with an approximation for the displacement and stress within the body.

1.2 LITERATURE REVIEW

As the elementary formulas of strength of materials are often not accurate enough, the theory of elasticity has found considerable application in the solution of modern engineering problems. Elementary methods of strength of materials were primary

tools for handling the problem of engineering structures. In many problems of practical concern the types of analysis examined so far are inadequate, i.e. the physical situation under investigation does not correspond to the assumptions introduced into the analyses. In such cases it is necessary to turn to a more comprehensive and more sophisticated type of analysis which does not involve as many simplifying assumptions. Such an analysis is to be found in the science termed as 'Theory of Elasticity'. Analysis based upon the theory of elasticity give much more detailed and more precise information about the state of stress, strain, and deformation at any point within the body. However, in general it is necessary to deal with much more complex mathematical situations in the theory of elasticity than are encountered in strength of materials.

In the recent years designing of machine components have demanded greater accuracy of stress analysis. Only in simple cases, rigorous mathematical solutions of elasticity problems were available. The modern trend in the theory is toward the use of various approximate methods. In cases where even approximate methods could not be developed, solutions have been obtained by using experimental methods. Photo-elastic methods, Soap-film methods, application of strain gages, moiré fringes are some of these experimental methods applied in the study of stress concentration.

The theories of elasticity had been established earlier. But, the solution of practical problem could be started after the introduction of the stress function. The success of the stress function developed earlier was very limited. The stress function had been developed and used for the solution of two-dimensional problems in the form of a polynomial and applied those results to several problems in bending of beams of narrow rectangular cross section. It had been found that the elementary formulas of strength of materials give correct values for normal and shearing stresses in a cantilever, loaded at its free end. The problem of stress in masonry dams is of great practical interest and has been attempted by various authors using polynomials expressions for the stress functions. But it should be noted that the solutions thus obtained do not satisfy the conditions at the bottom of the dam where it is connected with the foundation and would predict reasonable values of stress in the region far away from the foundation on account of Saint-Venant's Principle [1].

Stress function technique has been used by many researchers [1] for analyzing the stress around a circular hole in a plate, for stresses around a slender hole, for stresses around a concentrated load on a straight boundary, and for stresses around a concentrated load on a beam.

The stress distribution around a circular hole in a plate subjected to uniform tension in one direction was analyzed [1]. R.C.J. Howland considered the plate of finite width with a circular hole on the axis of symmetry. The case when the forces were applied to the boundary of the hole was treated by N. G. Bickley. The elliptic hole was investigated by G. V. Kolosoff. For complex boundary shape and also for restrained boundary, the difficulty of obtaining analytical solution becomes formidable. These difficulties were partially removed by extensometers and strain gauges or by the photo-elastic method. Many important two-dimensional problems of elasticity have been solved by using the functions of a complex variable.

Finite difference method is one of the numerical techniques of solving elastic problems. The use of stress function in numerical methods could not gain much more popularity. Because, as pointed out by Uddin [2], the boundary conditions in terms of restraints can not be discretized in terms of stress function. When two dimensional elastic problems are formulated in terms of displacement components u and v, the problem of discretization at the restrained boundaries are removed, but the resulting algebraic equations become ill-conditioned and often diverged in iterative method of solutions. Uddin [2] solved numerically two mixed boundary value problems using the displacement potential formulation and found that the algebraic equations resulting from discretization can only be solved if the restraint at the boundary is specified on more than one fourth of the boundary perimeter. It should also be pointed out that numerical method is unattractive when the effect of discontinuity is of major interest in the investigation which is always the case in mixed boundary value problems.

Stress analysis of two-dimensional rectangular plate with circular hole elastic body by finite difference technique is selected here as the research topic of this thesis. Though here rectangular plate is selected for analysis, this formulation can be used for other arbitrary shape of elastic body with arbitrary shape of internal hole. Always a body has three dimensions. But, most of the practical problems of stress analysis can be reduced to two dimensional problems under simplifying assumptions. The first of these two simplifying assumptions is that the loading on the body is confined in a plane and the dimension of the body in the direction perpendicular to this plane is relatively small compared to the other two directions lying on the plane. In such cases, the stresses in the body perpendicular to the plane of loading, is usually very small and thus neglected. As a result, these problems become two-dimensional, usually referred to as plane stress problem. The second simplifying assumption holds for bodies whose one of the three dimensions is relatively large or straining in a particular direction of the body is restrained. Under such circumstances, the problem of stress analysis becomes two-dimensional, referred to as plane strain problem. The present proposal thus includes a very wide range of problems of stress analysis of elastic bodies.

The term, "Mixed Boundary Conditions", to be dealt with in this thesis refers to the fact that boundaries are neither subjected to pure known loadings nor to pure known displacements. Rather, the known conditions to be satisfied on the boundaries are a mixture of known loadings and known displacements. These mixed boundary value problems are normally beyond the scope of any closed-form analytical solutions. The main problem to the stress analysis of elastic bodies is their boundary shapes. The problems of irregular boundary shapes are unquestionably beyond the scope of pure analytical methods of solutions. Most of the times, they are investigated experimentally and very recently attempts are made to solve them using finite element techniques. But, finite element techniques not only involve extensive computational efforts but also the solutions so obtained are not reliable if it is not verified by comparing to the solutions of other method.

The proposed analysis is computer based numerical solution of elastic body with internal hole using finite difference method. Finite difference schemes for the solution of elastic body have been developed so far are limited to the solution of solid elastic body only. In this thesis work a finite difference scheme has been developed to solve

elastic problem with internal hole. The effects of hole on the distribution of stress and displacement components have been analyzed. The effects of hole size also studied. Here a function is used, known as "Displacement Potential Function, ψ" introduced by Uddin [2] and extensively used by Idris [3-4], Ahmed [5-8] and Akanda [9-11] in solving mixed boundary value elastic problems.

1.3 OBJECTIVE OF THE THESIS

The objective of this thesis is to obtain numerical solutions for the stresses and deformations of a plate with internal circular hole and analysis the effect of circular hole in the plate. It is mixed boundary value elastic problem. The main objectives includes,

(a) Development of a scheme for the treatment of boundary conditions on arbitrary boundary shapes of elastic bodies for hole boundary in a two dimensional rectangular grid field.

(b) Development of a methodology for treating sharp changes in boundary contours and discontinuity in boundary conditions for hole boundary,

(c) Development of a scheme for accepting all possible mixed and changeable boundary conditions at any point on the arbitrary boundary of the hole.

(d) Establishment of the reliability of the methodology through its application on simple problems of known solutions.

(e) Solution for the effect of presence of circular hole and the effect of hole sizes on the distribution of stress in plate.

(f) Analyze the critical sections of the elastic body.

(g) Establishment of the credibility of the methodology by comparing the results obtained with the known solutions.

1.4 MATHEMATICAL STATEMENT OF THE OBJECTIVES

Stresses in elastic bodies are functions of three independent variables of Cartesian co-ordinates. They are simplified here to be independent of one of the three co-ordinates and thus become functions of two independent variables. Therefore, the problem is to solve the stresses in an arbitrary bounded plane region under all conceivable physical boundary conditions. Stated mathematically, the problem is reduced to the evaluation of a single function ψ of x, y from the bi-harmonic partial differential equation,

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{1.1}$$

The displacement potential function, ψ can be defined as,

$$u = \frac{\partial^2 \psi}{\partial x \partial y}$$

$$v = -\frac{1}{1+\mu}\left[(1-\mu)\frac{\partial^2 \psi}{\partial y^2} + 2\frac{\partial^2 \psi}{\partial x^2}\right] \tag{1.2}$$

The stress components can be written in the following forms,

$$\sigma_x = \frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^2\partial y} - \mu\frac{\partial^3\psi}{\partial y^3}\right]$$

$$\sigma_y = -\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial y^3} - (2+\mu)\frac{\partial^3\psi}{\partial x^2\partial y}\right] \qquad (1.3)$$

$$\sigma_{xy} = -\frac{E}{(1+\mu)^2}\left[\mu\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{\partial^3\psi}{\partial x^3}\right]$$

From the single potential function ψ the relevant stress and displacement components can be calculated all over the region within the body by using equations (1.2) and (1.3).

On the boundary of the body, the normal and tangential components of stress and displacement can be calculated from the following expressions,

$$u_n = ul + vm$$
$$= l\frac{\partial^2\psi}{\partial x\partial y} - m\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2\psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2\psi}{\partial x^2}\right] \qquad (1.4a)$$

$$u_t = vl - um$$
$$= -l\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2\psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2\psi}{\partial x^2}\right] + m\frac{\partial^2\psi}{\partial x\partial y} \qquad (1.4b)$$

$$\sigma_{n'} = \sigma_x l^2 + 2\sigma_{xy}lm + \sigma_y m^2$$
$$= \frac{E}{(1+\mu)^2}\left[(l^2 - m^2 - \mu m^2)\frac{\partial^3\psi}{\partial x^2\partial y} - (\mu l^2 + m^2)\frac{\partial^3\psi}{\partial y^3}\right] + \left[2lm\left\{\mu\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{\partial^3\psi}{\partial x^3}\right\}\right] \qquad (1.4c)$$

$$\sigma_t = \sigma_{xy}(l^2 - m^2) + (\sigma_y - \sigma_x)lm$$
$$= \frac{(l^2-m^2)E}{(1+\mu)}\left[\mu\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{\partial^3\psi}{\partial x^3}\right] + \frac{lmE}{1+\mu}\left[\frac{\partial^3\psi}{\partial x^2\partial y} - \frac{\partial^3\psi}{\partial y^3}\right] \qquad (1.4d)$$

In the above expressions l and m are the direction cosines of the normal to the boundary surface.

1.5 CHOICE OF SOLUTION TECHNIQUE

Analytical methods are limited in solving elastic body with some definite shape and some specific loading conditions. The boundary conditions should be uniform for analytical solutions. That is, the boundary conditions should be either in stresses only or in displacement only. But most of the practical problems rarely exist with such idealized boundary conditions. There have been some attempts to solve mixed boundary value problems when the degree of mixedness of boundary conditions is fairly low. But, these solutions have considerable shortcomings and are not valid all over the region concerned. In analytical approach, the solution is sought in terms of a

stress function, ψ which satisfies the bi-harmonic equation, and then the stress and the displacement components are calculated from the knowledge of this function. The usual practice is to assume a polynomial of some order or a series of two independent variables for ψ and then to match up the boundary conditions concerned. This is to some extent, a matter of trial and complexity of the analytical solution lead to the conclusion that a general program for numerical solution of all the problems of this class is highly desirable. Due to the development of high speed computing machines it is now easy to handle complicated problems numerically which is beyond the scope of analytical solutions. In this thesis a numerical formulation is used to solve elastic bodies under considerations.

1.6 SCOPE OF APPLICATIONS

This thesis work is limited to the analysis of stress and displacement components of two dimensional elastic body with internal hole. Although most of the practical problems are three dimensional, most of them are plane stress and plane strain problems and can be reduced to as two dimensional. So, a large number of practical problems can be reduced to one of these two dimensional problems. Examples of this kind of problems are, a retaining wall with lateral pressure, a culvert or tunnel, a cylindrical tube with internal pressure, a cylindrical roller compressed by forces in diametral plane as in case of roller bearing. In every case the loading must not vary along the length. Rectangular outer boundary and circular hole boundary are considered here in this thesis, this scheme and computer code is capable of solving any type of arbitrary shape outer boundary as well as hole boundary.

CHAPTER – 2
GENERAL THEOREMS AND DEFINITIONS

2.1 INTRODUCTION

The theory of elasticity is excellent for investigation the state of stress and deformation in the immediate vicinity of small hole, notches and cuts in an elastic body. Such effects cannot be treated by the strength of materials approach. In elastic body if any external forces producing deformation do not exceed a certain limit, the deformation disappears with the removal of the forces. Throughout this thesis work it has been assumed that all of the bodies undergoing action of external forces are perfectly elastic.

2.2 STRESS AND STRAIN

The intensity of internal forces for the application of external force in a body is called stress. Two types of external forces may act on a body. Force applied over the surface of the body, such as pressure of one body on other, hydrostatic pressure, concentrated or distributed types of load, are called surface force. Forces which are distributed over the volume of a body, such as gravitational forces, magnetic forces or, in case of a body with motion, inertia forces, are called body forces. As the effect of body forces compared to surface forces are very small, most of the cases body force can be neglected. In thesis work only surface forces are considered.

The surface forces per unit area are resolved into three components parallel to Cartesian coordinate axes. For normal stress, it is positive in sign when it produces tension and negative when it produces compression. For shearing stress, the sign is positive when acting clockwise and negative when anticlockwise.

The normal stress components acting along x, y and z axis are σ_x, σ_y and σ_z respectively. The shearing stresses in xy, yz and zx planes are τ_{xy}, τ_{yz} and τ_{zx} respectively. These six stress components are sufficient to describe the stress condition at any point within the body.

If u, v and w are the displacement of particles of deformed body along x, y and z axis respectively then the normal and shear strain in an elastic body can be written as,

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{2.1}$$

ε_x, ε_y and ε_z are the normal strain components parallel to coordinate axes and γ_{xy}, γ_{yz}, and γ_{zx} are the shear strain in xy, yz and zx planes respectively. Using Hook's Law and Poisson's ratio the strain components can be written as follows,

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)] \qquad (2.2)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)]$$

Where, E is the Young's Modulus and μ is Poisson's ratio.

2.3 SAINT-VENANT'S PRINCIPLE

Let us consider a number of statically equivalent force systems acting over a specified small portion of the surface of an elastic body. By statically equivalent it is implied that the systems all have the same force and moment resultants. Saint-Venant's principle states that although these various statically equivalent systems may have considerably different localized effects, all have essentially the same effect on stresses at any distance which is large compared to the dimension of the part of the surface on which these forces are applied.

2.4 PLANE STRESS AND PLANE STRAIN

If a thin plate is under the application of forces parallel to plane xy of the plate and distributed uniformly over the thickness of the plate, the stress components σ_z, τ_{xz} and τ_{yz} are zero on both faces of the plate. In such cases the state of stress can be specified only by σ_x, σ_y and τ_{xy}. These three components are independent of z. This type elastic problem can be termed as plane stress problem.

On the other hand for a body of which one dimension is very large or one dimension is restrained. Examples are retaining wall with lateral pressure, culvert or tunnel, cylindrical tube with internal pressure, cylindrical roller compressed by forces in a diametral plane as in a roller bearing. For such cases the loading must not vary along length. Similar to plane stress problem the state of stress of these types of problem can be specified by σ_x, σ_y and τ_{xy}. These types of problem are termed as plane strain problem.

So, for plane stress and plane strain problem the only unknowns are σ_x, σ_y and τ_{xy}. The displacement components are u (alone x axis) and v (alone y axis). The value of u, v, σ_x, σ_y and τ_{xy} are to be determined as a function of x and y. The stress acting on any plane passing through a point and inclined to x and y axis can be calculated from the equation of static. The normal and shear stress components and normal and tangential displacement component on a plane whose normal has the direction cosines (l, m) can be obtained from the following relations,

$$\sigma_n = \sigma_x l^2 + \sigma_y m^2 + 2\sigma_{xy} lm$$

$$\sigma_t = \tau_{xy}(l^2 - m^2) + (\sigma_x - \sigma_y) lm \qquad (2.3)$$

$$u_n = ul + vm$$
$$u_t = vl - um \tag{2.4}$$

σ_n and σ_t are positive normally outward on the boundary and tangential components are taken positive if they act anti-clockwise on the body. The displacement components, u an v are taken positive in the direction of x and y axes.

2.5 DIFFERENTIAL EQUATIONS

The solution of two-dimensional problems requires the integration of the differential equations of equilibrium together with the compatibility equations and the boundary conditions. Equations of equilibrium for two-dimensional problems are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + Y = 0 \tag{2.5}$$

Where, X and Y are the body forces per unit volume acting in the x and y direction. Throughout this thesis work body forces are assumed to absent. Neglecting the body forces equation (2.5) can be written as,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{2.6}$$

Equation (2.6) contains three variables but, there are only two equations. At least three equations are necessary to find three unknown. So, equation of compatibility is necessary. The equation compatibility can be obtained as follows,

The strain components ε_x, ε_y and γ_{xy} can be expressed in terms of displacement components, u and v,

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{2.7}$$
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

From (2.7) we can find,

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2.8}$$

Replacing strain components by stress components Equation (2.8) can be obtained as the following form,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0 \tag{2.9}$$

Equation (2.9) is known as the Equation of Compatibility for two dimensional plane stress and plane strain problem.

2.6 SOLUTION TECHNIQUE AND DISPLACEMENT POTENTIAL FUNCTION

The solution of two dimensional plane stress and plane strain problem requires integration of the differential equations of equilibrium (2.6) together with the compatibility equations (2.9) and the boundary conditions (2.3) and (2.4) [1].

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}=0$$
$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \sigma_{xy}}{\partial x}=0 \tag{2.6}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0 \tag{2.9}$$

$$\sigma_n=\sigma_x l^2+\sigma_y m^2+2\sigma_{xy} lm$$
$$\sigma_t=\tau_{xy}(l^2-m^2)+(\sigma_x-\sigma_y)lm \tag{2.3}$$

$$u_n=ul+vm$$
$$u_t=vl-um \tag{2.4}$$

Substitution of stress components by displacement components u and v into Eqs. (2.6) and (2.9) makes Eq. (2.9) redundant and Eqs. (2.6) transform to

$$\frac{\partial^2 u}{\partial x^2}+\left(\frac{1-\mu}{2}\right)\frac{\partial^2 u}{\partial y^2}+\left(\frac{1+\mu}{2}\right)\frac{\partial^2 v}{\partial x \partial y}=0 \tag{2.10}$$

$$\frac{\partial^2 v}{\partial y^2}+\left(\frac{1-\mu}{2}\right)\frac{\partial^2 v}{\partial x^2}+\left(\frac{1+\mu}{2}\right)\frac{\partial^2 u}{\partial x \partial y}=0 \tag{2.11}$$

Now the problem is to find u and v from a two dimensional field satisfying the two elliptical partial differential Eqs. (2.10) and (2.11).

Instead of determining the two functions u and v the problem can be reduced to solving a single function $\psi(x, y)$, which can be determined by satisfying Eqs. (2.10) and (2.11). The displacement potential function $\psi(x, y)$ can be defined as [2]

Most of the practical problems appear with mixed boundary conditions, i.e. the conditions on the boundary include known stresses, known displacements/strain or combined stress and displacements with different conditions on different segments of the boundary. To solve problem of such kind a function had been introduced by Uddin [5], known as Displacement Potential Function, $\psi(x, y)$, that can be defined as,

$$u = \frac{\partial^2 \psi}{\partial x \partial y}$$
$$v = -\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2 \psi}{\partial x^2}\right] \tag{2.12}$$

The function ψ, satisfies Eq (2.10), and the only equation ψ has to satisfy is Eq. (2.11) and can be expressed as,

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{2.13}$$

So, the problem has been reduced to the solution of a single function ψ from the bi-harmonic differential equation (2.13). The Eq (2.13) are applicable only within the internal part of the elastic body. For the above solution the boundary conditions of equation (2.3) and (2.4) should be expressed in terms of ψ. The stress components σ_x, σ_y and σ_{xy} can be expressed in terms of ψ as,

$$\sigma_x = \left(\frac{E}{1-\mu^2}\right)\left(\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y}\right) = \left(\frac{E}{1+\mu^2}\right)\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu\frac{\partial^3 \psi}{\partial y^3}\right)$$
$$\sigma_x = \left(\frac{E}{1-\mu^2}\right)\left(\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x}\right) = \left(\frac{E}{1+\mu^2}\right)\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + (2+\mu)\frac{\partial^3 \psi}{\partial x \partial y^2}\right) \tag{2.14}$$
$$\sigma_{xy} = \frac{E}{2(1+\mu)^2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{E}{(1+\mu)^2}\left(\mu\frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial^3 \psi}{\partial x^3}\right)$$

From the solution of ψ, the values of stress and displacement component can be obtained from equations (2.12) and (2.14).

2.7 BOUNDARY EQUATIONS

In practical problems the boundary values are known in terms of normal and tangential components of displacements and stresses. The boundary equations for the known conditions on the arbitrarily shaped boundary are,

$$u_n = ul + vm$$
$$= l\frac{\partial^2 \psi}{\partial x \partial y} - m\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2 \psi}{\partial x^2}\right] \tag{2.15}$$

$$u_t = vl - um$$
$$= -l\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2\psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2\psi}{\partial x^2}\right] - m\frac{\partial^2\psi}{\partial x \partial y} \tag{2.16}$$

$$\sigma_n = \sigma_x l^2 + \sigma_y m^2 + 2\sigma_{xy} lm$$
$$= \frac{E}{(1+\mu)^2}\left[(l^2 - 2m^2 - \mu m^2)\frac{\partial^3\psi}{\partial x^2 \partial y} - (\mu l^2 + m^2)\frac{\partial^3\psi}{\partial x^3} + 2lm\left\{\mu\frac{\partial^3\psi}{\partial x \partial y^2} - \frac{\partial^3\psi}{\partial y^3}\right\}\right] \tag{2.17}$$

$$\sigma_t = \tau_{xy}(l^2 - m^2) + (\sigma_x - \sigma_y).lm$$
$$= \frac{(l^2 - m^2)E}{(1+\mu)^2}\left[\mu\frac{\partial^3\psi}{\partial x \partial y^2} - \frac{\partial^3\psi}{\partial y^3}\right] + \frac{lmE}{(1+\mu)}\left[\frac{\partial^3\psi}{\partial y^3} + \frac{\partial^3\psi}{\partial x^2 \partial y}\right] \tag{2.18}$$

2.8 SELECTION OF BOUNDARY CONDITIONS

There may be any two out of four quantities, namely u_n, u_t, σ_n and σ_t applicable at any boundary points. The possible known boundary conditions of a problem on the boundary are,

(a) (1) Normal displacement component, u_n
(2) Tangential displacement component, u_t
or,
(b) (1) Normal displacement component, u_n
(2) Normal stress component, σ_n
or,
(c) (1) Normal displacement component, u_n
(2) Tangential stress component, σ_t
or,
(d) (1) Tangential displacement component, u_t
(2) Tangential stress component, σ_t
or,
(e) (1) Tangential displacement component, u_t
(2) Normal stress component, σ_n
or,
(f) (1) Normal stress component, σ_n
(2) Tangential stress component, σ_t

Among the above six sets of boundary conditions (b) and (d) do not occur in practical problems. Therefore, at any point on the boundary, the remaining four possible sets of boundary conditions are,

(1) $\{u_n , u_t\}$
(2) $\{u_n , \sigma_t\}$
(3) $\{u_t , \sigma_n\}$ and, (2.19)
(4) $\{\sigma_n , \sigma_t\}$

2.9 CONCLUSION

A formulation for the solution of two-dimensional elastic problems has been briefly described in this chapter. The governing equation (2.13) is applied only on the inner grid points of the body. The boundary equations (2.15) to (2.18) are used on the boundary of the body and the boundary of the hole. The procedure to use the formula to solve a particular problem is described in the following chapter.

CHAPTER – 3
DESCRIPTION OF THE PROBLEM

3.1 INTRODUCTION

The investigation of the state of stress and deformation in the immediate vicinity of small hole in an elastic body is an important study in the field of Engineering. In such problems the distribution of stresses and displacement components sharply change around the hole. Study of such effect is necessary to design a structure. Most of the structural and machine components contain plates with internal hole. The shape and size of the plate and hole and the loading condition are different in different problems. In this thesis work a solution scheme has been developed in finite difference technique to solve such type of problem. Solution of a body with internal hole by finite difference technique is a new work and first attempted in this thesis work. The validity and acceptance of the scheme have been checked with the help of known solution. The details of the finite difference solution scheme are described in Chapter 4. A rectangular plate with a circular hole inside the body has been selected to solve by using the developed scheme. The hole is placed in the middle of the plate. The geometry of the plate and the hole is also known. The results obtained have been verified with the known solution. The same problem has been solved for different hole sizes and the effect of hole size on the distribution of stress and displacement has been analyzed. The detail descriptions of the results are given in Chapter 5. Now the selected problem is described in the following sections.

3.2 GEOMETRY OF THE PROBLEM AND LOADING CONDITION

To examine the accuracy of the method of solution in comparison to the available theoretical and experimental results a rectangular plate having internal circular hole has been solved under tensile loading. The geometry of the problem is shown in Fig. 3.1.

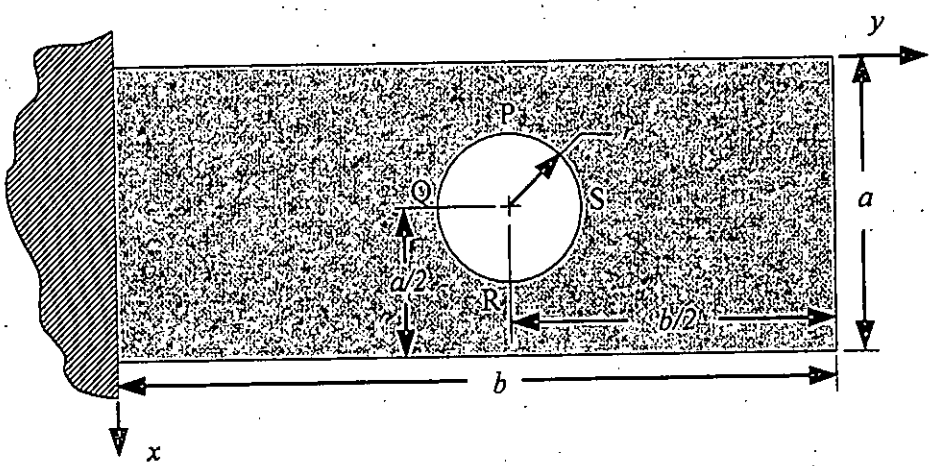


Fig. 3.1 Geometry of the problem

Where,

b = length of the body
a = width of the body
r = radius of circular hole
$b/a = 2.5$
r/a = 6/16, 7/16 and 8/16

The solution is known for $r/a = 8/16$

Although in this thesis work the problem of a rectangular elastic body with internal circular hole has been solved, this solution scheme is valid for solving any shape of body and hole. This solution technique is also valid for any type of loading condition and any combination of boundary conditions.

3.2.1 Material

The material taken assumed perfectly elastic and were given the properties of material with Poisson's ratio $\mu = 0.3$. Despite the choice, this procedure is also valid for any elastic material.

3.2.2 Boundary Conditions

The boundary AB as shown in Fig. 3.2 is considered rigidly fixed. So there will be no displacement in this part of the boundary and thus the boundary conditions are set as u_n=0.0, u_t=0.0. At the right boundary (CD) uniform tensile load is applied. For this boundary for every nodal points the boundary conditions are set as $\sigma_n/E = 3\times10^{-4}$, $\sigma_t/E = 0.0$, where the symbol E denotes modulus of elasticity.

The top and bottom boundaries (boundary AD and BC) are free from stress. The boundary conditions for the free boundary are set as $\sigma_n/E = 0.0$, $\sigma_t/E = 0.0$. The surface of the internal hole is free from any external load; boundary conditions are, therefore, assigned as $\sigma_n/E = 0.0$, $\sigma_t/E = 0.0$.

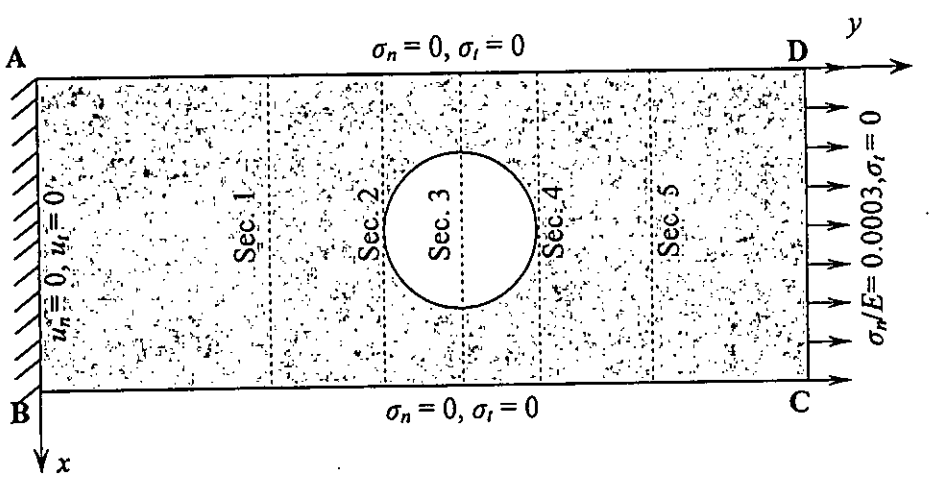


Fig 3.2: Boundary Conditions applied for the problem

For the above problems distribution of stress and displacement components are obtained from the output of the computer program. From that result stresses and displacements of five different sections (as shown in Fig 3.2) are analyzed to study the effect of hole in plate.

3.3 THEORETICAL SOLUTION

Figure 3.3 represents a plate submitted to a uniform tension of magnitude S in the x' direction. If a small circular hole is made in the middle of the plate, the stress distribution in the neighborhood of the hole will be changed [1].

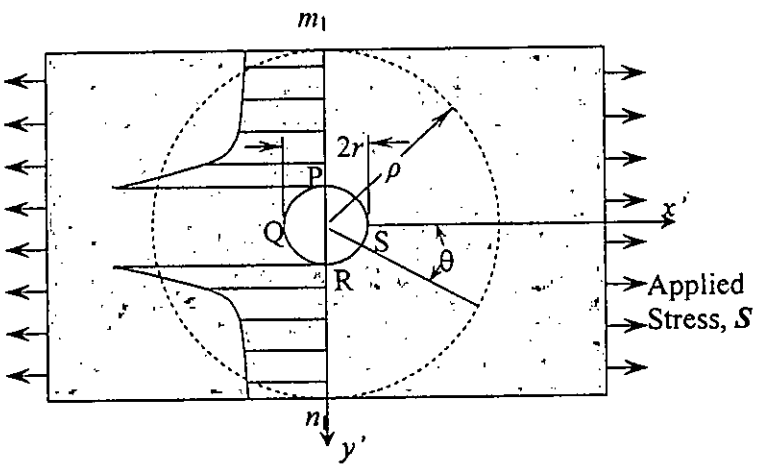


Fig. 3.3 Stress analysis of plate with internal hole

The stress component for the above problem can be written as

$$\sigma_\rho = \frac{S}{2}\left(1-\frac{r^2}{\rho^2}\right)+\frac{S}{2}\left(1+\frac{3r^4}{\rho^4}-\frac{4r^2}{\rho^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{S}{2}\left(1+\frac{r^2}{\rho^2}\right)-\frac{S}{2}\left(1+\frac{3r^4}{\rho^4}\right)\cos 2\theta \qquad (3.1)$$

$$\tau_{\rho\theta} = -\frac{S}{2}\left(1-\frac{3r^4}{\rho^4}+\frac{2r^2}{\rho^2}\right)\sin 2\theta$$

The above analytical solution has been well confirmed many times by strain measurements and by photoelastic method. The displacements u, v can be found from these, using the following Eqs.

$$\varepsilon_\rho = \frac{\partial u}{\partial \rho}$$

$$\varepsilon_\theta = \frac{u}{\rho}+\frac{\partial v}{\rho\partial\theta} \qquad (3.2)$$

$$\gamma_{\rho\theta} = \frac{\partial u}{\rho\partial\theta}+\frac{\partial v}{\partial\rho}-\frac{v}{\rho}$$

$$\varepsilon_\rho = \frac{1}{E}(\sigma_\rho - \mu\sigma_\theta)$$
$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_\rho) \qquad (3.3)$$
$$\gamma_{\rho\theta} = \frac{1}{G}\tau_{\rho\theta}$$

If ρ is very large, the values of σ_ρ and $\tau_{\rho\theta}$ are effectively the same as in the plate without the hole and are therefore given by,

$$\sigma_\rho = \tfrac{1}{2}S(1+\cos 2\theta)$$
$$\tau_{\rho\theta} = -\tfrac{1}{2}S\sin 2\theta \qquad (3.4)$$

At the edge of the hole, for $\rho = r$

$$\sigma_\rho = \tau_{\rho\theta} = 0$$
$$\sigma_\theta = S - 2S\cos 2\theta \qquad (3.5)$$

It can be seen that σ_θ is greatest when $\theta = \pi/2$ or $3\pi/2$, that is at the end P and R of the diameter perpendicular to the direction of the tension as shown in Fig. 3.3. At these points $\sigma_\theta = 3S$. At points Q and S, when $\theta = \pi$ or 0, $\sigma_\theta = -S$. So, there is a compressive stress in the tangential direction at these points.

The case of circular hole near the straight boundary of a semi-infinite plate under tension parallel to this boundary was analyzed by G.B. Jeffery [1]. A corrected result and a comparison with photoelastic tests were given later by R.D. Mindlin [1]. The stress at the hole, at the point P nearest the edge, becomes a very large multiple of the undisturbed tensile stress when the distance of hole boundary from the outer boundary is small compared with the diameter of the hole [1].

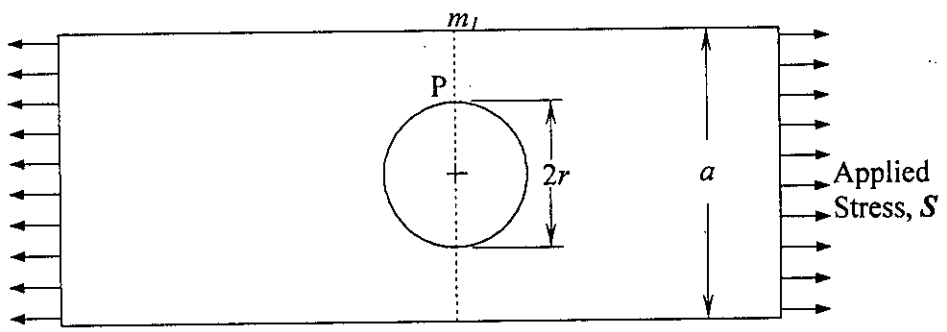


Fig 3.4

For the case, when the plate width is finite and a circular hole is present on the axis of symmetry as shown in Fig. 3.4, it is found that when $2r = 0.5a$, $\sigma_\theta = 4.3S$ at the point P and $\sigma_\theta = 0.75S$ at the point m_1 [1]. The experimental results presented here for the above particular case will be compared later with our numerical solutions.

3.4 CONCLUSION

The elastic problem described above has been studied by some researchers and their results are given above. Most of the solutions are analytical and the shape of the body and the hole are not arbitrary. Numerical solution is necessary to solve this type of elastic problem with arbitrary shape and for different loading conditions. To solve the above problem numerically by finite difference technique we need to develop finite difference expressions for the fourth order differential equations of the governing equation and second and third order differential equations of the boundary conditions. These will be described in the next chapter.

CHAPTER – 4
FORMULATIONS AND NUMERICAL SOLUTION OF THE PROBLEM

4.1 INTRODUCTION

Finite difference method is well known for numerical solution of differential equations. In this method, the related differential equation along with its boundary conditions is transformed into a set of algebraic equations. To do this, the whole field under consideration is divided into a number of rectangular meshes. The difference expressions for the differential equation applicable for every mesh points are obtained by using Taylor's series. In this thesis work, the governing differential equation (2.13) and the boundary conditions (2.15) to (2.18) are transformed into difference expressions, and hence a system of algebraic equations is obtained. Finite difference forms of the derivation of different order are described in the following sections.

4.2 FINITE DIFFERENCE FORMULAE OF THE DERIVATIVES

It will be appropriate to derive the finite difference formulae of the different derivatives to show the underlying principle as well as to obtain the truncation error of these expressions. To illustrate the derivation of the difference equations, a general pivotal point at co-ordinate (x, y) is taken and its neighboring points are designed at a finite distances. The neighboring points are at distances h and k in the x and y directions respectively. Assuming that a certain function $\psi(x, y)$ has continuous partial derivative of considerably higher order at (x, y) the value of the function at point $(x+h, y)$ can be written by Taylor's series expansion in terms of the values of the function and its derivatives at (x, y) as

$$f(x+h,y) = f(x,y) + hf_x(x,y) + \frac{h^2}{2!} f_{xx}(x,y) + \frac{h^3}{3!} f_{xxx}(x,y) + \ldots\ldots \tag{4.1}$$

The function at point $(x-h, y)$ can similarly be written as,

$$f(x-h,y) = f(x,y) - hf_x(x,y) + \frac{h^2}{2!} f_{xx}(x,y) - \frac{h^3}{3!} f_{xxx}(x,y) + \ldots\ldots \tag{4.2}$$

Subtracting one equation from other we get,

$$f_x(x,y) = \left(\frac{\partial f}{\partial x}\right)_{x,y} = \frac{f(x+h,y) - f(x-h,y)}{2h} + O(h^2)$$

or,

$$\left(\frac{\partial f}{\partial x}\right)_{i,j} = \frac{f(i+1,j) - f(i-1,j)}{2h} + O(h^2) \tag{4.3}$$

Adding the same equations we get,

$$f_{xx}(x,y)=\left(\frac{\partial^2 f}{\partial x^2}\right)_{x,y}=\frac{f(x+h,y)-2f(x,y)+f(x-h,y)}{h^2}+O(h^2)$$

or,

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j}=\frac{f(i+1,j)-2f(i,j)+f(i-1,j)}{h^2}+O(h^2) \tag{4.4}$$

Here $h=\Delta x$ and (i,j) stands for x and y respectively in such a way that $i+1 = x+h$ and $i-1 = x-h$, and so on. The term $O(h^2)$ indicates that the error in the omission of the term is of order of h^2.

The two point forward and backward difference formulae of first order derivative can be obtained from equations (4.1) and (4.2) respectively as,

$$\left(\frac{\partial f}{\partial x}\right)_{i,j}=\frac{f(i+1,j)-f(i,j)}{h}+O(h) \tag{4.5}$$

$$\left(\frac{\partial f}{\partial x}\right)_{i,j}=\frac{f(i,j)-f(i-1,j)}{h}+O(h) \tag{4.6}$$

Now if we take a Taylor series expansion for $x \pm 2h$, we obtain

$$f(x+2h,y)=f(x,y)+2hf_x(x,y)+2h^2 f_{xx}(x,y)+\frac{4h^3}{3}f_{xxx}(x,y)+\frac{2h^4}{3}f_{xxxx}(x,y)+\ldots \tag{4.7}$$

$$f(x-2h,y)=f(x,y)-2hf_x(x,y)+2h^2 f_{xx}(x,y)-\frac{4h^3}{3}f_{xxx}(x,y)+\frac{2h^4}{3}f_{xxxx}(x,y)+\ldots \tag{4.8}$$

If we combine these new equations with the previous equations, it is seen that the central difference formulas for first and second order derivatives are,

$$\left(\frac{\partial f}{\partial x}\right)_{i,j}=\frac{-f(i+2,j)+8f(i+1,j)-8f(i-1,j)+f(i-2,j)}{12h}+O(h^4) \tag{4.9}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j}=\frac{-f(i+2,j)+16f(i+1,j)-30f(i-1,j)+f(i-2,j)}{12h^2}+O(h^4) \tag{4.10}$$

Again combining equation (4.1) and (4.7), one can get three point forward difference equations for first and second order derivations as,

$$\left(\frac{\partial f}{\partial x}\right)_{i,j}=\frac{-3f(i,j)+4f(i+1,j)-f(i+2,j)}{2h}+O(h^2) \tag{4.11}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} = \frac{f(i,j) - 2f(i+1,j) - f(i+2,j)}{2h} + O(h) \tag{4.12}$$

Similarly, from equation (4.2) and (4.8), the three point backward difference equations are,

$$\left(\frac{\partial f}{\partial x}\right)_{i,j} = \frac{3f(i,j) - 4f(i-1,j) + f(i-2,j)}{2h} + O(h^2) \tag{4.13}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} = \frac{f(i-2,j) - 2f(i-1,j) + f(i,j)}{2h} + O(h) \tag{4.14}$$

The finite difference equations for higher order derivatives can be obtained by using the above expressions, as follows:

$$\left(\frac{\partial^3 f}{\partial x^3}\right)_{i,j} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j}$$

$$= \left(\frac{\partial z}{\partial x}\right)_{i,j}, \qquad \text{where } z = \frac{\partial^2 f}{\partial x^2}$$

$$= \frac{z(i+1,j) - z(i-1,j)}{2h}$$

$$= \frac{f(i+2,j) - 2f(i+1,j) + 2f(i-1,j) - f(i-2,j)}{2h^2} + O(h^2) \tag{4.15}$$

Similarly, a forward difference expression for the same derivative can be written as,

$$\left(\frac{\partial^3 f}{\partial x^3}\right)_{i,j} = \frac{-3z(i,j) - 4z(i+1,j) - z(i+2,j)}{2h}$$

$$= \frac{f(i+2,j) - 2f(i+1,j) + 2f(i-1,j) - f(i-2,j)}{2h^3} + O(h^2) \tag{4.16}$$

and the corresponding backward difference expression,

$$\left(\frac{\partial^3 f}{\partial x^3}\right)_{i,j} = \frac{3f(i+1,j) - 10f(i,j) + 12f(i-1,j) - 6f(i-2,j) + f(i-3,j)}{2h^3} + O(h^2) \tag{4.17}$$

For the case of a fourth order derivative, the central difference expression can be written as,

$$\left(\frac{\partial^4 f}{\partial x^4}\right)_{i,j} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j}$$

$$= \left(\frac{\partial^2 z}{\partial x^2}\right)_{i,j}, \qquad \text{where } z = \frac{\partial^2 f}{\partial x^2}$$

$$= \frac{z(i+1,j) - 2z(i,j) + z(i-1,j)}{h^2}, \text{ or}$$

$$\left(\frac{\partial^4 f}{\partial x^4}\right)_{i,j} = \frac{f(i+2,j) - 4f(i+1,j) + 6f(i,j) - 4f(i-1,j) + f(i-2,j)}{2h^4} + O(h^2) \quad (4.18)$$

The central difference expressions for fourth order cross derivatives can also be obtained in the similar manner, and the expressions is

$$\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)_{i,j} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\right)_{i,j}$$

$$= \left(\frac{\partial^2 z}{\partial x^2}\right)_{i,j}, \text{ where, } z = \frac{\partial^2 f}{\partial y^2}$$

$$= \frac{z(i+1,j) - 2z(i,j) + z(i-1,j)}{h^2}$$

$$= \frac{1}{h^2 k^2}[f(i+1,j+1) + f(k+1,j-1) + f(i-1,j+1) + f(i-1,j-1) \quad 4.17$$
$$+ 4f(i,j) - 2\{f(i+1,j) + f(i,j+1) + f(i,j-1) + f(i-2,j)\}] + O(h^2)$$

4.3 SIMULATION OF THE FINITE DIFFERENCE EQUATIONS IN RECTANGULAR FIELD GRIDS AND THE SOLUTION TECHNIQUE

In finite difference method, the region in which the dependent function is to be evaluated is divided into a desirable number of mesh or nodal points and the value of the function is sought only at those mesh points. The division into mesh points can be done in any regular or irregular manner, but considering the fact of solving the problem by a computer, regularity is desirable. Whether the problem is to solve a function within geometrically irregular region, it is divided into meshes with lines parallel to the rectangular co-ordinate axes, i.e., rectangular mesh/grid points are used here. So, the physical boundary may not match with the rectangular grid points. The physical problems are associated with the known boundary conditions at the boundary points. This requires a further treatment to relate the values on the boundary with the field grid points. The boundary of a problem is divided into four segments, namely (i) top-left, (ii) bottom-left, (iii) bottom-right, and (iv) bottom-left. For denoting any boundary grid points, a reference field grid point is used. For example, for a boundary grid point A, in Fig. 4.1 the reference field grid point is Q for outer boundary and T for internal-hole boundary. If any physical boundary grid point matches with the field grid point, like L, M, N of Fig. 4.1 then the boundary point itself is a reference grid point. The boundary value of point A can be found by linear interpolation of the four neighboring points P, Q, R, T. If a boundary point falls on field

grid point like L, M, N then no interpolation will be required. The boundary values of points like D can be found by linear interpolation of two points E, F.

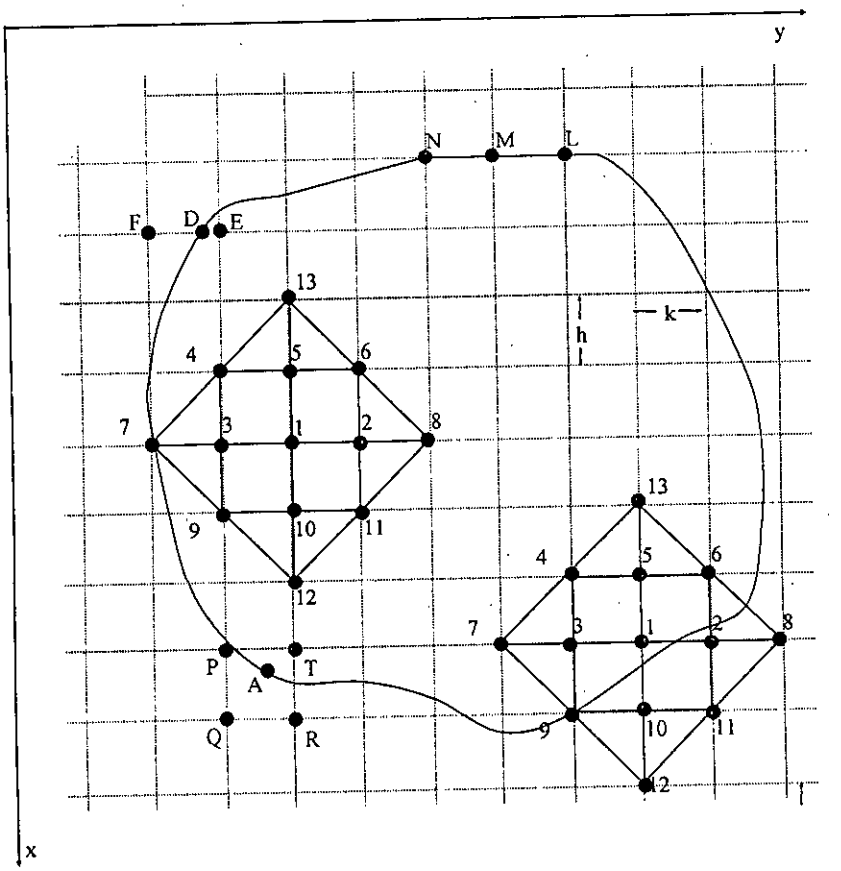


Fig 4.1 Physical Boundary of the body

The physical boundary points are selected in such a way that for each boundary point, there must be a unique reference field grid point. Having decided upon the pattern and the total number of mesh points, we must provide a sufficient number of equations to solve for all the "discretized variables" (the values of the function at the mesh points) that results from the division of the domain into mesh points. For an interior mesh point 1, the bi-harmonic equation in terms of ψ applied to this point will give rise to a single algebraic equation and, therefore, the single unknown concerning this point has been provided with a single equation for its evaluation. This algebraic equation will contain the discretized variable of the thirteen neighboring mesh points from 1 to 13 in Fig. 4.1, provided all the derivatives present in the bi-harmonic equation are replaced by their respective central difference formulae. This implies that when '1' becomes an immediate neighboring of the boundary mesh points, this equation will contain mesh points both interior and exterior to the boundary, as shown in Fig. 4.1. Among the exterior points some would be the boundary reference field grids and remaining would be the extra points, other than the reference grid points. Again the boundary conditions in terms of stress and displacement components contain 2nd and 3rd order derivative of ψ (Eqs. 2.15 to 2.18) and the application of the boundary conditions at an arbitrary point on the boundary will not be very easy without the involvement of exterior mesh points to the physical boundary of the domain concerned (Fig.A5 in annexure A). Considering an arbitrary point on the boundary, the boundary conditions may be specified as any one of

the four groups of equations (2.19). Therefore since the functions are not directly specified, there are always two conditions to be satisfied at an arbitrary point on the boundary and these two conditions are theoretically sufficient to determine two equations at this point. In this respect, if the boundary conditions are given either in terms of displacement or stress components, that is, in the form of differential equations of unknown function, these differential equations have to be expressed into difference equations.

So, one exterior field grid point is necessary to use central difference expressions of the bi-harmonic equation to the points in the immediate neighborhood of the physical boundary and to apply the finite difference expressions of the differential equations associated with the boundary conditions applied to the boundary mesh points. For this an imaginary boundary grid line is used exterior to the reference boundary line, and this boundary line is known as false boundary line.

Central difference expressions for the differential equations associated with the boundary conditions containing second and third order derivatives of the functions ψ can not be used, because most of the time points exterior to the false boundary are required. So, considering the position of the boundary points three-point forward or backward difference expressions are used so that no point exterior to the false boundary is needed. Two-point backward or forward difference expressions are not used due to higher error level (of the order h^2).

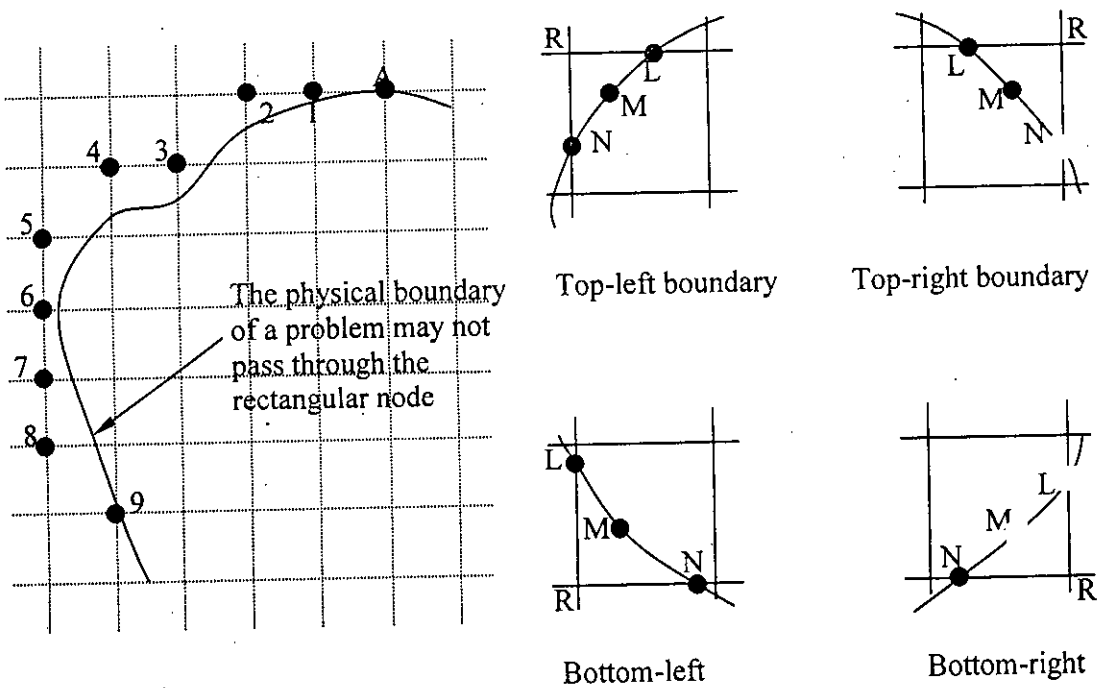
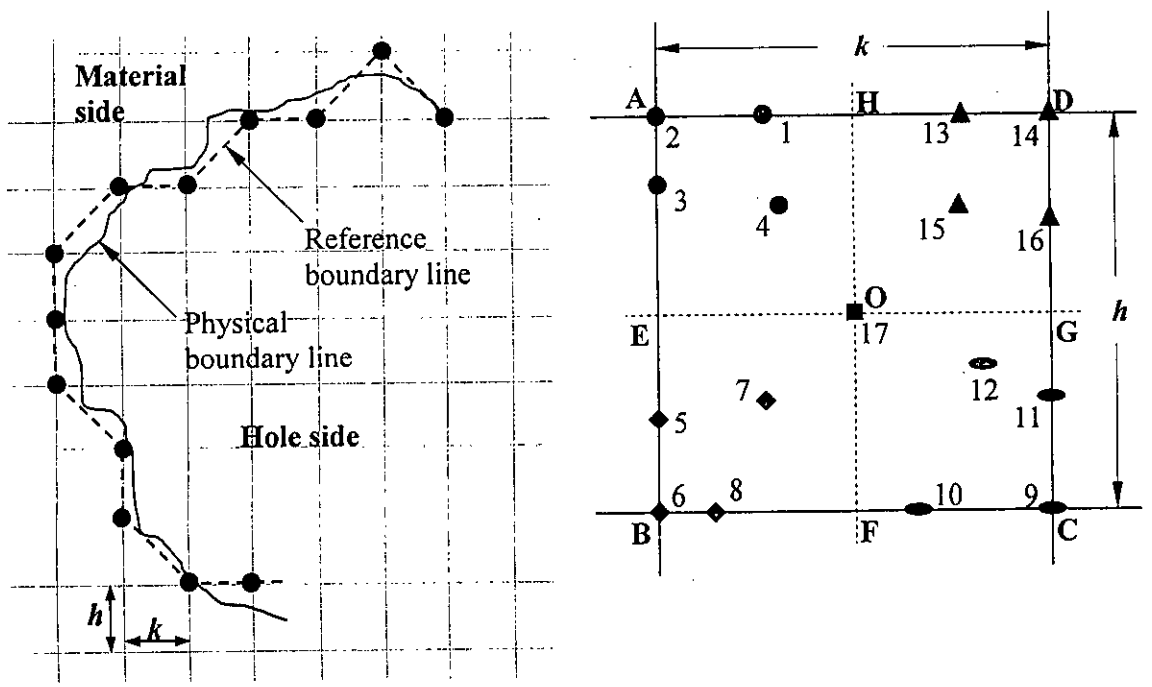


Fig 4.2 Boundary Management for outer boundary

The physical field is divided into rectangular meshes by lines parallel to co-ordinate axes x, y.

The management of external field boundary is shown in Fig. 4.2. It has been described that, if any physical boundary point matches with a rectangular grid point, then that grid point will be the reference boundary points for that boundary point. For any physical grid point not matching with the rectangular grid point, the exterior grid points, which is closer to the boundary (1,2,3,4,5,6,7,8,9) will be the reference point. As for each physical boundary points there should be only one reference grid point. For reference point R, the different possible cases of physical boundary segments are shown in Fig. 4.2(b), only one of the points denoted by L, M and N should be selected as physical boundary point.

The management of hole-boundary is shown in Fig. 4.3. The physical (actual) boundary line is shown by solid line and reference boundary line is shown by dotted line. Similar to the external boundary, if any boundary point matches with any rectangular grid point, then that grid point will be the reference boundary point for that boundary point. If any point does not match with any rectangular grid point then the nearest rectangular grid point will be the reference boundary point for that boundary point. To explain the selection of reference grid points a rectangular mesh ABCD is shown in Fig 4.3(b). Where AB=CD=h and AD=BC=k. The mesh is divided into four sections. If any boundary grid point do not match any rectangular grid point but lies in the zone AEOH (like points 1, 3 and 4) then the corresponding reference grid point will be 'A'. Similarly if any point lies in the zone EBFO (like 5, 8 and 7) the corresponding reference grid point will be 'B', if lies in the zone FCGO (like 10, 11and 12) then the corresponding reference grid point will be 'C' and if the point lies in the zone GDHO (like 13, 15 and 16) the corresponding reference grid point will be 'D'. If any boundary point lies in the middle of a mesh like point O then the corresponding reference grid point for that boundary points will be 'A'.



(a) Physical and reference boundary

(b) boundary points and corresponding reference grid points

Fig. 4.3 Hole-Boundary Management

As there are always two conditions to be satisfied at an arbitrary point on the physical boundary of the domain, two finite difference expressions of the differential equations associated with the boundary conditions are applied to the corresponding points on reference and false boundaries. A major problem is faced in formulating points on the sharp turning boundary, like the corners of the rectangle, where the total points on the reference boundary are not equal to the number of points on the false boundary as shown in Fig. 4.4.

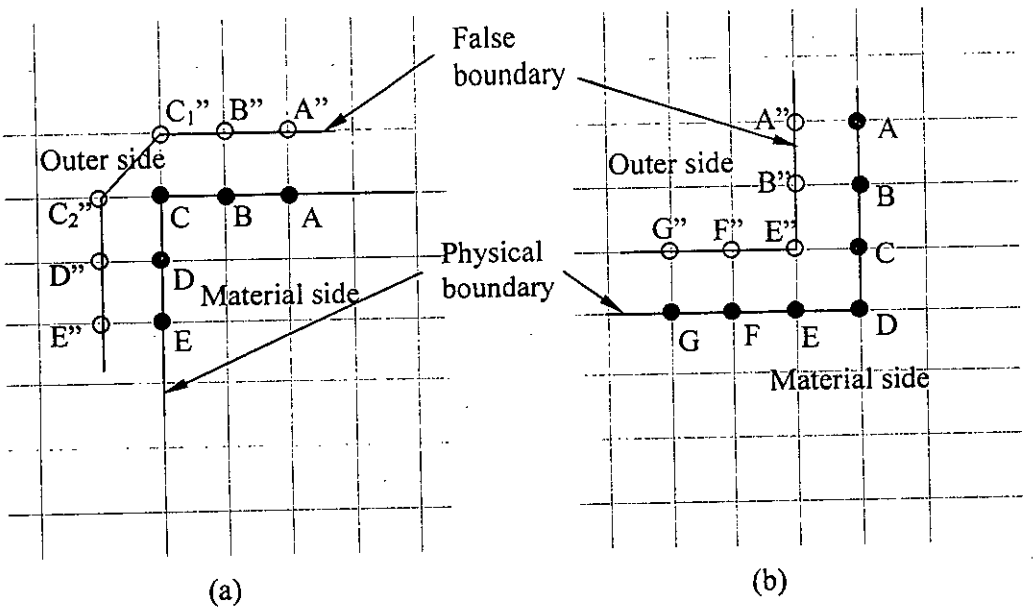


Fig 4.4 Boundary Management for sharp turning boundary

In Fig. 4.4 as the physical boundary passes through the rectangular mesh points, physical boundary and reference boundary are same line. In Fig. 4.4(a) reference boundary line is represented by line A-B-C-D-E and the corresponding false boundary line is A"-B"-C_1"-C_2"-D"-E". For physical grid point C there are two points (C_1" and C_2") on the false boundary. Each boundary point has two boundary conditions. One is used for point on reference boundary and one for point on false boundary. But here for points like C there are three points one on reference boundary (C) and two on false boundary (C_1" and C_2"). But for point C only two conditions are known. So, for points like C, number of unknown exceeds the number of available equations. The problem can be resolved by using (i) an additional boundary condition, (ii) by assigning an arbitrary value of the unknown function ψ at one of the three points, or (iii) omitting one of the two points (C_1" or C_2"). Among the above options, (iii) is not possible, because omitting of C_1" or C_2" restrict the use of finite difference expression for boundary value of point C. Option (ii) is applicable for a maximum of three similar points. Because, the function ψ indicates a surface and if the surface rotates as a whole or translate as a whole or tilts as a whole, then the configuration of the surface would not changed, thus desired results that depends only on the configuration of the surface would be the same. Option (i) can be satisfied by taking an additional boundary condition anywhere on the segment intercepted by the involved mesh boundary.

Some times the number of boundary conditions may be more than that of the available points where they can be applied. Figure 4.4(b) shows such a problem. For points C and

D there is no corresponding points on the false boundary. Here for points C and D the number of unknowns is less than the number of conditions to be satisfied, which is impractical. Here the excess conditions are declared redundant.

4.4 FINITE DIFFERENCE FORMS OF THE BI-HARMONIC EQUATION AND THE BOUNDARY EQUATIONS

The governing equation of elasticity in terms of displacement potential function can be written as,

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0$$

For a mesh point (i, j) the central difference expressions in finite difference method can be written as,

$$\frac{1}{h^4}[\psi(i-2,j)-4\psi(i-1,j)+6\psi(i,j)-4\psi(i+1,j)+\psi(i+2,j)]+$$
$$\frac{1}{h^2k^2}\begin{bmatrix}\psi(i+1,j+1)+\psi(i+1,j-1)+\psi(i-1,j+1)+\psi(i-1,j-1)+4\psi(i,j)\\-2\{\psi(i+1,j)+\psi(i,j+1)+\psi(i,j-1)+\psi(i-1,j)\}\end{bmatrix}+$$
$$\frac{1}{k^4}[\psi(i,j-2)-4\psi(i,j-1)+6\psi(i,j)-4\psi(i,j+1)+\psi(i,j+2)]=0$$

Multiplying both sides by k^4 and putting $k/h = r$ we get,

$$\begin{aligned}&r^4\{\psi(i-2,j)-4\psi(i-1,j)+6\psi(i,j)-4\psi(i+1,j)+\psi(i+2,j)\}\\&+2r^2\{\psi(i+1,j+1)+\psi(i+1,j-1)+\psi(i-1,j+1)+\psi(i-1,j-1)+4\psi(i,j)\\&-2\psi(i+1,j)-2\psi(i,j+1)-2\psi(i,j-1)-2\psi(i-1,j)\}+\{\psi(i,j-2)-4\psi(i,j-1)\\&+6\psi(i,j)-4\psi(i,j+1)+\psi(i,j+2)\}=0\end{aligned}\qquad(4.18)$$

The above equation is the difference approximation to the bi-harmonic partial differential equation and valid for all internal mesh points except for those on the boundary of hole and the plate. The structure of the above expressions can be found in Annexure A.

The concerned boundary conditions are given by equations (2.15) to (2.18),

$$u_n=l\frac{\partial^2\psi}{\partial x\partial y}-m\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2\psi}{\partial y^2}+\left(\frac{2}{1+\mu}\right)\frac{\partial^2\psi}{\partial x^2}\right]$$
$$u_t=-l\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2\psi}{\partial y^2}+\left(\frac{2}{1+\mu}\right)\frac{\partial^2\psi}{\partial x^2}\right]-m\frac{\partial^2\psi}{\partial x\partial y}$$
$$\sigma_n=\frac{E}{(1+\mu)^2}\left[(l^2-2m^2-\mu m^2)\frac{\partial^3\psi}{\partial x^2\partial y}-(\mu l^2+m^2)\frac{\partial^3\psi}{\partial x^3}+2lm\left\{\mu\frac{\partial^3\psi}{\partial x\partial y^2}-\frac{\partial^3\psi}{\partial y^3}\right\}\right]$$
$$\sigma_t=\frac{(l^2-m^2)E}{(1+\mu)^2}\left[\mu\frac{\partial^3\psi}{\partial x\partial y^2}-\frac{\partial^3\psi}{\partial y^3}\right]+\frac{lmE}{(1+\mu)}\left[\frac{\partial^3\psi}{\partial y^3}+\frac{\partial^3\psi}{\partial x^2\partial y}\right]$$

Central difference expressions for the above boundary conditions can not be used due to the fact that these expressions need mesh points exterior to the false grid line. So, forward, backward or a combination of forward and backward expressions are necessary for different sections of boundary points. Considering this, the boundary of the hole and the body are divided into four sections namely (i) top-left, (ii) bottom-left, (iii) bottom-right and (iv) top-right boundary.

For top-left boundary the final form of the finite difference expressions for boundary equations are,

$$u_n = l\frac{\partial^2\psi}{\partial x\partial y} - m\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2\psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2\psi}{\partial x^2}\right]$$

$$\begin{aligned}\therefore u_n(i,j) &= (9s1.l - m.s2).\psi(i,j) + zk7.s2.m.\psi(i-1,j) + zk6.m.s2.\psi(i,j-1)\\ &+ (zk6.m.s2 - 12s1.l).\psi(i,j+1) + 3s1.l.\psi(i,j+2) + 16.s1.l.\psi(i+1,j+1)\\ &+ (zk7.m.s2 - 12.s1.l).\psi(i+1,j) - 4.s1.l.\psi(i+1,j+2) + 3.s1.l.\psi(i+2,j)\\ &- 4.s1.l.\psi(i+2,j+1) + s1.l.\psi(i+2,j+2)\end{aligned} \tag{4.19}$$

$$u_t = -l\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2\psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2\psi}{\partial x^2}\right] - m\frac{\partial^2\psi}{\partial x\partial y}$$

$$\begin{aligned}\therefore u_t(i,j) &= (-9s1.m - l.s2).\psi(i,j) + zk7.s2.l.\psi(i-1,j) + zk6.l.s2.\psi(i,j-1)\\ &+ (zk6.l.s2 + 12s1.m).\psi(i,j+1) - 3s1.m.\psi(i,j+2) - 16.s1.m.\psi(i+1,j+1)\\ &+ (zk7.l.s2 + 12.s1.m).\psi(i+1,j) + 4.s1.m.\psi(i+1,j+2) - 3.s1.m.\psi(i+2,j)\\ &+ 4.s1.m.\psi(i+2,j+1) - s1.m.\psi(i+2,j+2)\end{aligned} \tag{4.20}$$

$$\sigma_n = \frac{E}{(1+\mu)^2}\left[(l^2 - 2m^2 - \mu m^2)\frac{\partial^3\psi}{\partial x^2\partial y} - (\mu l^2 + m^2)\frac{\partial^3\psi}{\partial x^3} + 2lm\left\{\mu\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{\partial^3\psi}{\partial y^3}\right\}\right]$$

$$\begin{aligned}\therefore \sigma_n(i,j) &= (6zk9.R.s3.l.m - 3zk9.s3.l^2 - 3zk10.s4.m^2)\psi(i-1,j)\\ &+ (4zk9.s3.l^2 + zk10.s4.m^2)\psi(i-1,j+1) + (-zk9.s3.l^2 - zk10.s4.m^2)\psi(i-1,j+2)\\ &+ 1.5(s3.l^2 - s4.m^2 - 2lm.R.s3)\psi(i,j-1)\\ &+ \{s4.m^2(5 + 6zk10) + s3.l^2(6zk9 - 5) + 2R.s3.lm(3 - 10zk9)\}\psi(i,j)\\ &+ \{s3.l^2(6 - 8zk9) - 3R.s3.lm - s4.m^2(6 + 8zk10)\}\psi(i,j+1)\\ &+ \{s4.m^2(3 + 2.zk10) + s3.l^2(2zk9 - 3)\}\psi(i,j+2) 0.5(s3.l^2 - s4.m^2)\psi(i,j+3)\\ &+ 4R.s3.lm.\psi(i+1,j-1) + \{2R.s3.lm(12.zk9 - 4) - 3(zk9.s3.l^2 + zk10.s4.m^2)\}\psi(i+1,j)\\ &+ 4(R.s3.lm + zk9.s3.l^2 + zk10.s4.m^2)\psi(i+1,j+1)\\ &+ (-zk9.s3.l^2 - zk10.s4.m^2)\psi(i+1,j+2) + (-R.s3.lm)\psi(i+2,j-1)\\ &+ 2R.s3.lm(1 - 6zk9)\psi(i+2,j) + (-R.s3.lm)\psi(i+2,j+1) + 2zk9.R.s3.lm\psi(i+3,j)\end{aligned} \tag{4.21}$$

$$\sigma_t = \frac{(l^2 - m^2)E}{(1+\mu)^2}\left[\mu\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{\partial^3\psi}{\partial y^3}\right] + \frac{lmE}{(1+\mu)}\left[\frac{\partial^3\psi}{\partial y^3} + \frac{\partial^3\psi}{\partial x^2\partial y}\right]$$

$$\therefore \sigma_n(i,j) = \{3lm(-zk10.s4 + zk9.s3) - 3zk9.R.s3(m^2 - l^2)\}\psi(i-1,j)$$

$$
\begin{aligned}
&+4lm(-zk9.s3+zk10.s4)\psi(i-1,j+1)+lm(zk9.s3-zk10.s4)\psi(i-1,j+2)\\
&+\{1.5.R.s3(m^2-l^2)-1.5lm(s3+s4)\}\psi(i,j-1)\\
&-\{s3.lm(6zk9-5)-s4.lm(5+6zk10)+R.s3(m^2-l^2)(3-10.zk9)\}\psi(i,j)\\
&-\{s3.lm(6-8.zk9)-s4.lm(6+8.zk10)-(1.5).R.s3(m^2-l^2)\}\psi(i,j+1)\\
&-\{s3.lm(2zk9-3)-s4.lm.(3+2.zk10)\}\psi(i,j+2)-0.5lm(s3+s4)\psi(i,j+3)\\
&-2R.s3(m^2-l^2)\}\psi(i+1,j-1)\\
&-\{3lm(s4.zk10-s3.zk9)+(12zk9-4)R.s3((m^2-l^2)\}\psi(i+1,j)\\
&-\{4lm(s3.zk9-s4.zk10)+2.R.s3((m^2-l^2)\}\psi(i+1,j+1)\\
&-\{lm(s4.zk10-s3.zk9)\}\psi(i+1,j+2)-\{-0.5R.s3((m^2-l^2)\}\psi(i+2,j-1)\\
&-\{(1-6zk9)R.s3(m^2-l^2)\}\psi(i+2,j)-\{-0.5R.s3((m^2-l^2)\}\psi(i+2,j+1)\\
&-zk9.R.s3(m^2-l^2)\psi(i+3,j)
\end{aligned}
\qquad (4.22)
$$

Where,

r or R $= k/h$

$zk1 = r^4$

$zk2 = 4(r^4+r^2)$

$zk3 = 4(1+r^2)$

$zk4 = 6+8r^2+6r^4$

$zk5 = 2r^2$

$$zk6 = \frac{1-\mu}{2(1-\mu+2/r^2)}$$

$$zk7 = \frac{r^2}{2(1-\mu+2/r^2)}$$

$$zk8 = \frac{r^2h^2/(1+\mu)}{2(1-\mu+2/r^2)}$$

$zk9 = r^2/2\mu$

$$zk10 = \frac{r^2}{2(2+\mu)}$$

$$P2 = \frac{r^3h^3}{(1+\mu).2\mu}$$

$s1 = 1.0/(4rh^2)$

$s2 = -1.0/zk8$

$s3 = -1.0/P2$

$s4 = -1.0/\mu P2$

The formulae structure for the above expressions are shown in Annexure A. The computer code for handling such expressions are also shown in Annexure B.

The same set of expressions can be used to other three boundaries with very small editing which are being described in the following sections.

The above expressions are applicable for a boundary points those match with rectangular grid points. But if a boundary point does not match with a rectangular grid point like

point P as shown in Fig. 4.5 a new formula is used which is developed by interpolating the weight of four neighboring points A, B, C and D. If the reference point corresponding to P is A for outer boundary or hole boundary then the locations of P with respect to A can be described by a, s, b and t.
a=PF, b = PH, s=PE, t=PG.

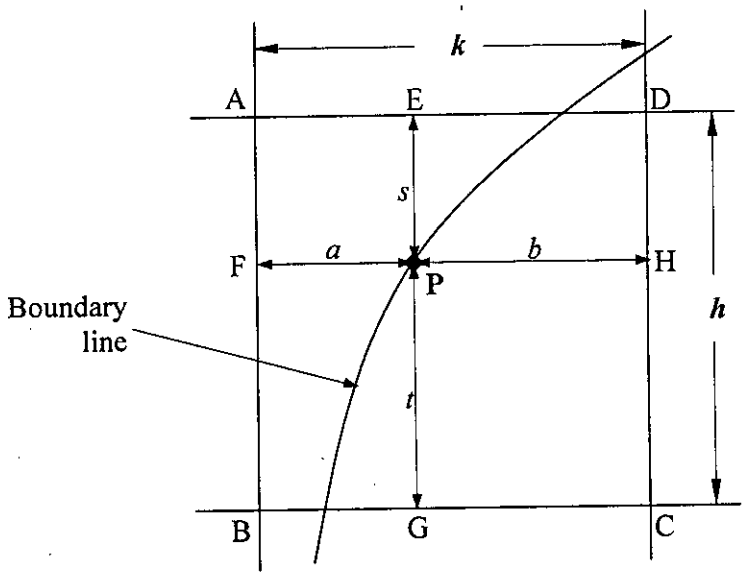


Fig 4.5 Locator of a, b, s and t of the boundary point P with respect to reference point A

If B is the reference point corresponding to boundary point P then
a=PF, b = PH, s=PG, t=PE.
If C is the reference point corresponding to boundary point P then
a=PH, b = PF, s=PG, t=PE.
If C is the reference point corresponding to boundary point P then
a=PH, b = PF, s=PE, t=PG

The following expressions are used for the calculation of u_n, u_t, σ_n and σ_t for points like P.

$$u_n(at\ P) = \frac{1}{s+t}\left[s.\frac{au_n(i+1,j+1)+bu_n(i+1,j)}{a+b} + t.\frac{au_n(i,j+1)+bu_n(i,j)}{a+b}\right] \quad (4.23)$$

$$u_t(at\ P) = \frac{1}{s+t}\left[s.\frac{au_t(i+1,j+1)+bu_t(i+1,j)}{a+b} + t.\frac{au_t(i,j+1)+bu_t(i,j)}{a+b}\right] \quad (4.24)$$

$$\sigma_n(at\ P) = \frac{1}{s+t}\left[s.\frac{a\sigma_n(i+1,j+1)+b\sigma_n(i+1,j)}{a+b} + t.\frac{a\sigma_n(i,j+1)+b\sigma_n(i,j)}{a+b}\right] \quad (4.25)$$

$$\sigma_t(at\ P) = \frac{1}{s+t}\left[s.\frac{a\sigma_t(i+1,j+1)+b\sigma_t(i+1,j)}{a+b} + t.\frac{a\sigma_t(i,j+1)+b\sigma_t(i,j)}{a+b}\right] \quad (4.26)$$

The finite difference expressions for Eqs. (4.23) to (4.26) can be obtained by putting the values of Eqs. (4.19) to (4.22).

Four sets of expressions have been developed for the four sections of boundary of an arbitrary shaped outer boundary. The same four sets of finite difference expressions have been used for the internal hole boundary.

4.4.1 Set 1: Both *i* and *j* forward

The expressions can be found in reference [12]. These formulae have been developed for the boundary points of the top-left sections of the outer boundary. To avoid the presence of mesh points exterior to the false/pseudo boundary line, forward difference expressions is used in both x (i) and $y(j)$ directions. In this thesis work these expressions are used for the bottom-right sections of the hole boundary.

4.4.2 Set 2: *i* backward and *j* forward

The expressions can be found in reference [12]. These formulae are developed for the boundary points of the bottom-left sections of the outer boundary. To avoid the presence of mesh points exterior to the false/pseudo boundary line backward difference expressions is used in x (i) direction and forward difference expressions is used in $y(j)$ direction. In this thesis work these expressions are used for the top-right sections of the hole boundary.

4.4.3 Set 3: *i* backward and *j* backward

The expressions can be found in reference [12]. These formulae are developed for the boundary points of the bottom-right sections of the outer boundary. To avoid the presence of mesh points exterior to the false/pseudo boundary line backward difference expressions is used in x (i) direction and backward difference expressions is used in $y(j)$ direction. In this thesis work these expressions are used for the top-left sections of the hole boundary.

4.4.4 Set 3: *i* forward and *j* backward

The expressions can be found in reference [12]. These formulae are developed for the boundary points of the top-right sections of the outer boundary. To avoid the presence of mesh points exterior to the false/pseudo boundary line forward difference expressions is used in x (i) direction and backward difference expressions is used in $y(j)$ direction. In this thesis work these expressions are used for the bottom-left sections of the hole boundary.

4.5 MATHEMATICAL MODELING FOR COMPUTER PROGRAMMING

4.5.1 Similarity of the different boundary equations

To avoid the existence of any mesh point exterior to the false boundary four sets of boundary equations are developed for four part of the boundary of the hole and the body as explained in 4.4.1 to 4.4.4. It has been observed that some changes occur in the expressions of the coefficient as the function of a, b, l, m and some constants r, $s1$, $s2$, $s3$, $s4$, defined in Annexure B. The mesh generation program is developed in such a way that the variables a, b, s, t, l and m satisfy all four sets of boundary equations without any change. But some constants change its sign when they are considered in different sets of equations. It has been found that the boundary equations for bottom-left, bottom-right and top-right boundaries can be obtained from the boundary equations of top-left boundary with the following substitutions.

(1) Conversion from top-left boundary bottom-left boundary expressions by replacing

R	by	-R	
S1	by	-S1	
i-*	by	i+*	
i+*	by	i-*	* = digit 1 or 2 or 3

(2) Conversion from top-left boundary bottom-right boundary expressions by replacing

S3	by	-S3	
S4	by	-S4	
i-*	by	i+*	
i+*	by	i-*	
j-*	by	j+*	
j+*	by	j-*	* = digit 1 or 2 or 3

(3) Conversion from top-left boundary top-right boundary expressions by replacing

R	by	-R	
S1	by	-S1	
S3	by	-S3	
S4	by	-S4	
j±*	by	j+*	* = digit 1 or 2 or 3

So, the only one set of boundary equations can be repeatedly used for all the four sets parts of the boundary.

Another similarity is found in the expressions of u_n and u_t. The expression of u_n converts to that of u_t by the substitution of l, m by $-m$ and l respectively.

The expressions of boundary equations and stress and displacement components for every points are given in Annexure B.

4.6 GENERATION OF COEFFICIENT MATRIX

The coefficient matrix for the solution of the system of algebraic equation was made from the expressions of boundary equations for the boundary points and governing equations for the internal nodal points are given in Annexure B.

4.7 SOLUTION OF ALGEBRAIC EQUATION

A system of simultaneous equations can be solved by direct method and the method of iteration. Iterative method is not always converging and sometimes the rate of convergence is very slow. For this direct method of solution is used to solve the system of equations. "Cholesky's" method is used here. "Cholesky's" method, also known as "Crout's Method", the method of matrix decomposition, and the method of matrix factorization, is more economical in terms of computer time than other elimination

methods. In this thesis work L-U decomposition method is used. The solutions have been obtained by use of the back substitution.

4.8 CONCLUSIONS

The above formulations are for the solution of elastic problem with internal hole. The above scheme is also applicable for the solution of the solid elastic problem. The scheme is a generalized one, that is, it is open for the solution of arbitrary shaped elastic problem with internal hole of arbitrary shaped. Here in this research work a rectangular plate with internal circular hole of different sizes have been solved and the results obtained from this solution scheme are being described in the following chapter.

CHAPTER – 5
RESULTS AND DISCUSSIONS

5.1 INTRODUCTION

The main aim of this thesis work is to analyze the effect of holes inside an elastic body. Some analytical solution is available for such type of problem with very simple shape of the body as well as the hole, such as, rectangular body with a circular hole. Stress concentration occurs at the vicinity of the hole boundary. The high stress concentration at the boundary of the hole is of great practical importance. Elastic problems with different hole sizes have been solved and the results are compared with the known solution. To describe the state of stress and displacement of the body, the distribution of stress and displacement components for five selected sections are plotted against distance from the top boundary and are described in the following sections.

5.2 PRESENTATION OF THE RESULTS

Distributions of stress and displacement components are being described with the help of some figures which are the direct output from the computer program developed in this thesis work. Although the numerical values of stress and displacement components for each nodal point are calculated, the computer program has been developed in such a way that an output file can be obtained which gives symbolic representation of stress and displacement components for every nodal points of the whole body. Twelve specific symbols are used to show these results. The symbols are $, 1, 2, 3, 4, 5, 6, 7, 8, 9, #, and *. The weight of each symbol is defined by a specific range of numerical values. As for example if a symbol '$' is shown in a nodal point for stress or displacement then the value of stress or displacement component for that nodal point will be in the range 0.0000 to 0.00005. Similarly if a symbol '5' is shown in a nodal point then the value of stress or displacement component for that nodal point will be in between 0.00045 to 0.00055. There is a provision to change the weights of the symbols. The weights of each symbol are presented as follows.

$$
\begin{aligned}
0.00 &\le \text{"\$"} < 0.50\times10^{-4} \\
0.50\times10^{-4} &\le \text{"1"} < 1.50\times10^{-4} \\
1.50\times10^{-4} &\le \text{"2"} < 2.50\times10^{-4} \\
2.50\times10^{-4} &\le \text{"3"} < 3.50\times10^{-4} \\
3.50\times10^{-4} &\le \text{"4"} < 4.50\times10^{-4} \\
4.50\times10^{-4} &\le \text{"5"} < 5.50\times10^{-4} \\
5.50\times10^{-4} &\le \text{"6"} < 6.50\times10^{-4}
\end{aligned}
$$

$$6.50\times10^{-4} \leq \text{"7"} < 7.50\times10^{-4}$$
$$7.50\times10^{-4} \leq \text{"8"} < 8.50\times10^{-4}$$
$$8.50\times10^{-4} \leq \text{"9"} < 9.50\times10^{-4}$$
$$9.50\times10^{-4} \leq \text{"\#"} < 10.5\times10^{-4}$$
$$10.5\times10^{-4} \leq \text{"*"}$$

Though this result gives symbolic representation not the exact values of stress or displacement components for any nodal points, from symbolic presentation of the full body it can be easily observed the state of stress and displacement for the body as a whole. And the pattern of the distribution of stresses and displacements can be realized from these results.

5.2.1 Distribution of displacement component along *x* axis, *u*

The symbolic representation of the distributions of displacement component along x axis, u for different hole sizes are shown in Figs. 5.1(a), 5.1(b) and 5.1(c). These figures show that the distribution of u is anti-symmetric with respect to horizontal centerline AA in Fig. 5.1(a). Fig 5.1 shows that the displacement component u for the upper part of the body is positive and that for the lower part is negative. It indicates the contraction of the body along x axis. Figure 5.1 also shows that the contraction is higher at the section where hole is present. The values of u for every nodal points of the centerline are represented by the symbol '$'. So, no movement of the horizontal centerline will occur due to loading. This result satisfies theoretical expectations.

In Fig. 5.1(a) some parts of the body have been shown. Displacement is found higher in sections shown by the area PQRS, upper part of the hole. Similar section found in the lower part of the hole. Similar pattern of displacement component is found in Fig. 5.1(b) and in Fig. 5.1(c). Figure 5.1 shows that with the increase in hole size the area of the zone in which the value of u is higher increases.

The solutions presented in Figs. 5.1(a), 5.1(b) and 5.1(c) clearly show that the values of displacement component u for the left boundaries for every hole sizes are imitated mostly by symbol '$'. The weight of the symbol '$' is the lowest. The displacement components of these boundaries are already declared zero as boundary condition. Therefore, the solution reproduces the boundary condition assigned for this boundary.

Figure 5.1 shows that in some area in the left and right part of the side of the hole there is no displacement along x axis, i.e., the values of u presented by symbol '$' . In Fig. 5.1(a) this zones in which no displacement along x axis will occur are shown by DEFAD and MNOM. Figure 5.1(b) shows that the area of this zone is higher than that of Fig 5.1(a), and in Fig 5.1(c) the area of this zone is higher than that of Fig 5.1(a) and 5.1b. There is no displacement along x axis in this region is due to the presence of hole inside the body. And the area of this region will increase with increase in hole size.

5.2.2 Distribution of displacement component along *y* axis, *v*

The symbolic representation of the distributions of displacement component along y axis, v for different hole sizes are shown in Figs. 5.2(a), 5.2(b) and 5.2(c). The values of displacement increase from the fixed end to the loading end, i.e., minimum at the fixed end and maximum at the loading end. The values of v for most of the grid points are represented by the symbol '*'. So, the values of displacement component for those points are greater than 10.5×10^{-4}. The result presented in Fig. 5.2 shows that fact of lengthening the body along y axis.

Similar to the distribution of u, Figs. 5.2(a), 5.2(b) and 5.2(c) show that the values of displacement component v for the left boundaries for every hole sizes are mostly presented by symbol '$'. The displacement components v for these boundaries are already declared zero as boundary condition. Therefore, the solution reproduces the boundary condition assigned for this boundary.

5.2.3 Distribution of stress component along *x* axis, σ_x

The symbolic representation of the distributions of stress component for every nodal points of the whole body along x axis, σ_x for different hole sizes are shown in Figs. 5.3(a), 5.3(b) and 5.3(c). Figure 5.3 shows that the values ofσ_x are higher for the elastic body with bigger hole. In Fig 5.3(c) it can be found that the boundary of the hole at point Q is zero as a free surface. In Fig 5.3(c), a zone near the fixed end (shown by area the MNOM) experiences tension. Similar zone can be found in the right side of the hole (shown by the area ABCA). In Figs. 5.3(a) and 5.3(b) similar zone can be found but the area of those zones is smaller in Fig. 5.3(b) and smallest in Fig. 5.3(a).

The distribution of σ_x is very typical. From theoretical knowledge we know that without hole such a body with similar loading conditions will suffer compression along x axis. But here we found from Fig. 5.3 that some portion of the body suffers tension and some portion of the body suffers compression along x axis. This is due to the presence of hole inside the body. Though the dimension of the body decreases along x axis some portion near the hole of the body experiences tension due to flattening of the hole. If there is no hole inside the body, the whole body will experience compression along x axis. Figure 5.3 shows that the effect of hole on the distribution of σ_x disappears at a point far from the boundary of the hole. So, the solution satisfies Saint-Venant's Principle.

Figure 5.3 shows that some area in the upper part of the hole experiences tension. This is due to the fact of flattening the hole due to loading. Some zone of the elastic body around the hole (indicated by hexagon in Fig. 5.3(c)) experiences compression. Similar zone can be found in Fig 5.3(a) and 5.3(b), but this area is smaller in Fig 5.3(b) and smallest in Fig 5.3(a).

Figure 5.3 shows the symmetrical distribution of stress component σ_x around the hole. The values of stress componentσ_x is higher around the vicinity of the internal hole. It proves the stress concentration phenomena to the presence of hole. The values of σ_x at the top and bottom boundaries of the body (x/a = 0.0 and 1.0) the values of stress

component, σ_x are shown by the symbol '$'. i.e. the values are almost zero. As a free surface the values of σ_x for those boundaries are also set zero as boundary condition.

5.2.4 Distribution of stress component along *y* axis, σ_y

Symbolic presentation of the distributions of stress components along *y*-axis, σ_y for different hole sizes are shown in Fig. 5.4. Figure 5.4(c) shown the distribution for hole diameter and width of the plate ratio equals to 8/16 (0.50). It is found that the distribution is symmetric around the hole. Stress concentration occurs around the boundary of the hole. Especially top and bottom boundaries of the hole (near Q and S of Fig. 5.4(c)) suffer stress concentration. The values of σ_y are higher in Fig. 5.4(c) than that of Fig. 5.4(b), and the values of Fig. 5.4(b) is higher than that of Fig. 5.4(a). It proves that a bigger hole experiences more stresses around the vicinity of the hole-boundary. For finite width of the plate this happens, because, as hole size increases the actual width of the plate to carry the applied stress decreases.

A uniform stress equals to $S = \sigma_n / E = 0.0003$ is applied at the right boundary of the body. If there is no hole inside the body the stress component along y axis σ_y for all nodal points will be equals to 0.0003 (the representing symbol for corresponding value is '3'). But due the presence of hole the distribution of σ_y has been changed. Around the vicinity of the hole the values of σ_y are not presented by symbol '3'. Somewhere the values are more than 0.0003 (presented by symbols '4', '5', '6', '7', '8', '9', '#' and '*') and somewhere smaller than 0.0003 (presented by symbol '$', '1' and '2'). This result satisfies theoretical expectations. It is found that, the values of stress component σ_y changed around the vicinity of the hole and the change is negligible at distances large compared with the radius of the hole. This satisfies Saint-Venant's principle. Fig. 5.4(c) shows that the values of σ_y for every nodal point of a section perpendicular to the loading are equal to the applied stress can be found from 18 mesh distance from the hole boundary (shown by line AB). In Fig. 5.4(b) it can be found from 16 mesh distance from the hole boundary (shown by line CD) and in Fig. 5.4(a) it can be found from 14 mesh distance from the hole boundary (shown by line EF). So, the distance from the hole at which there is no effect of hole on the distribution of σ_y varies with hole size.

No stress along *y* axis will develop near the boundary P and R of Fig. 5.4(c). Similar condition is found in Fig 5.4(a) and Fig 5.4(b). The zones of these boundaries are free and the values of normal stress components for those points also declared zero as boundary conditions.

5.3 DISTRIBUTION OF STRESS AND DISPLACEMENT FOR SOME CRITICAL SECTIONS

Figure 5.1 to 5.4 give the symbolic representation of the stress and displacement components for the whole body. These results are obtained from the exact value calculated for each nodal point. The computer program developed in this thesis work calculates the exact values of u, v, σ_x, σ_y and σ_{xy} for every nodal points. The stress and displacement components for some sections are plotted against the distance from the top boundary for analysis and are shown in Figs. 5.5, 5.6 and 5.7. Five sections

perpendicular to the direction of loading are taken for discussion. The position of the five sections can be described as,

Section -1: $y / b = 0.25$
Section -2: Touching the left boundary of the hole
Section -3: Thorough the center of the hole. Also at the middle of the plate. $y / b = 0.50$
Section-4: Touching the right boundary of the hole
Section-5: $y / b = 0.75$
Where, y is distance of the section from the fixed end and
b is the length of the plate.

The unit of u and v is as obtained from the computer program is unit of the length. It may be meter, cm, mm, feet or inch as set by the users.
The unit of σ_x/E and σ_y/E is as obtained from the computer program is unit less.

The detail descriptions of the results are described in the following.

5.3.1 Distribution of *u* in different sections

The distributions of u for the five sections described above for d/a=8/16 are plotted against the distance from the top boundary and are shown in Fig. 5.7(a). The distribution of u for section 1 and 5 and 2 and 4 shows similar. This proves the symmetrical distribution of u around the hole. It is also found that for same loading condition the values of u are higher in section 3 (through the center of the hole). This has occurred due to the flattening of the body.

Similar to Fig. 5.1, Fig 5.7(a) shows that the values of u at the horizontal centerline (x/a = 0.5) are found zero. So, the centerline of the body will not move along x direction after deformation. Only the shape of the body will change.

5.3.2 Distribution of *v* in different sections

The distributions of v for five different sections for the body with hole size d/a=8/16 are plotted against the distance from the top boundary and are shown in Fig. 5.7(b). This figure shows the symmetrical distribution of v around the horizontal centerline (x/a = 0.5). For any section between the hole and the loading boundary (section 5) the value of v is larger at the horizontal centerline. And for section in the other zone i.e. between the fixed boundary and hole (section 1), the value of v, is minimum at the horizontal centerline. This occurs for flattening of the hole due to loading. All over the body the values of v are positive. The values of v are higher at sections apart from the fixed end and nearer to the loading end.

In section 3 the values of v for every nodal point are almost same. But for sections 2 and 4 the values are different in each nodal point. The variation is higher in sections 2 and 4, and negligible in section 1 and 5. This is due to the presence of the hole. If there is no hole present then for a section perpendicular to the direction of loading, the values of v for every point will be same. This effect will be diminishes at a distance large compared to the radius of the hole.

5.3.3 Distribution of σ_x in different sections

The distributions of σ_x for five different sections for the body with hole size d/a=8/16 are plotted against the distance from the top boundary and are shown in Fig. 5.7(c). The values of stress component for sections apart from the hole are positive and that for some sections at the vicinity of the hole are negative. If there is no hole the whole body will experience compression along x axis, i.e., the values of σ_x for every nodal point will be negative. Fig. 5.7(c) shows that the values of stress component at left and right boundary point of hole is nearly equal to the value of applied uniform stress. The figures show that for section 3, the values of σ_x increases towards the boundary of the hole from the boundary of the body. Values of σ_x for sections 1 and 5, and 2 and 4 for similar points are almost same. This proves the symmetrical distribution of σ_x around the hole. The values of σ_x are higher in sections nearer to the hole.

5.3.4 Distribution of σ_y in different sections

The distributions of σ_x for five different sections for the body with hole size d/a=8/16 are plotted against the distance from the top boundary and are shown in Fig. 5.7(d). This figure shows that the values of σ_y for similar nodal points are same at sections 1 and 5. . The distribution of σ_y for sections 2 and 4 are also same. The values of stress components for sections 2 and 4 are nearly equal to the value of S (applied uniform tension at right boundary of the body). So, the effect of hole on the distribution of σ_y in sections 2 and 4 is negligible. It satisfies Saint-Venant's Principle. The values of σ_y along the boundary of the hole are higher than other points of a section, because of the fact that stress concentration occurs around the vicinity of the boundary of the hole.

The values of stress component σ_y at the left and right boundary of the hole are found equals to zero. i.e., in sections 2 and 4 the values of σ_y at the grid points near the boundary of the hole are zero. The the values of σ_y at the top and bottom boundaries for different sections except section 3 are found larger than values of the applied stress. In section 3, the values of σ_y at grid points on the upper and lower boundaries of the hole are nearly equals to 4.2S. This is the highest values of stress developed in the body. In section 3, the values of σ_y at grid points on the upper and lower boundaries of the rectangular plate are nearly equal to S.

Figure 5.5d shows that among the sections, the values of stress component are higher in section 3 (section through the center of the hole and perpendicular to the direction of loading). So, this is the most critical section of the plate.

Similar solutions for the distribution of u, v, σ_x and σ_y for 5 different sections for the same problem having two other size of hole (d/a=6/16 and d/a=7/16) are presented in Figs. 5.5 and 5.6.

5.4 EFFECT OF HOLE SIZES

Figure 5.8 to 5.11 show the effect of hole size on the distribution of stresses and displacement components in a rectangular plate. The details descriptions are given below. Section through the center of the hole (section 3) and sections touching the boundary of the hole (sections 2 and 4) are more critical under loading. Here effect of hole sizes for sections 3 and 4 are described in the following.

5.4.1 Effect of hole sizes on the distribution of *u*

The effect of hole sizes on the distribution of displacement component u for sections 2 and 4 are shown in Figs. 5.8(a) and 5.8(b). It is observed that the values of u are higher for the presence of bigger hole. The horizontal centerline ($x/a = 0.5$) will not change for any hole size. There is no effect of hole size on the position of horizontal centerline, it remains unchanged.

5.4.2 Effect of hole sizes on the distribution of *u*

The effect of hole sizes on the distribution of displacement component v for different sections are shown in Figs. 5.9(a) and 5.9(b). In section 3 (passing through the center of the hole) the values of v increases with increase in hole sizes. Section 4 shows the opposite results, the values of v increases with decrease in hole sizes.

5.4.3 Effect of hole sizes on the distribution of *u*

The effect of hole sizes on the distribution of σ_x for sections 2 and 4 are shown in Figs. 5.10(a) and 5.10(b). It is found that the values of σ_x are higher at any sections with bigger hole. So, if the size of the hole increases the body experiences more stress along x direction.

5.4.4 Effect of hole sizes on the distribution of *u*

The effect of hole sizes on the distribution of σ_y for sections 2 and 4 are shown in Figs. 5.11(a) and 5.11(b). It is found that the values of σ_y are higher at any sections with bigger hole. So, if the size of the hole increases the body experiences more stress along x direction. In section 4 in some points the values of σ_y are higher with smaller hole, because of the fact of uneven stress distribution due to flattening of the hole under loading.

5.5 SOLUTION OF SOME OTHER PROBLEM

Using the same formulation and the developed computer program following elastic problems have been solved.

(i) a rectangular plate with a rectangular hole with the same geometry and same loading conditions,

(ii) a rectangular plate with circular hole but the position of the hole is different and same loading conditions,

(iii) a rectangular plate with circular hole at different position and different loading conditions, and

(iv) a arbitrary shaped plate with arbitrary shaped hole.

The symbolic representations of the stress and displacement components have been given in Annexure D.

5.6 CONCLUSIONS

Plates with internal hole have much more application in the field of engineering. So, stress analysis of such plate is necessary. The present work is mainly focused on the analysis of stress and displacement in a plate with internal hole. In this thesis work the analysis of a rectangular plate with circular hole under the action of uniform tension are done. But this scheme is capable to analyze any arbitrary shaped outer boundary and hole boundary and for any type of loading conditions.

From this thesis work we can conclude the following,

(a) The distribution of stress components change around the vicinity of the hole boundary and the change is negligible at a distance large compared with the radius of the hole.

(b) Due to the presence of hole, stress concentration occurs at the vicinity of the hole boundary.

(c) A plate will become weak with the presence of any internal hole and it is weaker with bigger hole.

(d) The most critical section of a rectangular plate under uniform tensile loading with internal circular hole is a section through the center of the hole and perpendicular to the direction of loading.

(e) Most critical points, the points which is more vulnerable to failure is the point on the boundary of the hole on the section passing through the center of the hole and perpendicular to the direction of loading.

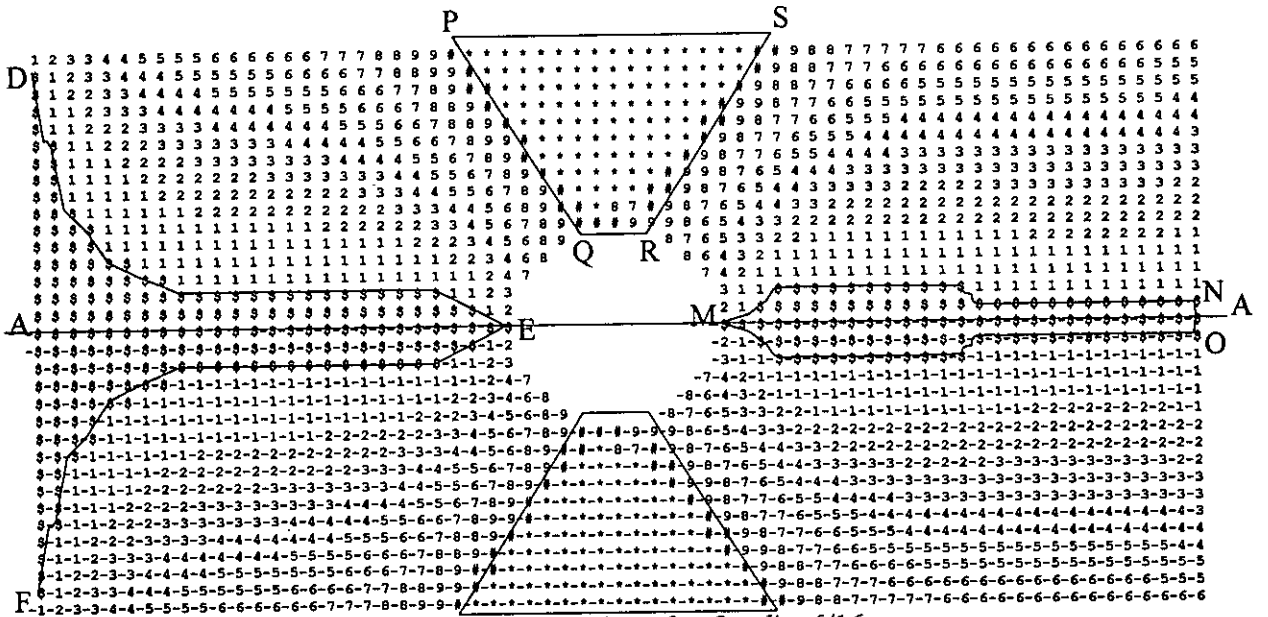


Fig 5.1(a): Distribution of u, for d/a=6/16

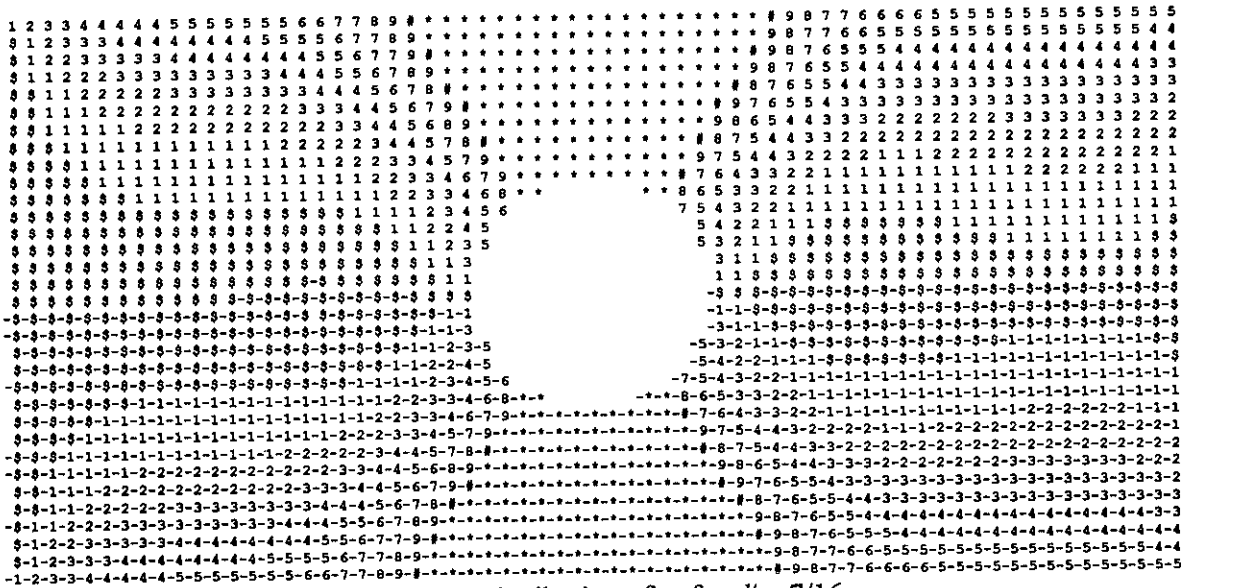


Fig 5.1(b): Distribution of u, for d/a=7/16

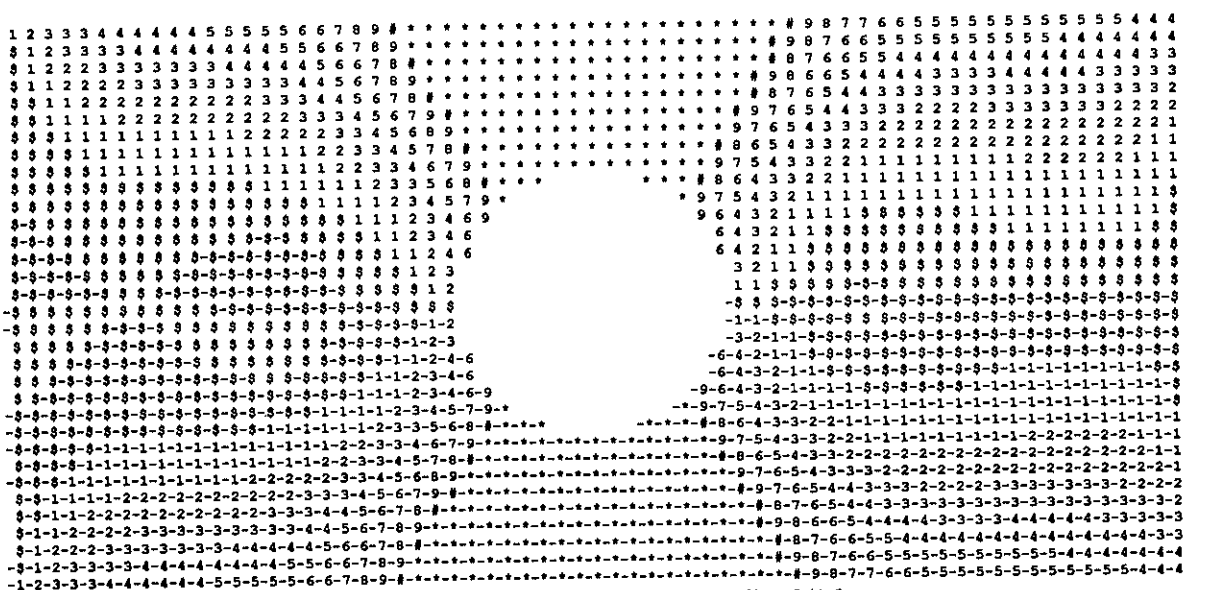


Fig 5.1(c): Distribution of u, for d/a=8/16

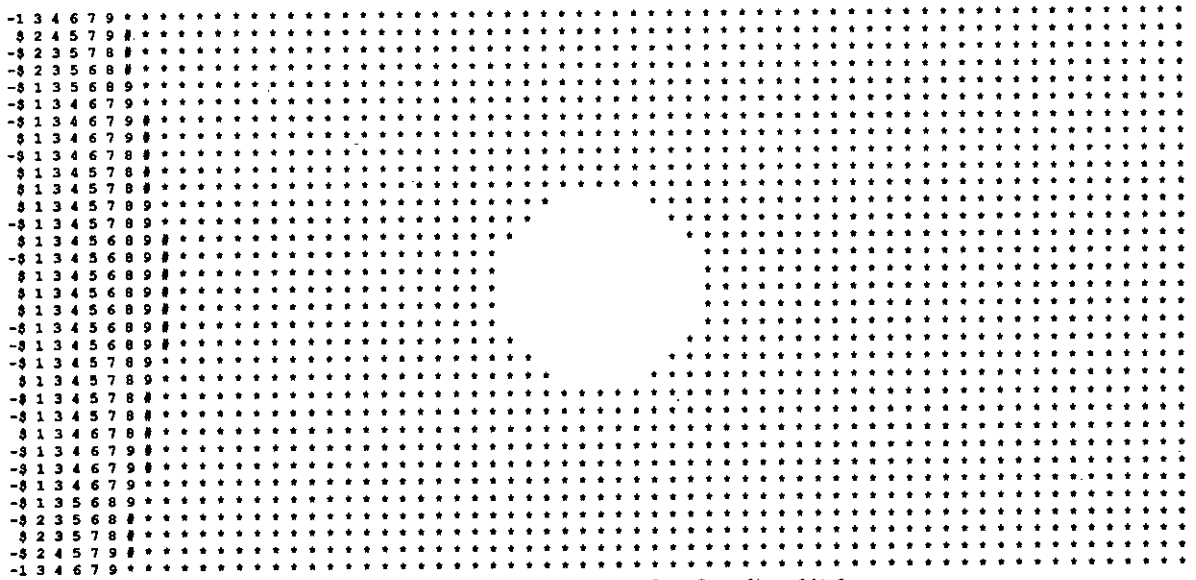


Fig 5.2(a): Distribution of v, for d/a=6/16

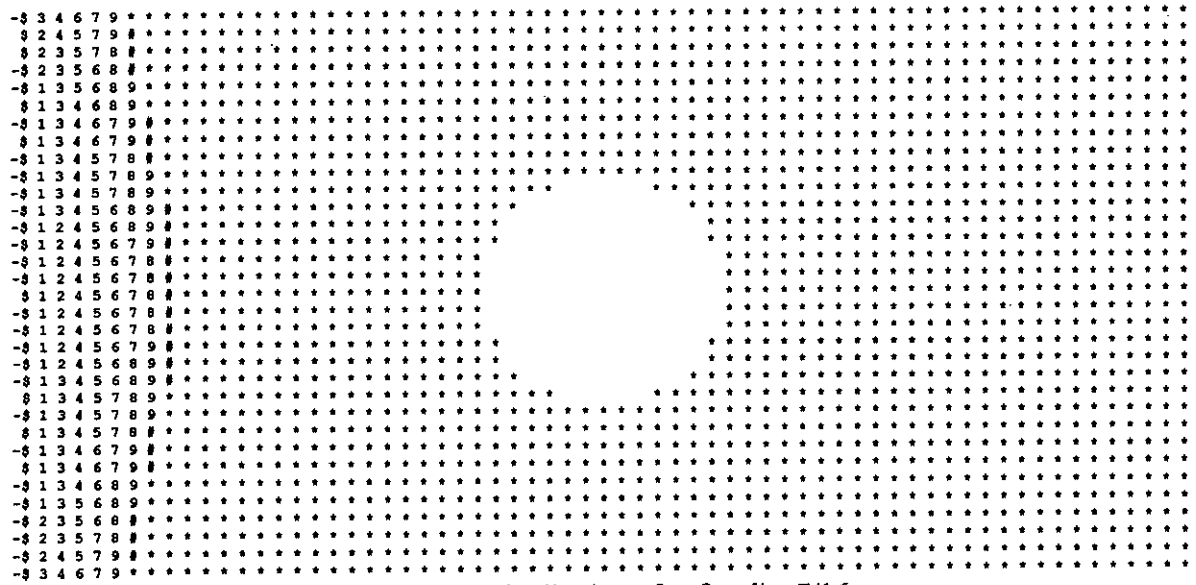


Fig 5.2(b): Distribution of v, for d/a=7/16

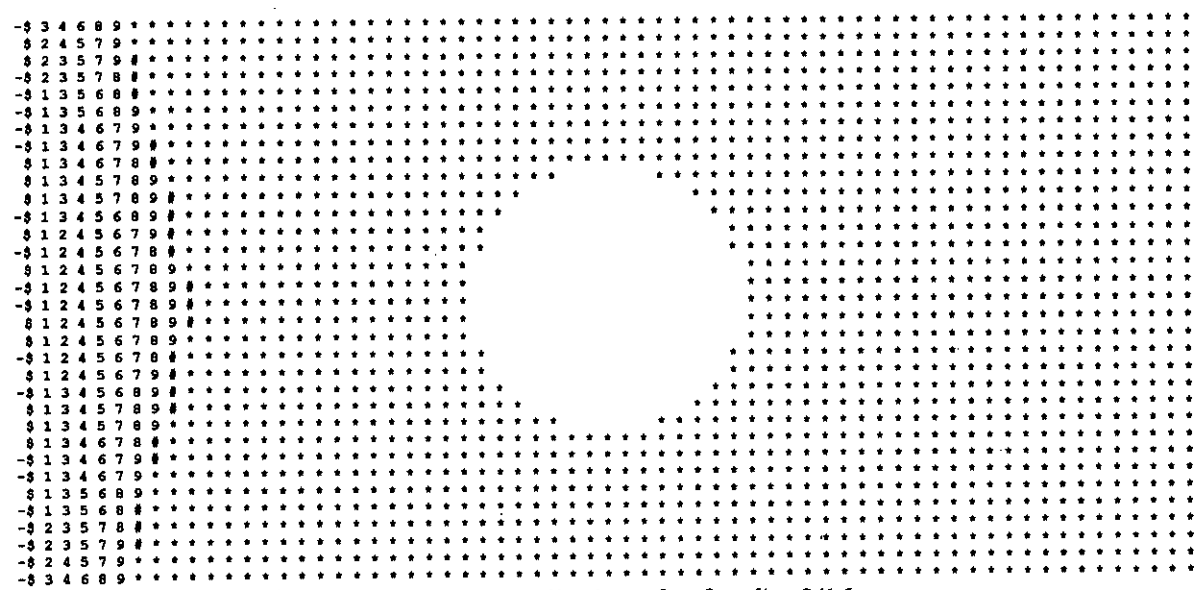


Fig 5.2(c): Distribution of v, for d/a=8/16

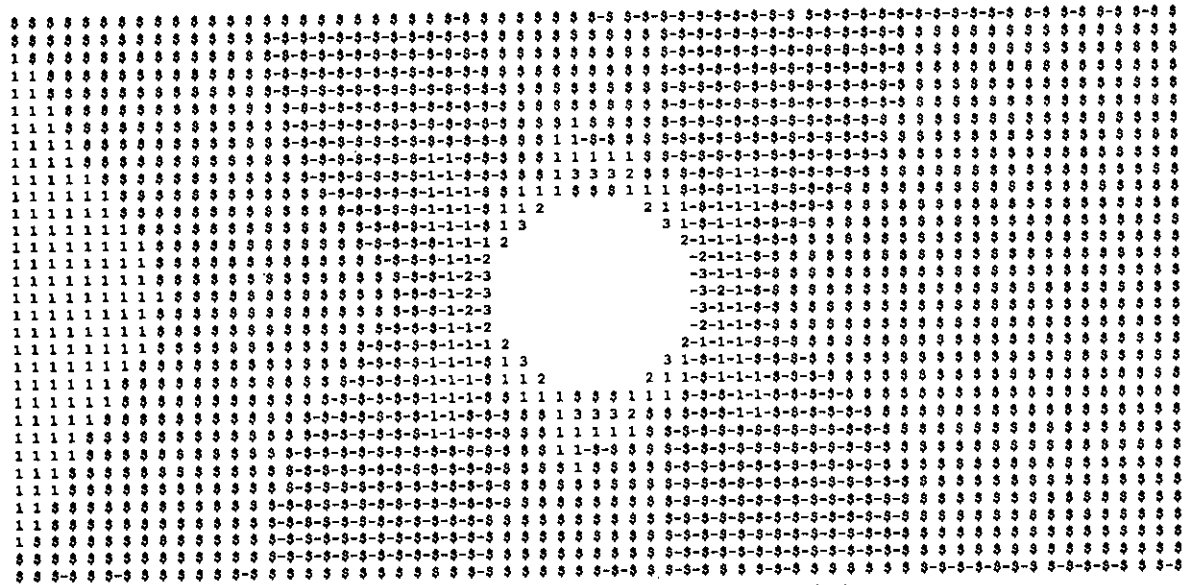


Fig 5.3(a): Distribution of σ_x, for d/a=6/16

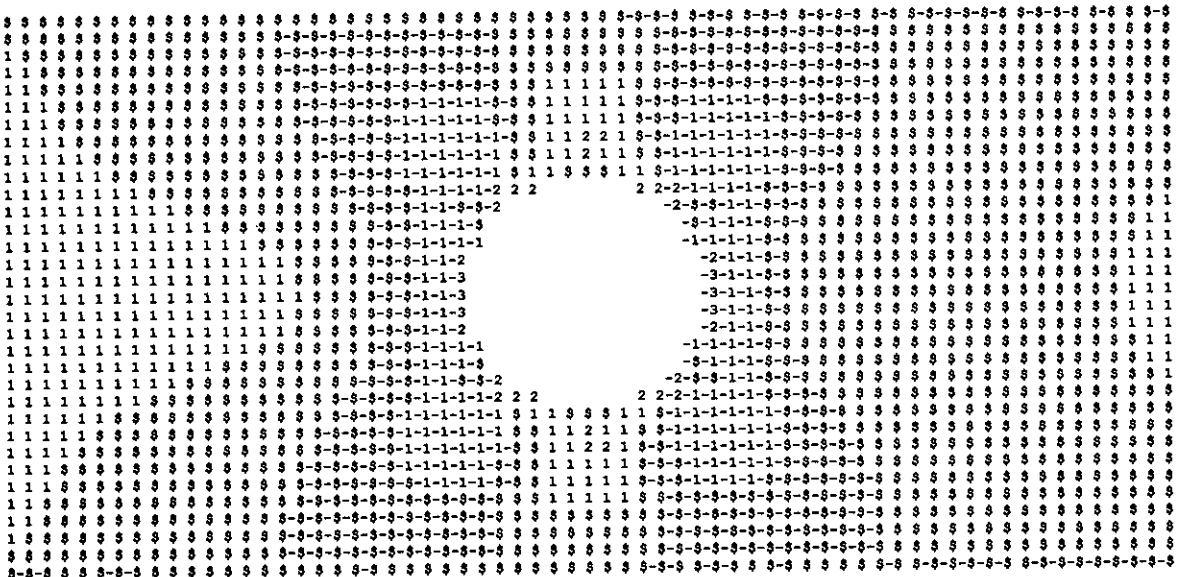


Fig 5.3(b): Distribution of σ_x, for d/a=7/16

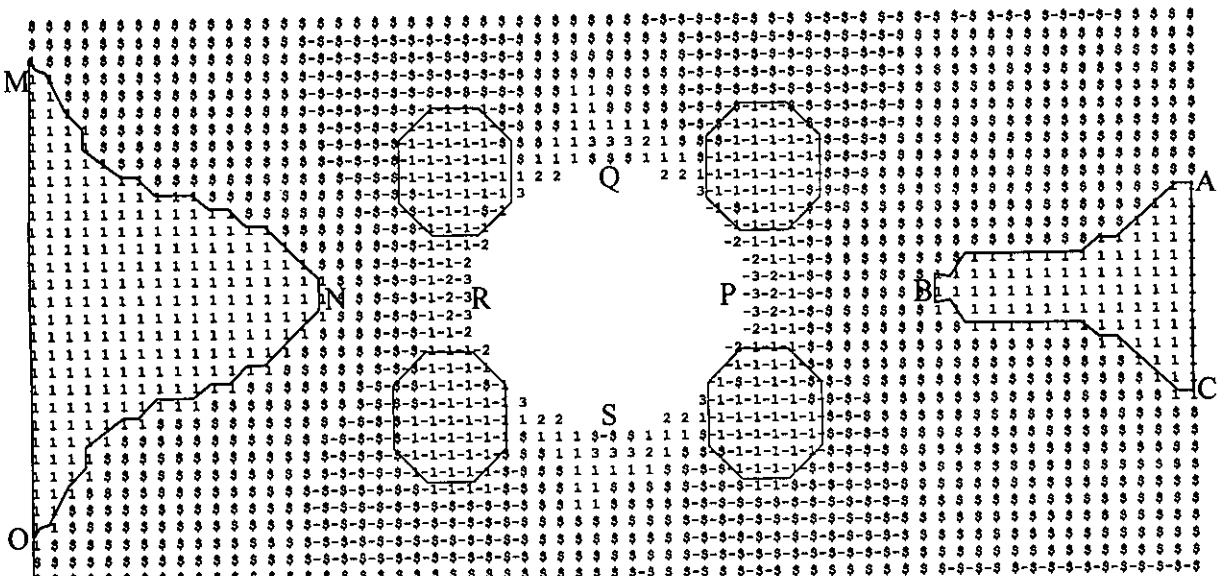


Fig 5.3(c): Distribution of σ_x, for d/a=8/16

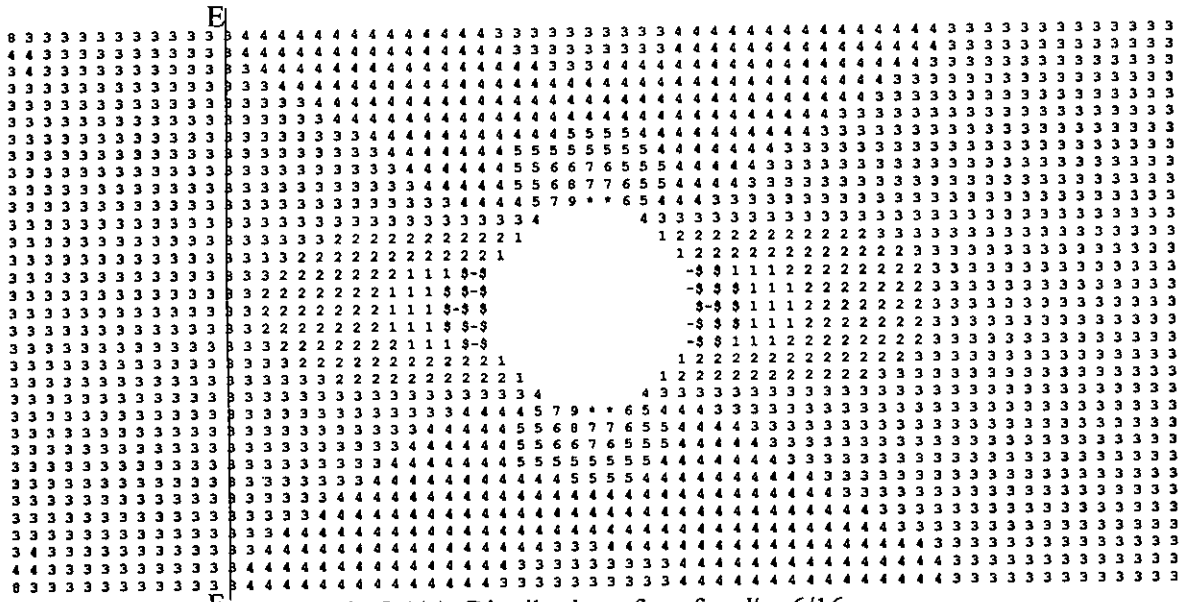


Fig 5.4(a): Distribution of σ_y, for d/a=6/16

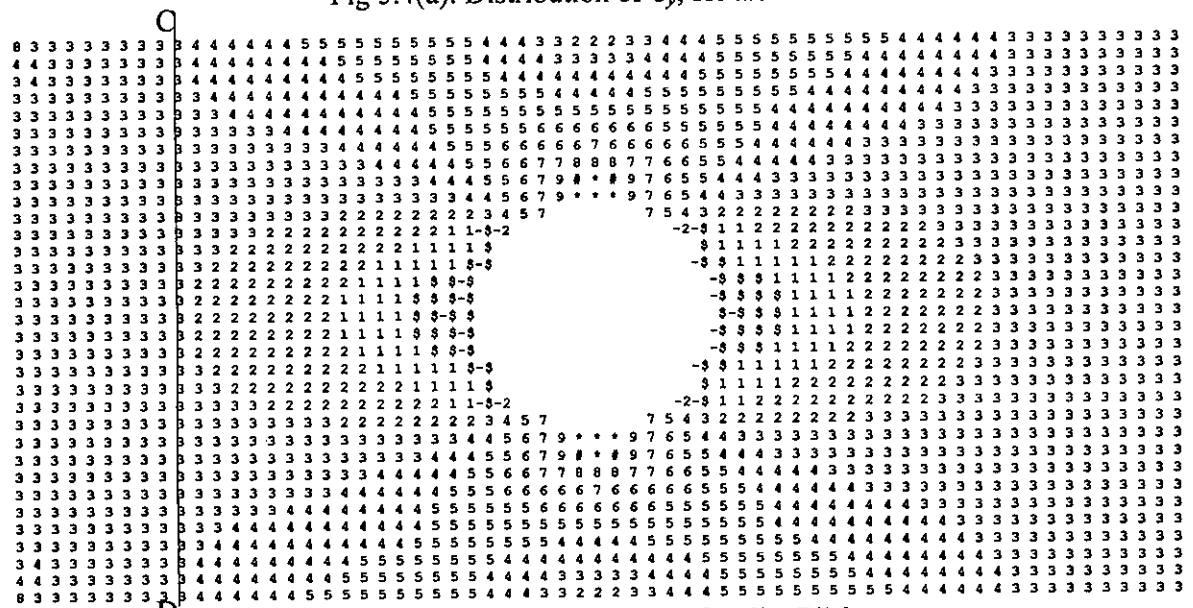


Fig 5.4(b): Distribution of σ_y, for d/a=7/16

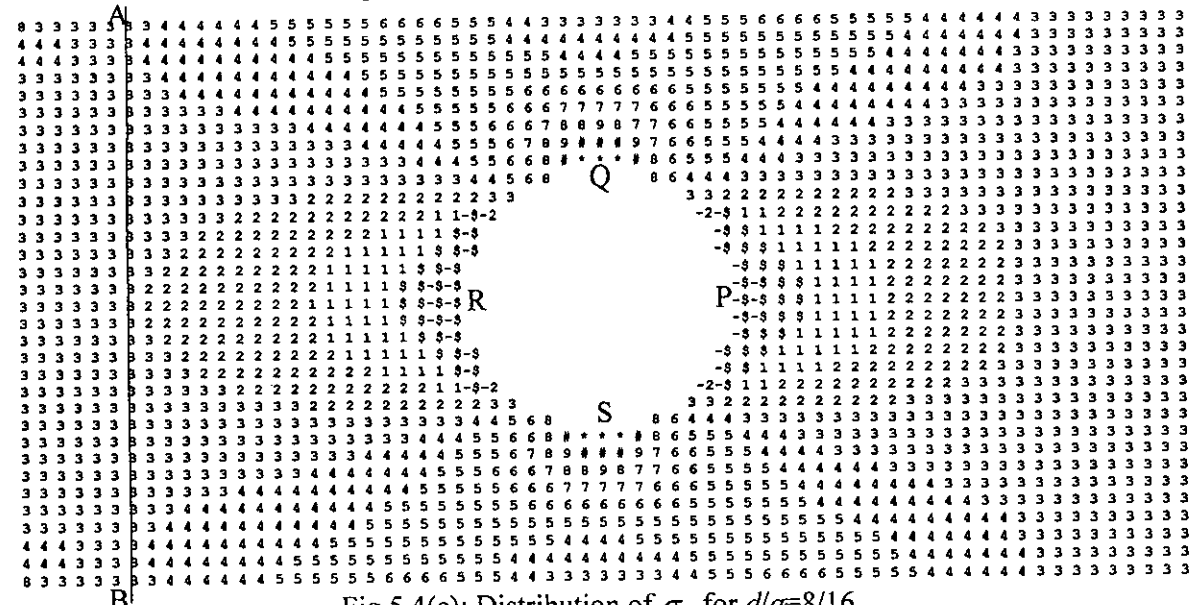
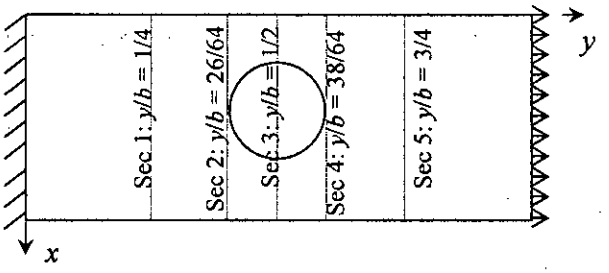


Fig 5.4(c): Distribution of σ_y, for d/a=8/16



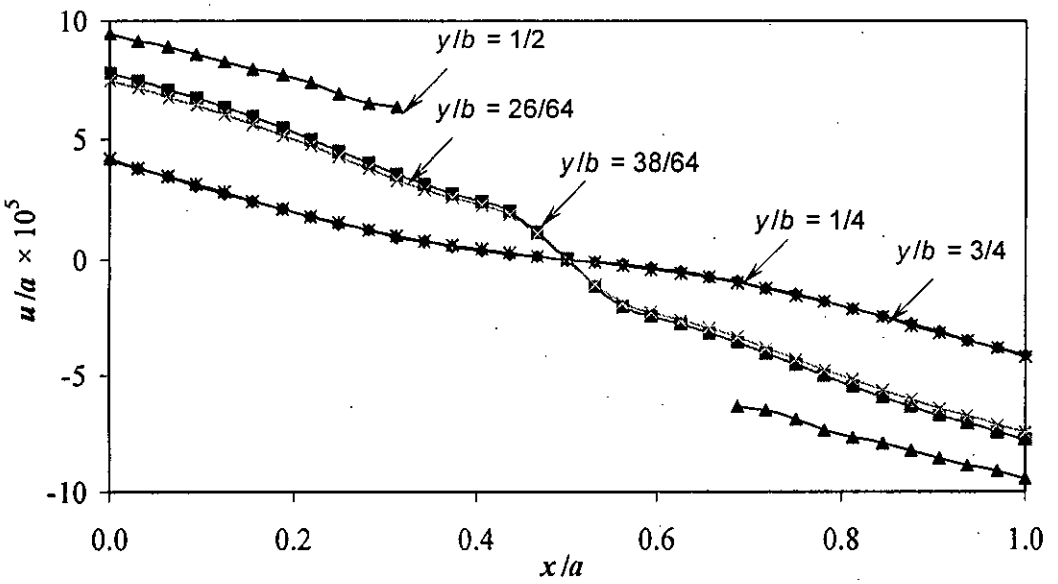


Figure 5.5(a): Distribution of u for d/a=6/16

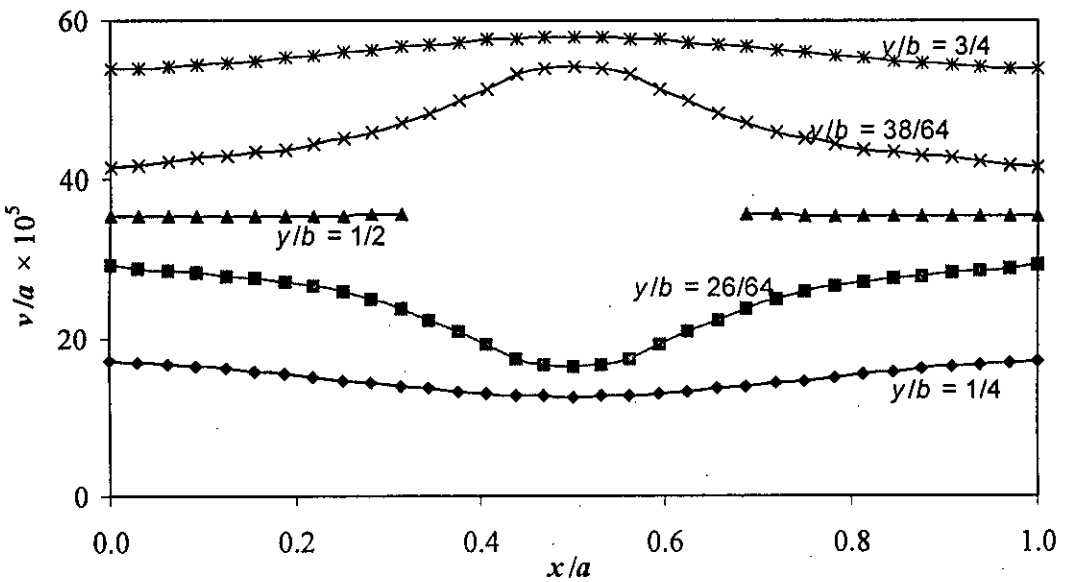
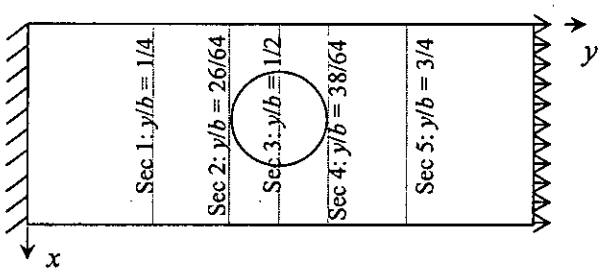


Figure 5.5(b): Distribution of v for d/a=6/16



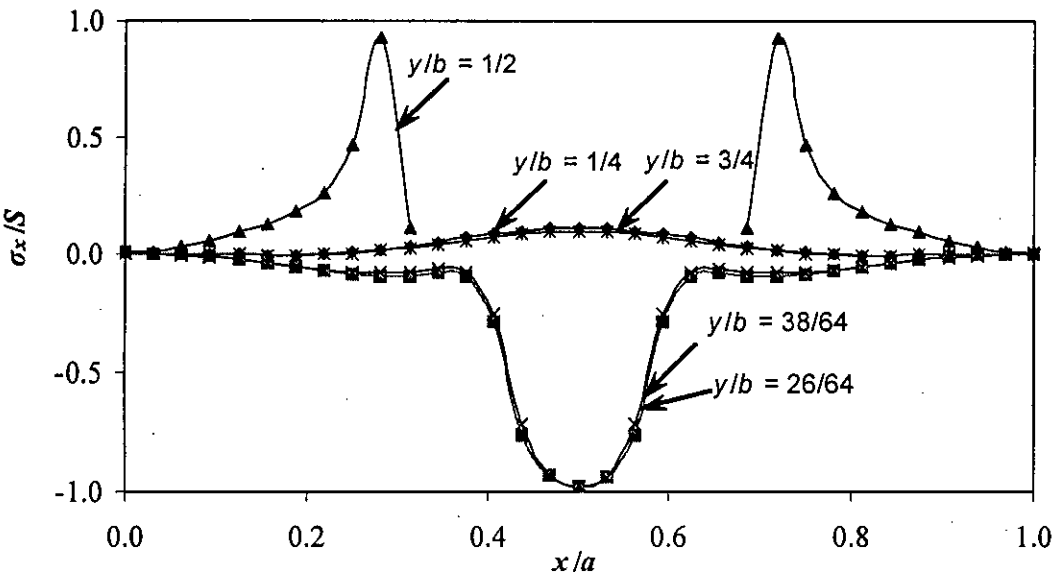


Figure 5.5(c): Distribution of σ_x for d/a=6/16

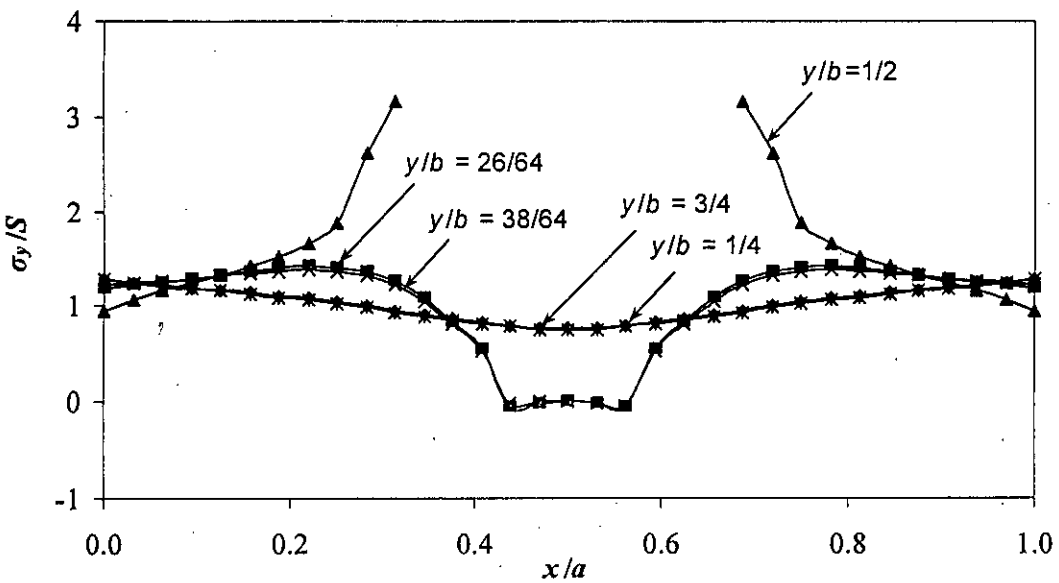
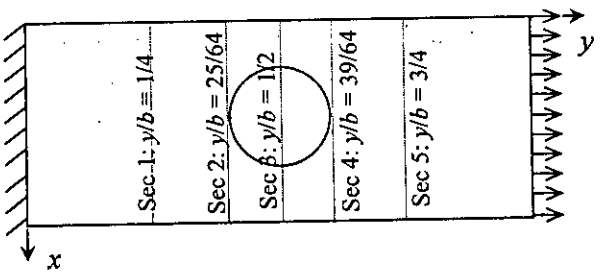


Figure 5.5(d): Distribution of σ_y for d/a=6/16



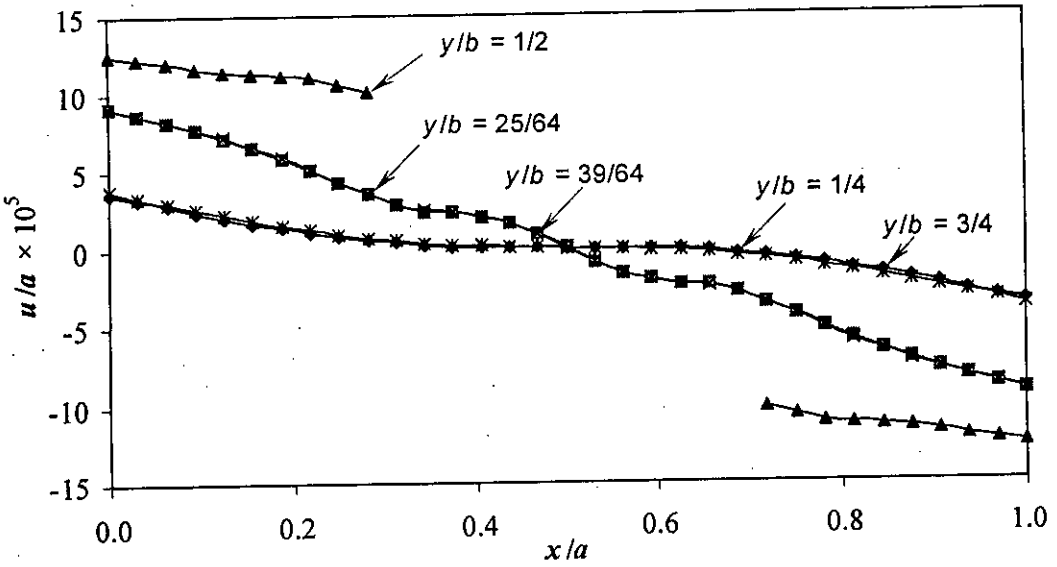


Fig. 5.6(a): Distribution of u for d/a=7/16

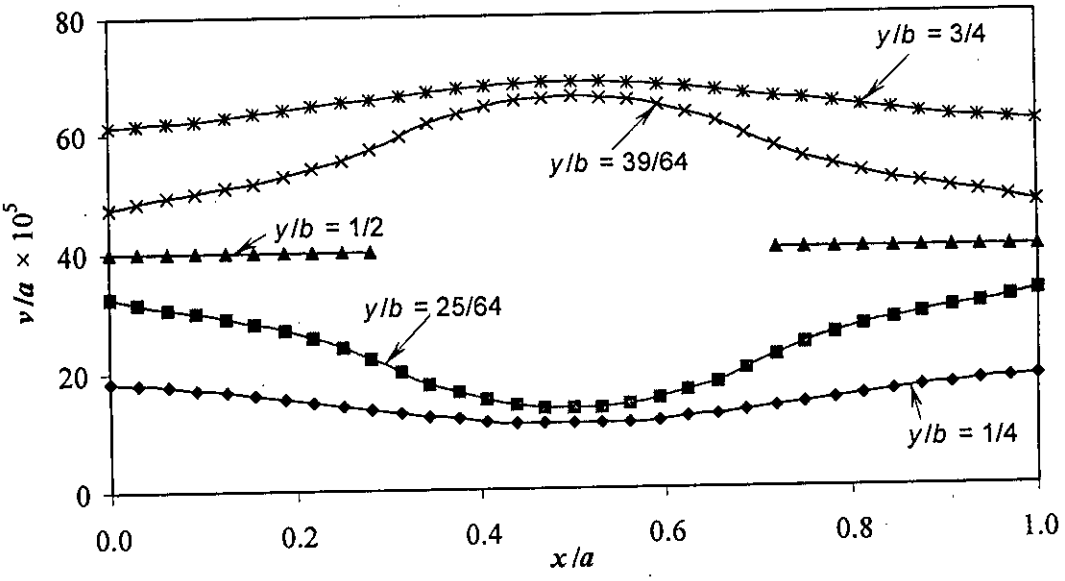


Fig. 5.6(b): Distribution of v for d/a=7/16

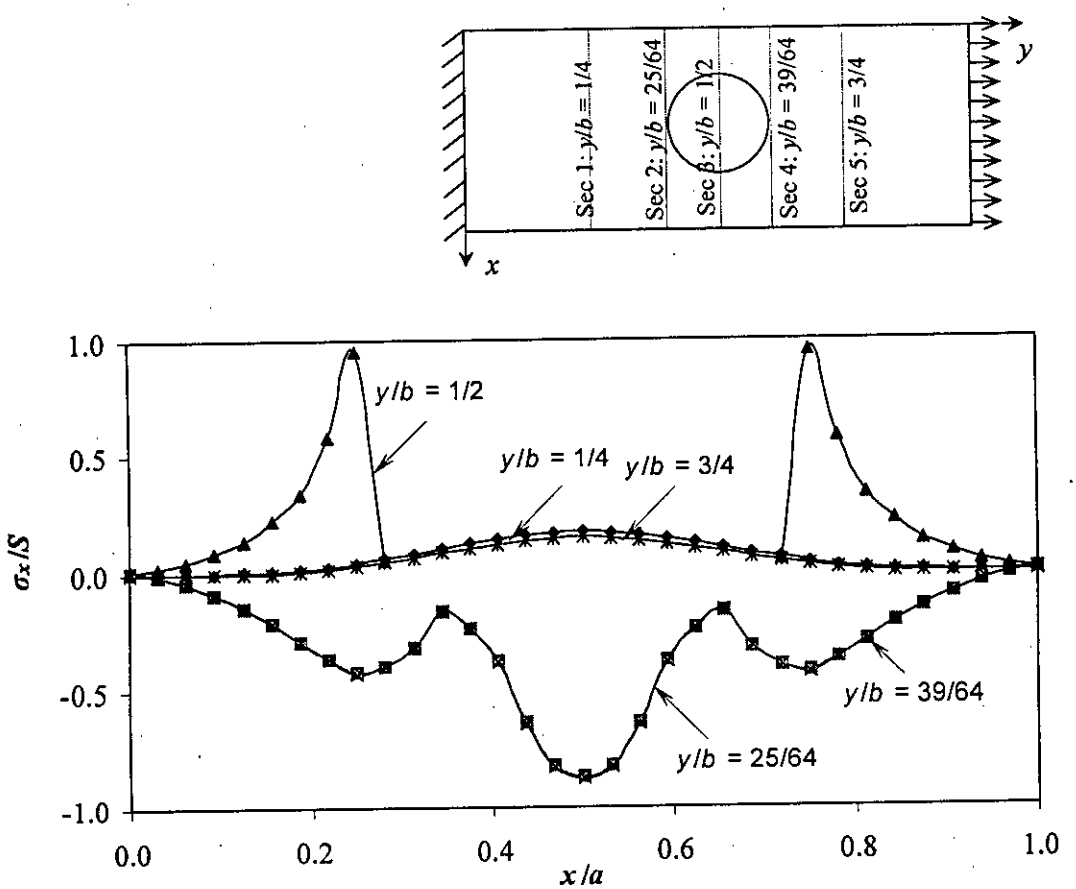


Fig. 5.6(c): Distribution of σ_x for d/a=7/16

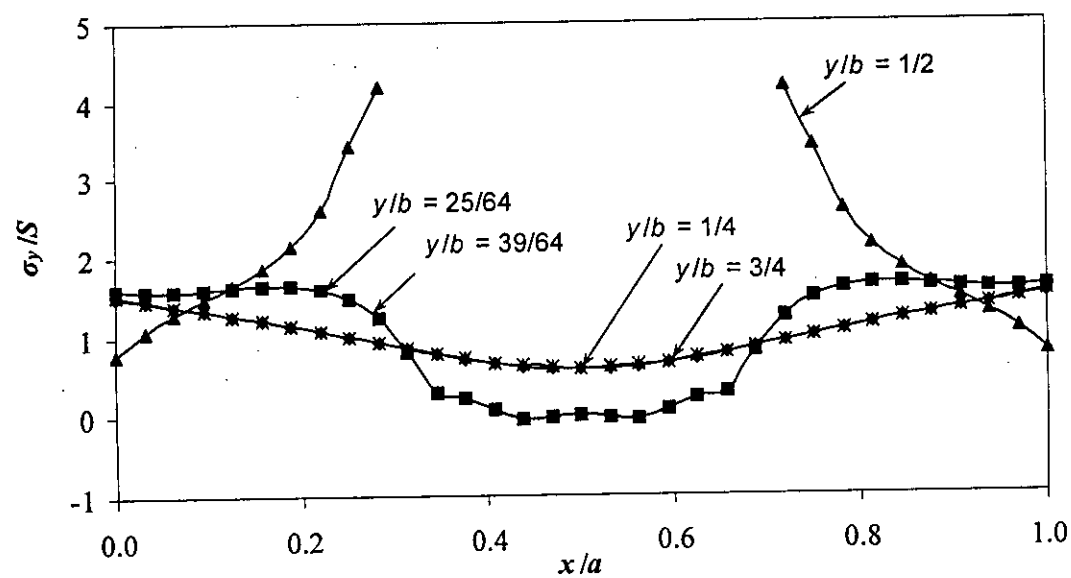
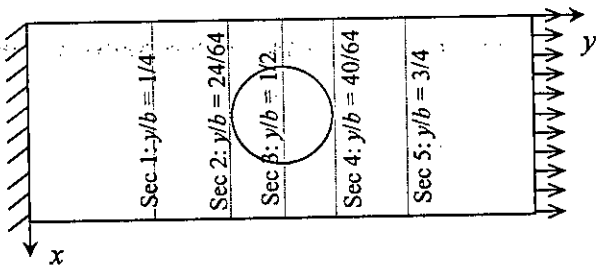


Fig. 5.6(d): Distribution of σ_y for d/a=7/16



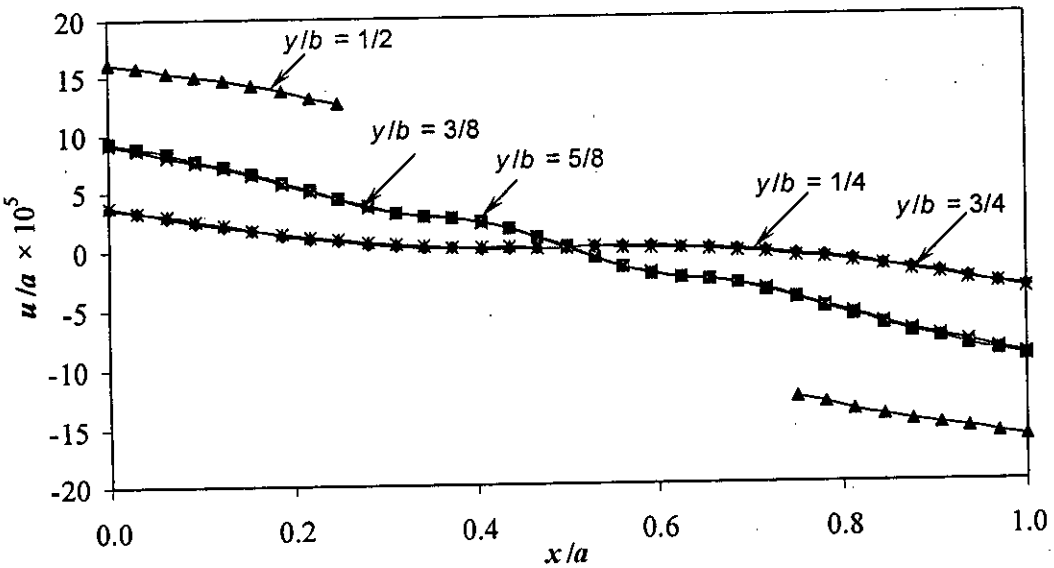


Figure 5.7(a): Distribution of u for d/a=8/16

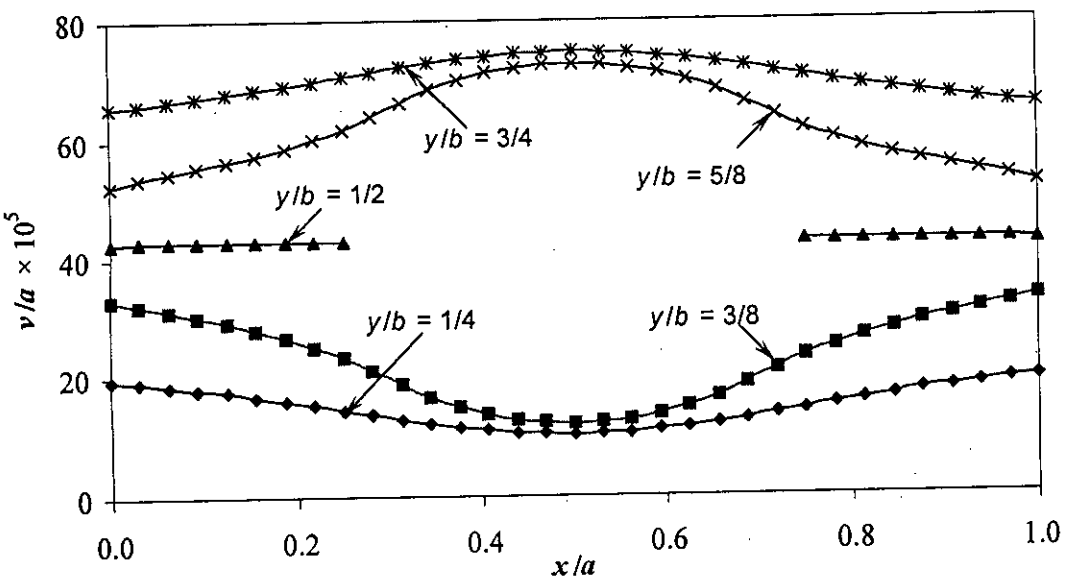
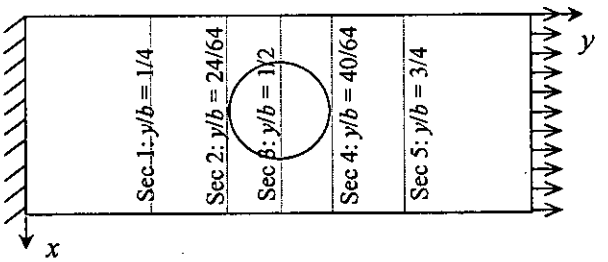


Figure 5.7(b): Distribution of v for d/a=8/16



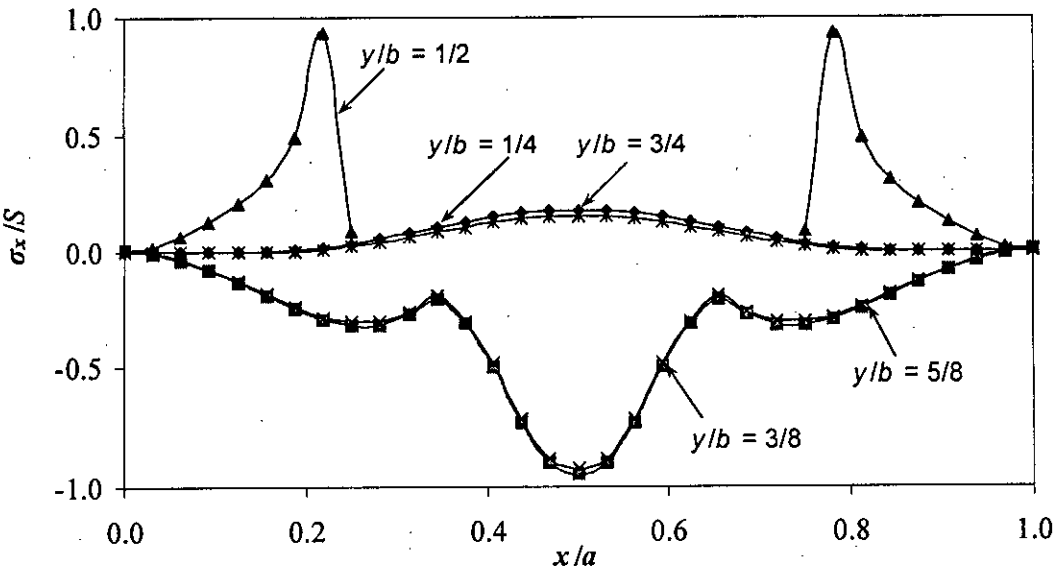


Figure 5.7(c): Distribution of σ_x for d/a=0.5

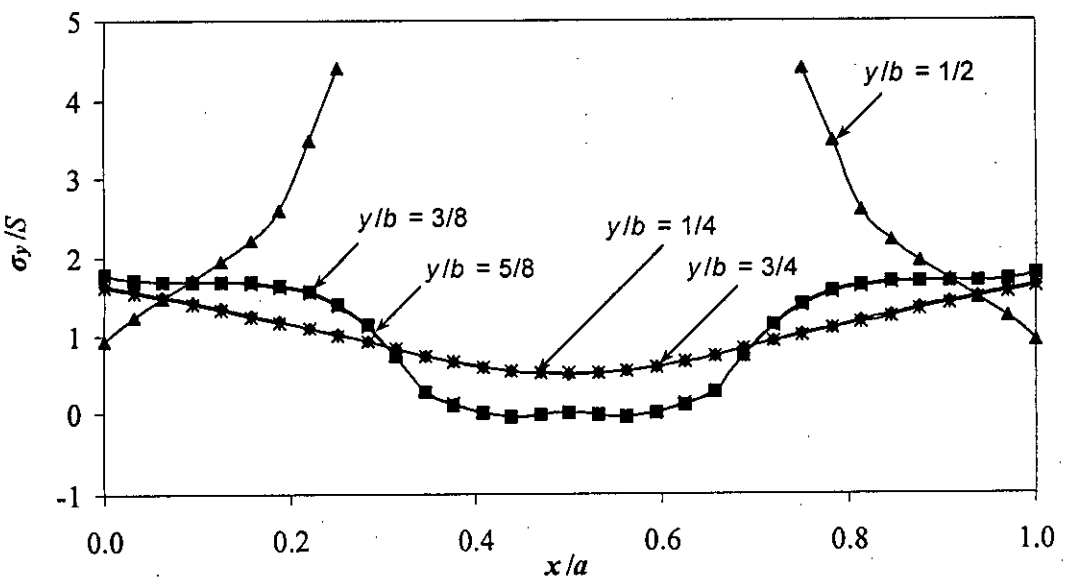


Figure 5.7(d): Distribution of σ_y for d/a=0.5

Effect of hole sizes on the distribution *u*

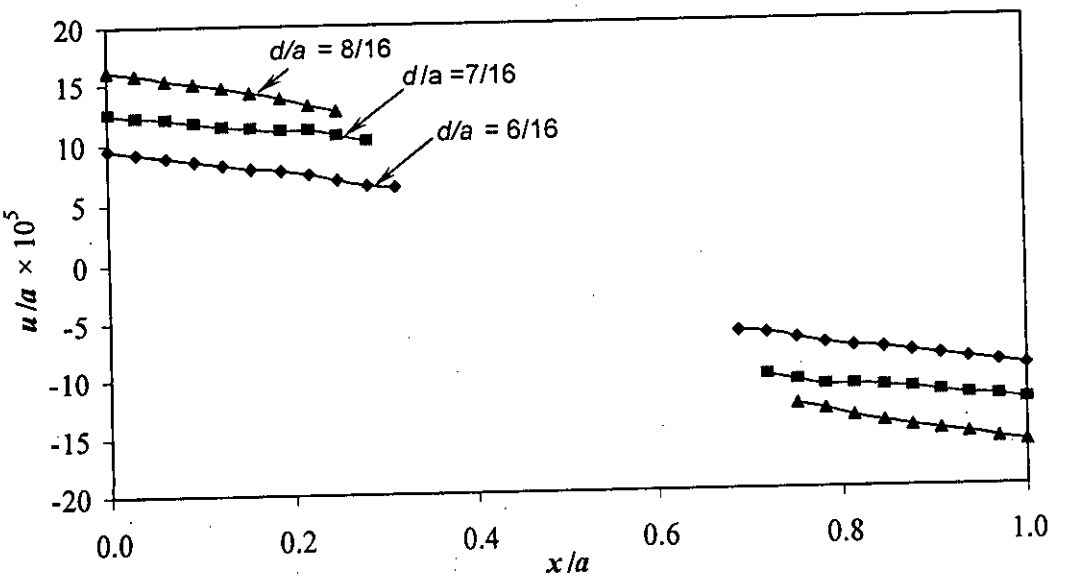


Figure 5.8(a): Effect of hole size on the distributions of displacement component, u in section 3

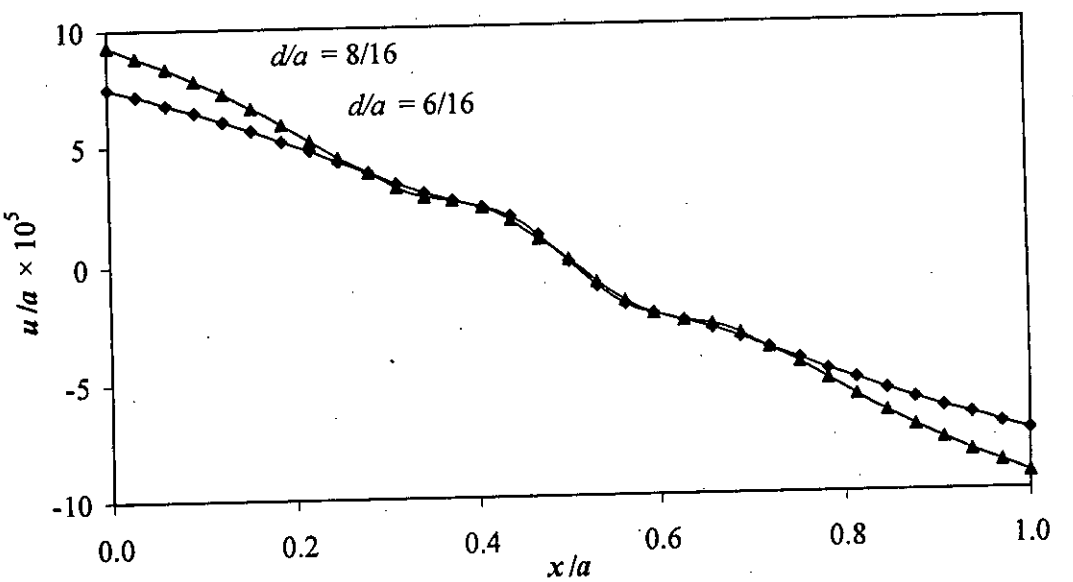


Figure 5.8(b): Effect of hole size on the distributions of displacement component, u in section 4

Effect of hole sizes on the distribution v

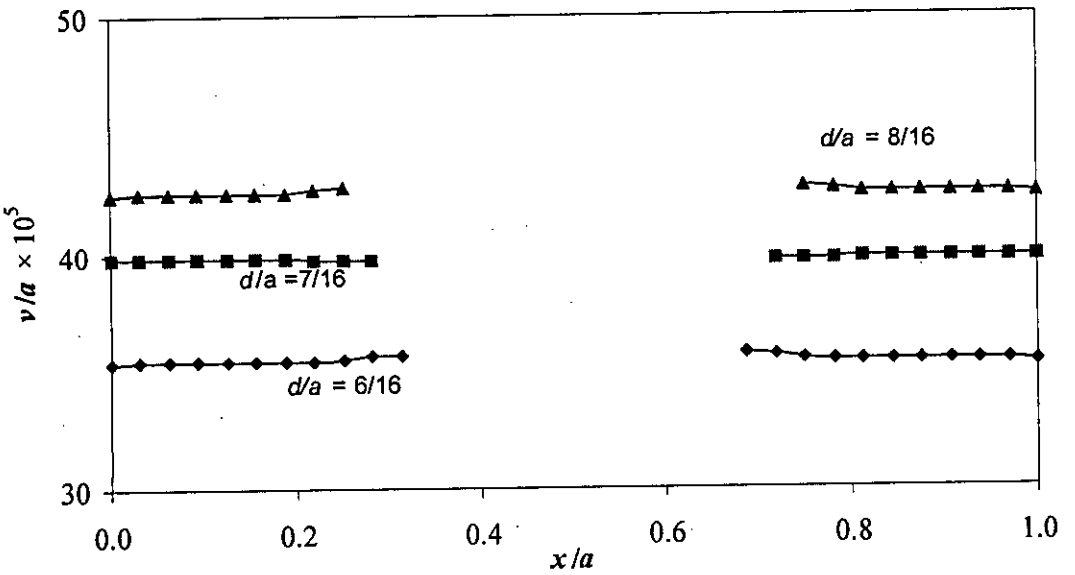


Figure 5.9(a): Effect of hole size on the distributions of displacement component, v in a section through the center of the hole (section 3)

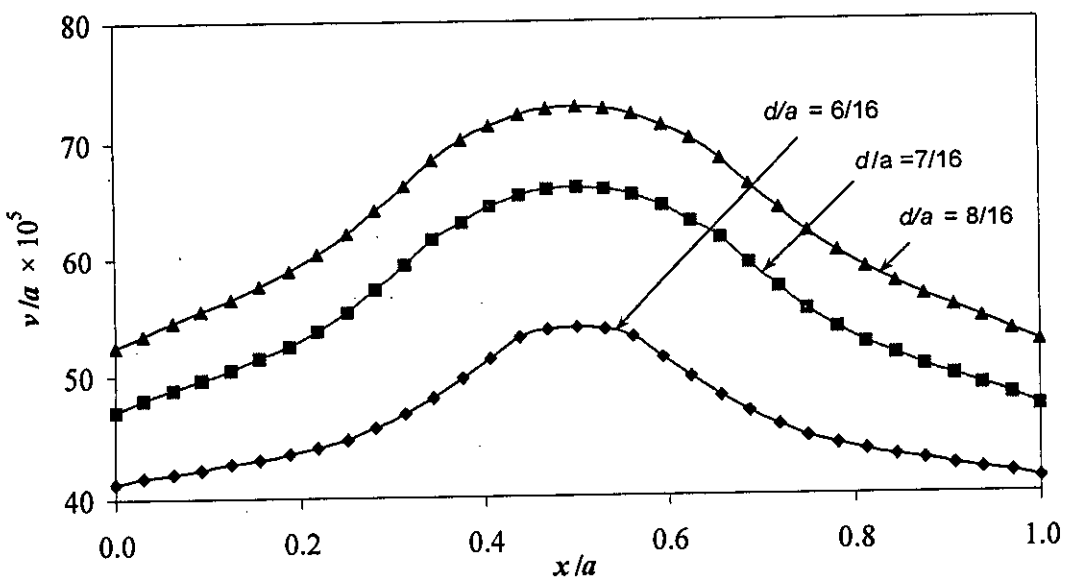


Figure 5.9(b): Effect of hole size on the distributions of displacement component, v in section 4

Effect of hole sizes on the distribution σ_x

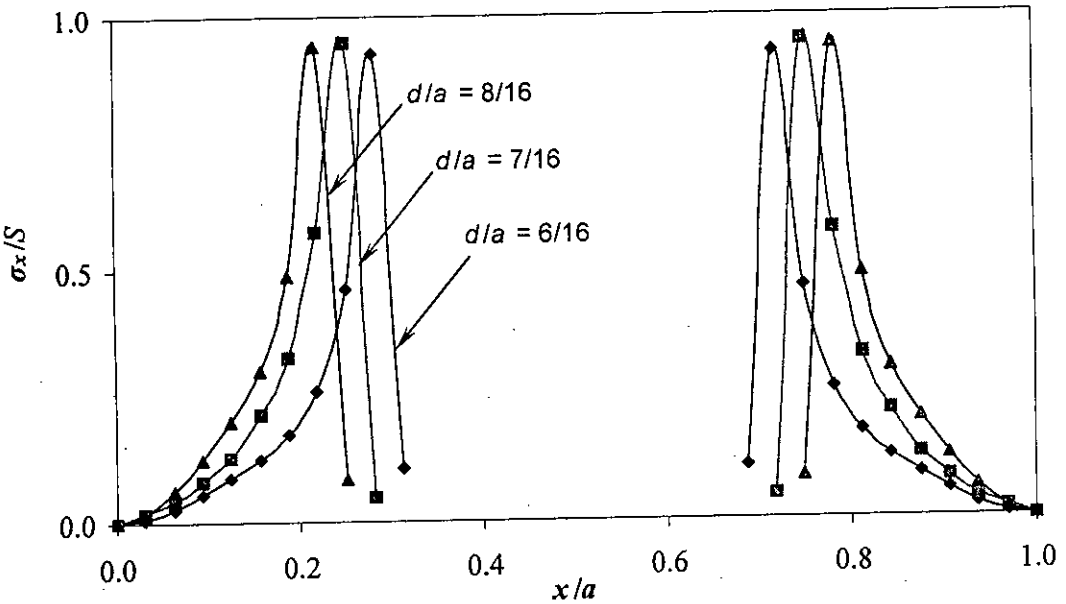


Fig. 5.10(a): Effect of hole size on the distributions of stress component, σ_x in section 3

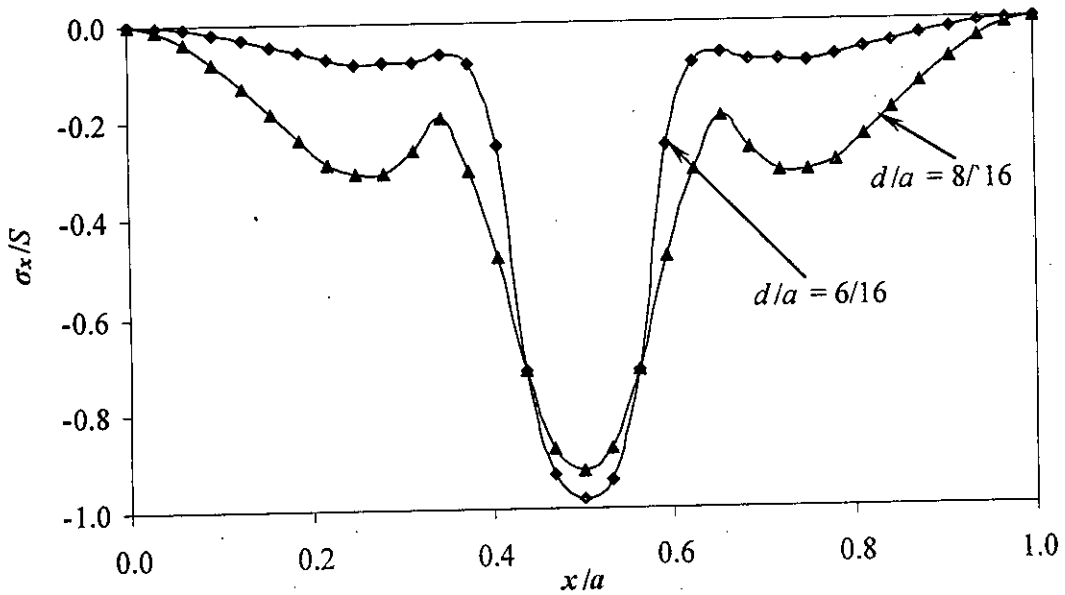


Fig. 5.10(b): Effect of hole size on the distributions of stress component, σ_x in section 4

Effect of hole sizes on the distribution σ_y

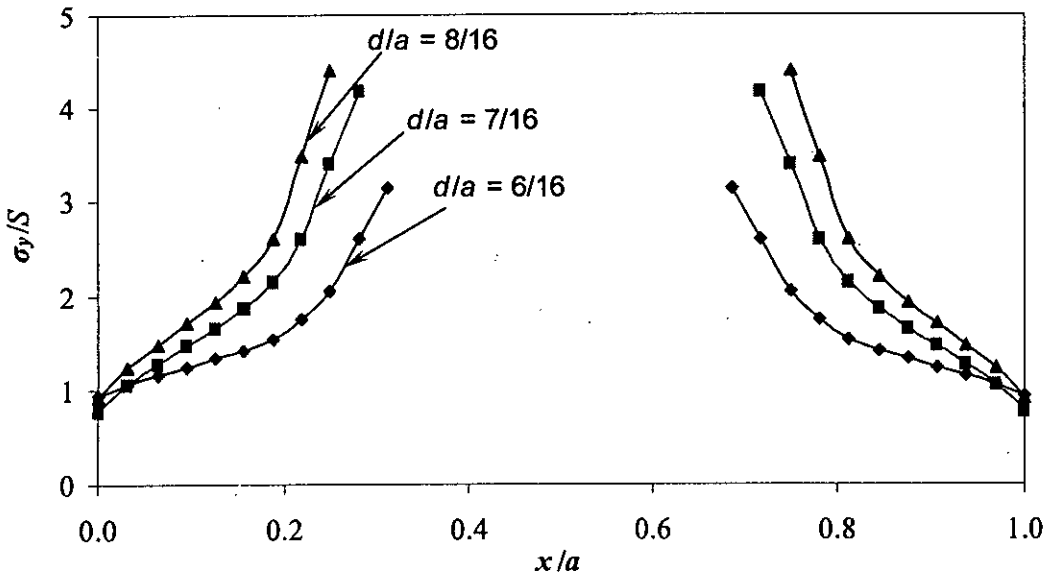


Fig. 5.11(a): Effect of hole size on the distributions of stress component, σ_y in section 3

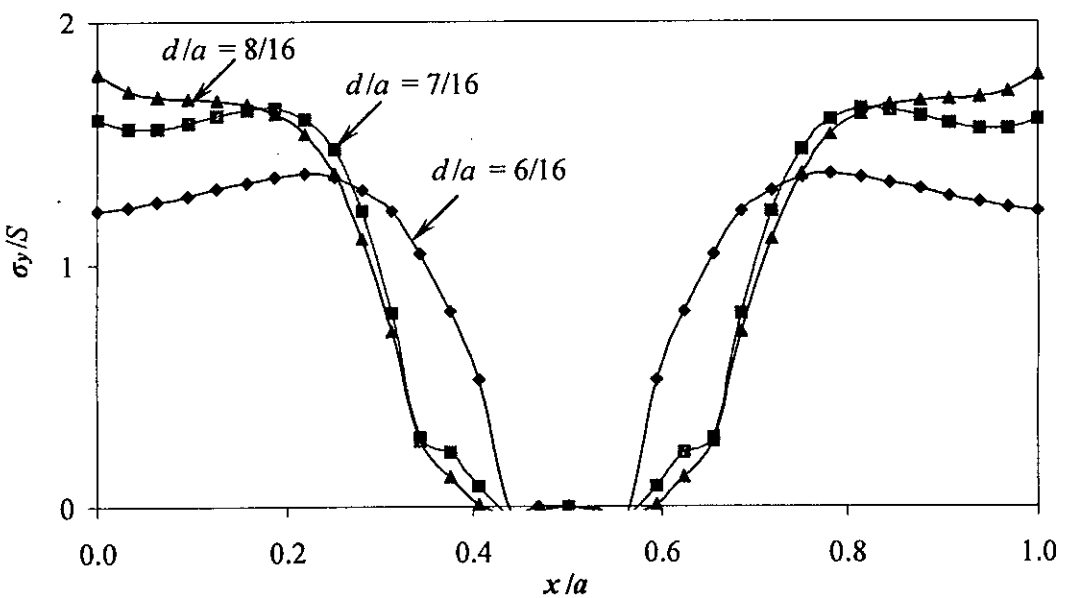


Fig. 5.11(b): Effect of hole size on the distributions of stress component, σ_y in section 4

CHAPTER – 6
CONCLUSION AND RECOMMENDATIONS

CONCLUSIONS

This thesis work has been dealt with the solution of two dimensional elastic problems with internal hole. To solve such type of problem we use two dimensional equilibrium equation of elasticity along with the compatibility equation and the boundary conditions, which have been described in chapter 2. A solution scheme has been developed to solve such type of problem in finite difference technique. The scheme has been described in chapter 4. We take a simple elastic problem (described in chapter 3) and solve it with the developed solution scheme and the computer code written in FORTRAN programming language. The numerical results from this scheme have been compared with the known solutions. The results and comparison with the known solutions are described in chapter 5.

The conclusion of the thesis work can be summarized as follows,

(a) This thesis work is a finite difference solution scheme for two dimensional elastic bodies with internal hole.

(b) This is a first attempt to solve elastic problem with internal hole by using finite difference technique.

(c) The results obtained are in good agreement with the known solution.

(d) This solution scheme is capable to accept any mixed and changeable boundary conditions. The conditions on the boundary may be in terms of displacement or in terms of stress components or in combination of the two, and the conditions may change along the boundary.

(e) Although in our thesis work we solve a rectangular plate with internal hole, this numerical scheme can be used for the solution of other arbitrary shaped of the plate and hole.

RECOMMENDATIONS FOR FURTHER RESEARCH WORK

Following recommendations are for further research work in numerical analysis.

(a) Here in this thesis work the effect of hole and its sizes are analyzed. Further research work can be carried out to analyze the effect of the shape of the hole and position of the hole in the plate.

(b) Though the stress analysis of three dimensional plane stress and plane strain problem can be treated as two-dimensional problem, but some problem can not treat in this way. For such cases three dimensional analyses of stresses is necessary. So, further work can be carried out to solve three dimensional elastic problems with internal hole.

(c) The scheme developed here is for the solution of elastic bodies with single internal hole. But elastic problems may have more than one hole. So, further work on the solution of elastic problems with more than one hole is recommended.

(d) An optimum mesh size may be found for the solution of a specific type of elastic problem.

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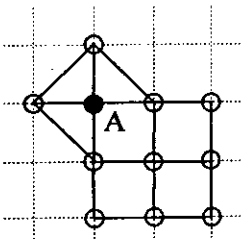
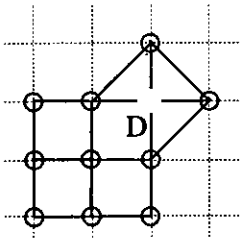
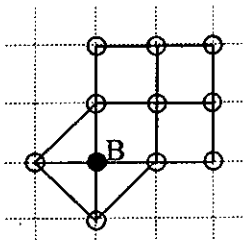
ANNEXURE – A



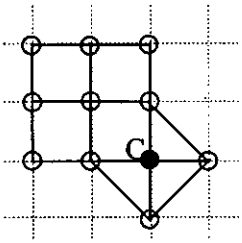
a: top-left boundary of body and bottom right boundary of the hole



b: top-right boundary of body and bottom-left boundary of the hole

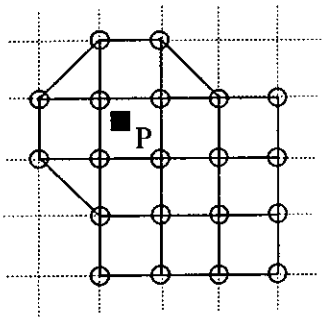


c: bottom-left boundary of body and top right boundary of the hole

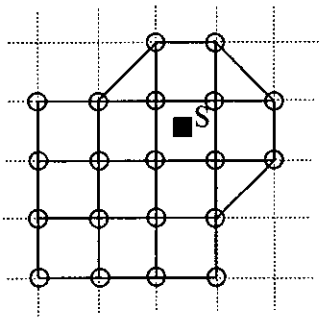


c: bottom-right boundary of body and top-left boundary of the hole

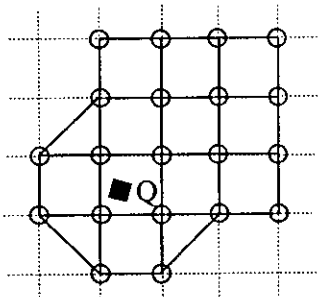
Fig. A1: Structure of finite difference expressions of u_n and u_t for pivotal points match with rectangular grid points



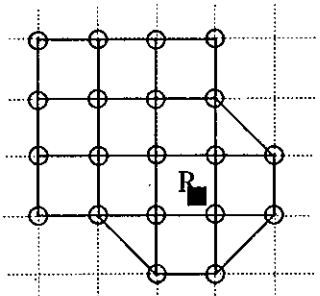
a: top-left boundary of body and bottom right boundary of the hole



b: top-right boundary of body and bottom-left boundary of the hole



c: bottom-left boundary of body and top right boundary of the hole



c: bottom-right boundary of body and top-left boundary of the hole

Fig. A2: Structure of finite difference expressions of u_n and u_t for pivotal points do not match with rectangular grid points

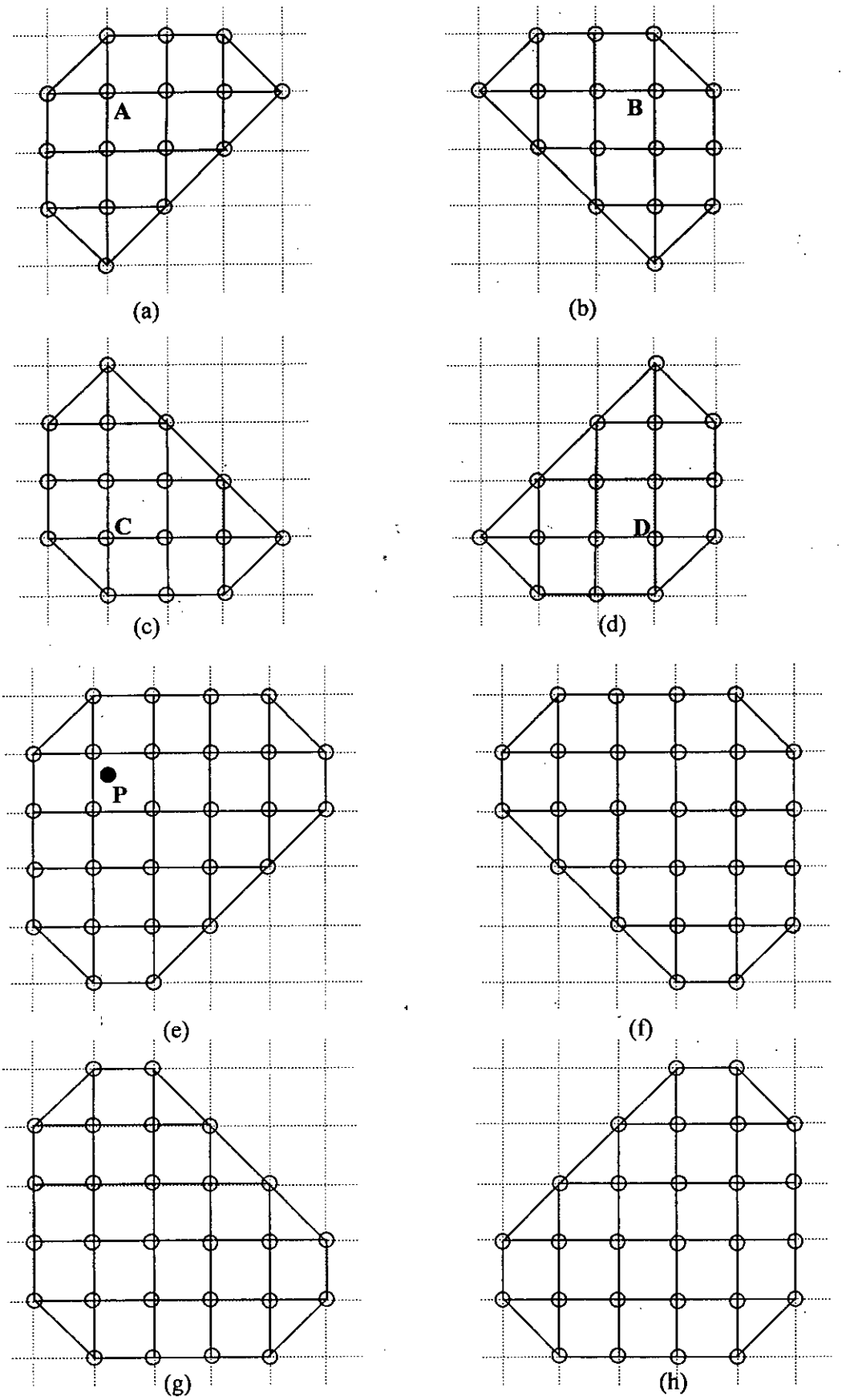
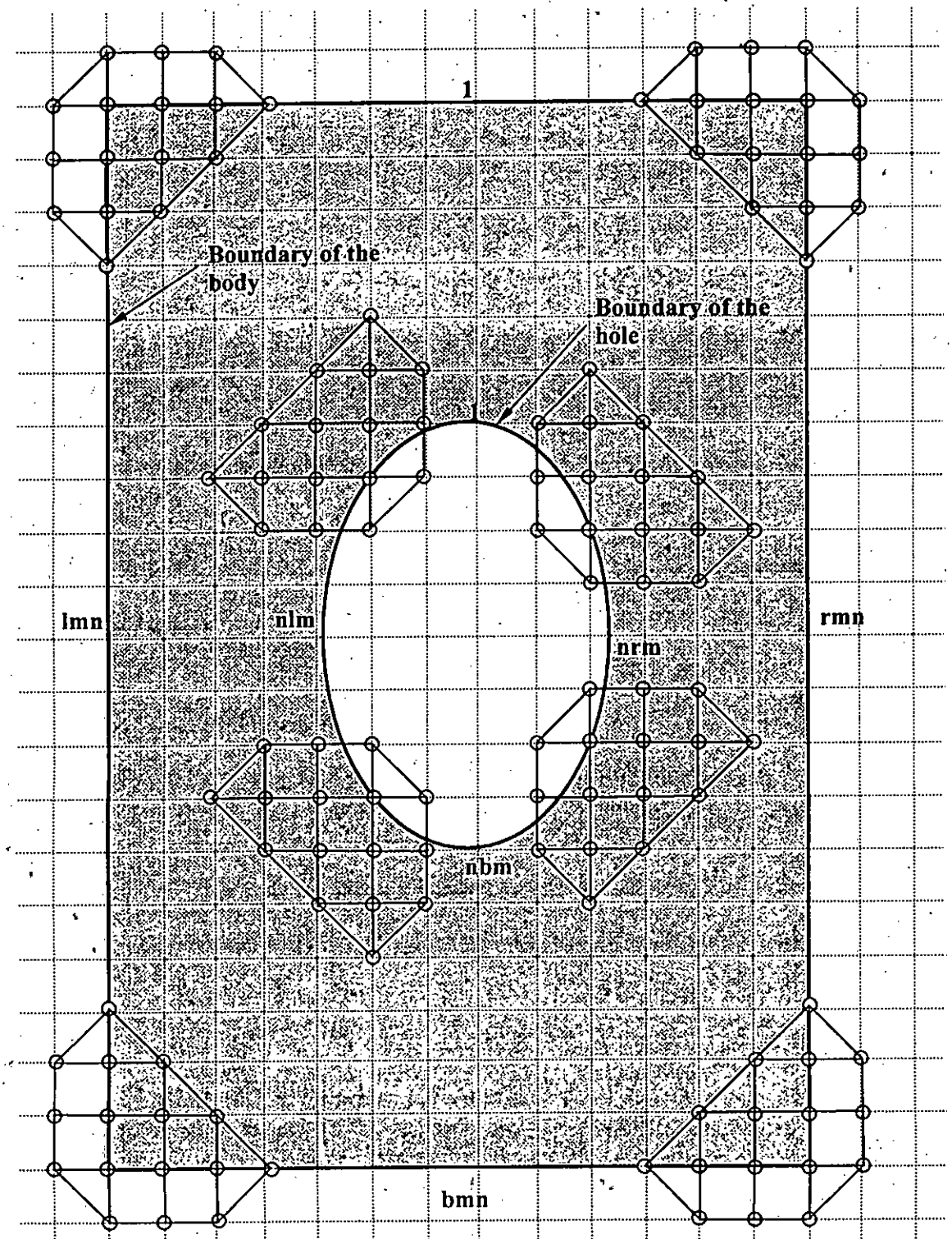


Fig A2 Structure of Finite difference expressions for stress components



The boundary equations are placed on the boundary in such a way so that most of the required grid points lie within the body.

Fig. A5. Application of boundary conditions

ANNEXURE B

Difference Equation for u_n (uun)

i = i2(jj),
j = j2(jj)

where i2 is the mesh distance from the false boundary line along x direction
j2 is the mesh distance from the false boundary line along y direction

aa=a(jj)	aa = a (described in Fig 4.5)
bb=k-aa	bb = b (described in Fig 4.5)
ss=s(jj)	ss = s (described in Fig 4.5)
tt=h-ss	tt = t (described in Fig 4.5)
lp=l(jj)	lp = l (direction cosine)
mp=m(jj)	mp =m

For top -left boundary of the body and bottom -right boundary of the hole

i0=1
j0=1
x0 = 1.0
x1 = 1.0
x3 = 1.0
x4 = 1.0

For bottom -left boundary of the body and top -right boundary of the hole

i0=-1
j0=1
x0 = -1.0
x1 = -1.0
x3 = 1.0
x4 = 1.0

For bottom -right boundary of the body and top -left boundary of the hole

i0=-1
j0=-1
x0 = 1.0
x1 = 1.0
x3 = -1.0
x4 = -1.0

For top -right boundary of the body and bottom -left boundary of the hole

i0=1
j0=-1
x0 = -1.0
x1 = -1.0
x3 = -1.0
x4 = -1.0

```
m1=a1(i,j)
m2=a1(i-i0,j)
m3=a1(i,j-j0)
m4=a1(i,j+j0)
m5=a1(i+i0,j)
m6=a1(i-i0,j+j0)
m7=a1(i+i0,j-j0)
m8=a1(i-i0,j+2*j0)
m9=a1(i+2*i0,j-j0)
```

ANNEXURE B

Difference Equation for u_n **(uun)**

```
m10=a1(i,j+2*j0)
m11=a1(i+i0,j+j0)
m12=a1(i+2*i0,j)
m21=a1(i+2*i0,j+2*j0)
m14=a1(i,j+3*j0)
m15=a1(i+i0,j+2*j0)
m16=a1(i+2*i0,j+j0)
m17=a1(i+3*i0,j)
    m13=a1(i-i0,j+3*j0)
    m18=a1(i+3*i0,j-j0)
    m19=a1(i,j+4*j0)
    m20=a1(i+i0,j+3*j0)
    m22=a1(i+3*i0,j+j0)
    m23=a1(i+4*i0,j)
    m24=a1(i+i0,j+4*j0)
    m25=a1(i+2*i0,j+3*j0)
    m26=a1(i+3*i0,j+2*j0)
    m27=a1(i+4*i0,j+j0)
    m28=a1(i+3*i0,j+3*j0)

sk1=aa*tt*zk6*mp*s2+bb*tt*(9.*s1*x1*lp-mp*s2)+bb*ss*zk7*mp*s2
sk2=bb*tt*zk7*mp*s2
sk3=bb*tt*zk6*mp*s2
sk4=mp*s2*(bb*tt*zk6+aa*ss*zk7-aa*tt)+tt*s1*x1*lp*(9.*aa-bb*12.)
sk5=mp*s2*(bb*tt*zk7+aa*ss*zk6-bb*ss)+bb*s1*x1*lp*(9.*ss-12.*tt)
sk6=aa*tt*zk7*mp*s2
sk7=bb*ss*zk6*mp*s2
sk10=3.*tt*s1*x1*lp*(bb-4.*aa)+aa*tt*zk6*mp*s2
sk11=s1*x1*lp*(16.*bb*tt-12.*aa*tt+9.*aa*ss-12.*bb*ss)+mp*s2*(aa*tt*zk7-
        aa*ss+bb*ss*zk6)
sk12=3.*bb*s1*x1*lp*(tt-4.*ss)+mp*s2*bb*ss*zk7
sk14=aa*tt*3.*s1*x1*lp
sk15=s1*x1*lp*(16.*aa*tt-4.*bb*tt-12.*aa*ss+3.*bb*ss)+ aa*ss*zk6*mp*s2
sk16=s1*x1*lp*(3.*aa*tt-4.*bb*tt-12.*aa*ss+16.*bb*ss)+aa*ss*zk7*mp*s2
sk17=bb*ss*3.*s1*x1*lp
sk21=s1*x1*lp*(bb*tt-4.*aa*tt+16.*aa*ss-4.*bb*ss)
    sk20=aa*s1*x1*lp*(3.*ss-4.*tt)
    sk22=ss*s1*x1*lp*(3.*aa-4.*bb)
    sk25=aa*s1*x1*lp*(tt-4.*ss)
    sk26=ss*s1*x1*lp*(bb-4.*aa)
    sk28=aa*ss*s1*x1*lp

uun(jj) =   sk1*si(m1)+sk2*si(m2)+sk3*si(m3)+sk4*si(m4)+sk5*si(m5)+
            sk6*si(m6)+sk7*si(m7)+sk10*si(m10)+sk11*si(m11)+sk12*si(m12)+
            sk14*si(m14)+sk15*si(m15)+sk16*si(m16)+sk17*si(m17)+sk21*si(m21)
```

For boundary points do not match with a rectangular grid point then,

```
uun(jj) =   uun(jj)+sk20*si(m20)+sk22*si(m22)+sk25*si(m25)+
            sk26*si(m26)+sk28*si(m28)
```

ANNEXURE B

Difference Equation for u_t(uut)

To transform the difference equation of u_n to u_t ,
Change direction cosine *l* with *-m* and *m* with *l*

```
ifloat=lp
lp = -mp
mp = ifloat
```

```
sk1 =   aa*tt*zk6*mp*s2+bb*tt*(9.*s1*x1*lp-mp*s2)+bb*ss*zk7*mp*s2
sk2 =   bb*tt*zk7*mp*s2
sk3 =   bb*tt*zk6*mp*s2
sk4 =   mp*s2*(bb*tt*zk6+aa*ss*zk7-aa*tt)+tt*s1*x1*lp*(9.*aa-bb*12.)
sk5 =   mp*s2*(bb*tt*zk7+aa*ss*zk6-bb*ss)+bb*s1*x1*lp*(9.*ss-12.*tt)
sk6 =   aa*tt*zk7*mp*s2
sk7 =   bb*ss*zk6*mp*s2
sk10 =  3.*tt*s1*x1*lp*(bb-4.*aa)+aa*tt*zk6*mp*s2
sk11 =  s1*x1*lp*(16.*bb*tt-12.*aa*tt+9.*aa*ss-12.*bb*ss)+mp*s2*(aa*tt*zk7-
        aa*ss+bb*ss*zk6)
sk12 =  3.*bb*s1*x1*lp*(tt-4.*ss)+mp*s2*bb*ss*zk7
sk14 =  aa*tt*3.*s1*x1*lp
sk15 =  s1*x1*lp*(16.*aa*tt-4.*bb*tt-12.*aa*ss+3.*bb*ss)+ aa*ss*zk6*mp*s2
sk16 =  s1*x1*lp*(3.*aa*tt-4.*bb*tt-12.*aa*ss+16.*bb*ss)+aa*ss*zk7*mp*s2
sk17 =  bb*ss*3.*s1*x1*lp
sk21 =  s1*x1*lp*(bb*tt-4.*aa*tt+16.*aa*ss-4.*bb*ss)
    sk20=aa*s1*x1*lp*(3.*ss-4.*tt)
    sk22=ss*s1*x1*lp*(3.*aa-4.*bb)
    sk25=aa*s1*x1*lp*(tt-4.*ss)
    sk26=ss*s1*x1*lp*(bb-4.*aa)
    sk28=aa*ss*s1*x1*lp
```

```
uut(jj) =   sk1*si(m1)+sk2*si(m2)+sk3*si(m3)+sk4*si(m4)+sk5*si(m5)+
            sk6*si(m6)+sk7*si(m7)+sk10*si(m10)+sk11*si(m11)+sk12*si(m12)+
            sk14*si(m14)+sk15*si(m15)+sk16*si(m16)+sk17*si(m17)+sk21*si(m21)
```

If any boundary point does not match with a rectangular grid point then,

```
uut(jj) =   uut(jj)+sk20*si(m20)+sk22*si(m22)+sk25*si(m25)+
            sk26*si(m26)+sk28*si(m28)
```

Annexure B

Difference Equation for σ_n(ssgn)

```
sai1 =   ((5.+6.*zk10)*s4*x4*mp*mp+(6.*zk9-5.)*s3*x3*lp*lp+
         (3.-10.*zk9)*2.*r*x0*s3*x3*lp*mp)
sai2 =   (3.*zk9*s3*x3*(2.*lp*mp*r*x0-lp*lp)-3.*zk10*s4*x4*mp*mp)
sai3 =   1.5*(s3*x3*lp*lp-s4*x4*mp*mp-2.*r*x0*s3*x3*lp*mp)
sai4 =   ((6.-8.*zk9)*s3*x3*lp*lp-1.5*r*x0*s3*x3*2.*lp*mp-(6.+8.*zk10)*s4*x4*mp*mp)
sai6 =   4.*(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai5 =   2.*r*x0*s3*x3*lp*mp*(12.*zk9-4.)-3.*(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai7 =   r*x0*s3*x3*4.*lp*mp
sai8 =   -(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai9 =   -r*x0*s3*x3*lp*mp
sai10 =  ((3.+2.*zk10)*s4*x4*mp*mp+(2.*zk9-3.)*s3*x3*lp*lp)
sai11 =  4.*(zk10*s4*x4*mp*mp+zk9*s3*x3*lp*lp+r*x0*s3*x3*lp*mp)
sai12 =  2.*(1.-6.*zk9)*r*x0*s3*x3*lp*mp
sai14 =  0.5*(s3*x3*lp*lp-s4*x4*mp*mp)
sai15 =  -(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai16 =  -r*x0*s3*x3*lp*mp
sai17 =  2.*lp*mp*zk9*r*x0*s3*x3
sk1 =    bb*ss*sai2+bb*tt*sai1+aa*tt*sai3
sk2 =    bb*tt*sai2
sk3 =    bb*tt*sai3
sk4 =    aa*ss*sai2+bb*ss*sai6+bb*tt*sai4+aa*tt*sai1
sk5 =    bb*ss*sai1+aa*tt*sai7+bb*tt*sai5+aa*ss*sai3
sk6 =    bb*tt*sai6+aa*tt*sai2
sk7 =    bb*ss*sai3+bb*tt*sai7
sk8 =    bb*tt*sai8+aa*tt*sai6
sk9 =    bb*ss*sai7+bb*tt*sai9
sk10 =   bb*ss*sai8+aa*ss*sai6+bb*tt*sai10+aa*tt*sai4
sk11 =   bb*ss*sai4+bb*tt*sai11+aa*tt*sai5+aa*ss*sai1
sk12 =   bb*ss*sai5+bb*tt*sai12+aa*tt*sai9+aa*ss*sai7
sk14 =   aa*ss*sai8+bb*tt*sai14+aa*tt*sai10
sk15 =   bb*ss*sai10+bb*tt*sai15+aa*tt*sai11+aa*ss*sai4
sk16 =   bb*ss*sai11+bb*tt*sai16+aa*tt*sai12+aa*ss*sai5
sk17 =   bb*ss*sai12+bb*tt*sai17+aa*ss*sai9
    sk13 =    aa*tt*sai8
     sk18 =   bb*ss*sai9
     sk19 =   aa*tt*sai14
     sk20 =   aa*ss*sai10+bb*ss*sai14+aa*tt*sai15
     sk21 =   aa*tt*sai16+bb*ss*sai15+aa*ss*sai11
     sk22 =   bb*ss*sai16+aa*tt*sai17+aa*ss*sai12
     sk23 =   bb*ss*sai17
     sk24 =   aa*ss*sai14
     sk25 =   aa*ss*sai15
     sk26 =   aa*ss*sai16
     sk27 =   aa*ss*sai17
ssgn(jj) =  sk1*si(m1)+sk2*si(m2)+sk3*si(m3)+sk4*si(m4)+sk5*si(m5)+
            sk6*si(m6)+sk7*si(m7)+sk8*si(m8)+sk9*si(m9)+sk10*si(m10)+sk11*
            si(m11)+sk12*si(m12)+sk14*si(m14)+sk15*si(m15)+sk16*si(m16)+sk17*si(m17)
```

If any boundary point does not match with a rectangular grid point then,

```
ssgn(jj) =ssgn(jj)+sk13*si(m13)+ sk18*si(m18)+sk19*si(m19)+sk20*+si(m20)+sk21*si(m21)+
   sk22*si(m22)+sk23*si(m23)+sk24*si(m24)+sk25*si(m25)+sk26*si(m26)+sk27*si(m27)
```

ANNEXURE B

Difference Equation for σ_t(ssgt)

```
udd1= lp*mp*(s3*x3*(6.*zk9-5.)-s4*x4*(5.+6.*zk10))+r*x0*s3*x3*
      (mp*mp-lp*lp)*(3.-10.*zk9)
udd2= 3.*lp*mp*(zk10*s4*x4-zk9*s3*x3)+3.*zk9*r*x0*s3*x3*(mp*mp-lp*lp)
udd3=1.5*(lp*mp*(s4*x4+s3*x3)-(r*x0*s3*x3*(mp*mp-lp*lp)))
udd4=lp*mp*(s4*x4*(6.+8.*zk10)+s3*x3*(6.-8.*zk9))-1.5*r*x0*s3*x3*(mp*mp-lp*lp)
udd5=3.*lp*mp*(s4*x4*zk10-s3*x3*zk9)+r*x0*s3*x3*(mp*mp-lp*lp)*(12.*zk9-4.)
udd6=4.*lp*mp*(zk9*s3*x3-zk10*s4*x4)
udd7=2.*r*x0*s3*x3*(mp*mp-lp*lp)
udd8=lp*mp*(zk10*s4*x4-zk9*s3*x3)
udd9=-0.5*r*x0*s3*x3*(mp*mp-lp*lp)
udd10=lp*mp*(s3*x3*(2.*zk9-3.)-s4*x4*(3.+2.*zk10))
udd11=4.*lp*mp*(s3*x3*zk9-s4*x4*zk10)+2.*r*x0*s3*x3*(mp*mp-lp*lp)
udd12=r*x0*s3*x3*(1.-6.*zk9)*(mp*mp-lp*lp)
udd14=0.5*lp*mp*(s4*x4+s3*x3)
udd15=lp*mp*(s4*x4*zk10-s3*x3*zk9)
udd16=-0.5*r*x0*s3*x3*(mp*mp-lp*lp)
udd17=r*x0*s3*x3*zk9*(mp*mp-lp*lp)
sk1=bb*ss*udd2+aa*tt*udd3+bb*tt*udd1
sk2=bb*tt*udd2
sk3=bb*tt*udd3
sk4=aa*ss*udd2+bb*ss*udd6+bb*tt*udd4+aa*tt*udd1
sk5=aa*ss*udd3+bb*ss*udd1+bb*tt*udd5+aa*tt*udd7
sk6=bb*tt*udd6+aa*tt*udd2
sk7=bb*tt*udd7+bb*ss*udd3
sk8=bb*tt*udd8+aa*tt*udd6
sk9=bb*ss*udd7+bb*tt*udd9
sk10=aa*ss*udd6+bb*ss*udd8+bb*tt*udd10+aa*tt*udd4
sk11=aa*ss*udd1+bb*ss*udd4+bb*tt*udd11+aa*tt*udd5
sk12=aa*ss*udd7+bb*ss*udd5+bb*tt*udd12+aa*tt*udd9
sk14=aa*ss*udd8+bb*tt*udd14+aa*tt*udd10
sk15=aa*ss*udd4+bb*ss*udd10+bb*tt*udd15+aa*tt*udd11
sk16=bb*ss*udd11+bb*tt*udd16+aa*tt*udd12+aa*ss*udd5
sk17=bb*ss*udd12+bb*tt*udd17+aa*ss*udd9
    sk13=aa*tt*udd8
    sk18=bb*ss*udd9
    sk19=aa*tt*udd14
    sk20=aa*ss*udd10+bb*ss*udd14+aa*tt*udd15
    sk21=bb*ss*udd15+aa*tt*udd16+aa*ss*udd11
    sk22=bb*ss*udd16+aa*tt*udd17+aa*ss*udd12
    sk23=bb*ss*udd17
    sk24=aa*ss*udd14
    sk25=aa*ss*udd15
    sk26=aa*ss*udd16
    sk27=aa*ss*udd17

ssgt(jj) =  sk1*si(m1)+sk2*si(m2)+sk3*si(m3)+sk4*si(m4)+sk5*si(m5)+
            sk6*si(m6)+sk7*si(m7)+sk8*si(m8)+sk9*si(m9)+sk10*si(m10)+sk11*
            si(m11)+sk12*si(m12)+sk14*si(m14)+sk15*si(m15)+
            sk16*si(m16)+sk17*si(m17)
```

If any boundary point does not match with a rectangular grid point then,

```
ssgt(jj) =  ssgt(jj)+sk13*si(m13)+sk18*si(m18)+sk19*si(m19)+sk20*
            si(m20)+sk21*si(m21)+sk22*si(m22)+sk23*si(m23)+sk24*si(m24)+
            sk25*si(m25)+sk26*si(m26)+sk27*si(m27)
```

```
rs = r*r          where r=k/h
hs = h*h
c1 = 1.0-c
c2 = 1.0+c
zk1=rs*rs
zk2=4.0*(rs*rs+rs)
zk3=4.0*(1.0+rs)
zk4=6.0+8.0*rs+6.0*rs*rs
zk5=2.0*rs
zk6=(c1/2.0)/(c1+2.0*rs)
zk66=(c1/2)/(c1-rs)
zk7=rs/(c1+2.0*rs)
zk77=rs/(c1-rs)
zk8=(rs*hs*c2)/(2.0*(c1+2.0*rs))
zk88=(hs*rs*c2)/(2.0*(c1-rs))
zk9=rs/(2.0*c)
zk10=rs*(2.0+c)/2.0
p2=c2*c2*r*rs*hs*h/c
s1=1.0/(4.0*r*h*h)
s2=-1.0/zk8
s22=-1.0/zk88
s3=1.0/p2
s4=-1.0/(c*p2)
```

Taking,

```
k1 = ktb(jj)
i = i2(jj)
j = j2(jj)
aa=a(jj)
bb=k-aa
ss=s(jj)
tt=h-ss

lp=l(jj+kdc(jj))
mp=m(jj+kdc(jj))
```

For top left boundary of body and bottom –right boundary of hole

```
i0=1
j0=1
x0 = 1.0
x1 = 1.0
x3 = 1.0
x4 = 1.0
```

For bottom left boundary of body and top –right boundary of hole

```
i0=-1
j0=1
x0 = -1.0
x1 = -1.0
x3 = 1.0
x4 = 1.0
```

ANNEXURE B1
Coefficient Matrix Generation

For bottom right boundary of body and top –left boundary of hole

i0=-1
j0=-1
x0 = 1.0
x1 = 1.0
x3 = -1.0
x4 = -1.0

For top right boundary of body and bottom –left boundary of hole

i0=1
j0=-1
x0 = -1.0
x1 = -1.0
x3 = -1.0
x4 = -1.0

m1=a1(i,j)
m2=a1(i-i0,j)
m3=a1(i,j-j0)
m4=a1(i,j+j0)
m5=a1(i+i0,j)
m6=a1(i-i0,j+j0)
m7=a1(i+i0,j-j0)
m8=a1(i-i0,j+2*j0)
m9=a1(i+2*i0,j-j0)
m10=a1(i,j+2*j0)
m11=a1(i+i0,j+j0)
m12=a1(i+2*i0,j)
m21=a1(i+2*i0,j+2*j0)
m14=a1(i,j+3*j0)
m15=a1(i+i0,j+2*j0)
m16=a1(i+2*i0,j+j0)
m17=a1(i+3*i0,j)
m13=a1(i-i0,j+3*j0)
m18=a1(i+3*i0,j-j0)
m19=a1(i,j+4*j0)
m20=a1(i+i0,j+3*j0)
m22=a1(i+3*i0,j+j0)
m23=a1(i+4*i0,j)
m24=a1(i+i0,j+4*j0)
m25=a1(i+2*i0,j+3*j0)
m26=a1(i+3*i0,j+2*j0)
m27=a1(i+4*i0,j+j0)
m28=a1(i+3*i0,j+3*j0)

For the arrangement of boundary values on the boundary following settings are made,

Normal component on the boundary and tangential on the false boundary and the orientation is up or down as shown in Table B1

k2 = m2
k3 = m1

Normal component on the boundary and tangential on the false boundary and the orientation is left or right shown in Table B1

k2 = m3
k3 = m1

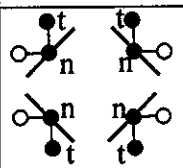
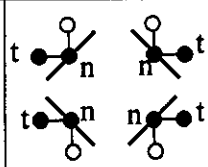
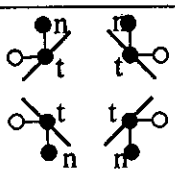
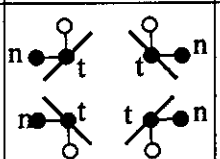
Normal component on the false boundary and tangential on the boundary and the orientation is up or down as shown in Table B1

k2 = m1
k3 = m2

Normal component on the false boundary and tangential on the boundary and the orientation is left or right shown in Table B1

k2 = m1
k3 = m3

Table B1: Selection of boundary conditions[12]

ktb		kpc		kdc		ktc	
Value	Control	Value	Control	Value	Control	Value	Control
1	u_n, u_t	1		0	lp = l(ij+0) mp = m(ij+0)	-1	n
2	u_n, σ_t	2		1	lp = l(ij+1) mp = m(ij+1)	1	t
3	σ_n, u_t	3		-1	lp = l(ij-1) mp = m(ij-1)	0	n,t
4	σ_n, σ_t	4				2	(skip)

ANNEXURE B1
Coefficient Matrix Generation

FOR BOUNDARY VALUE OF u_n

```
sik(k3,m1) =  aa*tt*zk6*mp*s2+bb*tt*(9.*s1*x1*lp-mp*s2)+bb*ss*zk7*mp*s2
sik(k3,m2) =  bb*tt*zk7*mp*s2
sik(k3,m3) =  bb*tt*zk6*mp*s2
sik(k3,m4) =  mp*s2*(bb*tt*zk6+aa*ss*zk7-aa* tt)+tt*s1*x1*lp*(9.*aa-bb*12.)
sik(k3,m5) =  mp*s2*(bb*tt*zk7+aa*ss*zk6-bb*ss)+bb*s1*x1*lp*(9.*ss-12.*tt)
sik(k3,m6) =  aa*tt*zk7*mp*s2
sik(k3,m7) =  bb*ss*zk6*mp*s2
sik(k3,m10) = 3.*tt*s1*x1*lp*(bb-4.*aa)+aa*tt*zk6*mp*s2
sik(k3,m11) = s1*x1*lp*(16.*bb*tt-12.*aa*tt+9.*aa*ss-12.*bb*ss)+
              mp*s2*(aa*tt*zk7-aa*ss+bb*ss*zk6)
sik(k3,m12) = 3.*bb*s1*x1*lp*(tt-4.*ss)+mp*s2*bb*ss*zk7
sik(k3,m14) = aa*tt*3.*s1*x1*lp
sik(k3,m15) = s1*x1*lp*(16.*aa*tt-4.*bb*tt-12.*aa*ss+3.*bb*ss)+aa*ss*zk6*mp*s2
sik(k3,m16) = s1*x1*lp*(3.*aa*tt-4.*bb*tt-12.*aa*ss+16.*bb*ss)+aa*ss*zk7*mp*s2
sik(k3,m17) = bb*ss*3.*s1*x1*lp
sik(k3,m21) = s1*x1*lp*(bb*tt-4.*aa*tt+16.*aa*ss-4.*bb*ss)
```

If boundary points do not match with node points then,

```
sik(k3,m20) = aa*s1*x1*lp*(3.*ss-4.*tt)
sik(k3,m22) = ss*s1*x1*lp*(3.*aa-4.*bb)
sik(k3,m25) = aa*s1*x1*lp*(tt-4.*ss)
sik(k3,m26) = ss*s1*x1*lp*(bb-4.*aa)
sik(k3,m28) = aa*ss*s1*x1*lp
```

All coefficient from* sik(k3,1) to sik(k3,nog) *except the above coefficients are zero.

Where, **nog** = number of grid points on the whole body.

```
ck(k3) = un(jj)*h*k
```

ANNEXURE B1
Coefficient Matrix Generation

FOR BOUNDARY VALUE OF u_t

```
ifloat=lp
 lp = -mp
 mp = ifloat
 sik(k2,m1)=aa*tt*zk6*mp*s2+bb*tt*(9.*s1*x1*lp-mp*s2)+bb*ss*zk7*mp*s2
 sik(k2,m2)=bb*tt*zk7*mp*s2
 sik(k2,m3)=bb*tt*zk6*mp*s2
 sik(k2,m4)=mp*s2*(bb*tt*zk6+aa*ss*zk7-aa*tt)+tt*s1*x1*lp*(9.*aa-bb*12.)
 sik(k2,m5)=mp*s2*(bb*tt*zk7+aa*ss*zk6-bb*ss)+bb*s1*x1*lp*(9.*ss-12.*tt)
 sik(k2,m6)=aa*tt*zk7*mp*s2
 sik(k2,m7)=bb*ss*zk6*mp*s2
 sik(k2,m10)=3.*tt*s1*x1*lp*(bb-4.*aa)+aa*tt*zk6*mp*s2
 sik(k2,m11)=s1*x1*lp*(16.*bb*tt-12.*aa*tt+9.*aa*ss-12.*bb*ss)+mp*s2*
    (aa*tt*zk7- aa*ss+bb*ss*zk6)
 sik(k2,m12)=3.*bb*s1*x1*lp*(tt-4.*ss)+mp*s2*bb*ss*zk7
 sik(k2,m14)=aa*tt*3.*s1*x1*lp
 sik(k2,m15)=s1*x1*lp*(16.*aa*tt-4.*bb*tt-12.*aa* ss+3.*bb*ss)+aa*ss*zk6*mp*s2
 sik(k2,m16)=s1*x1*lp*(3.*aa*tt-4.*bb*tt-12.* aa*ss+16.*bb*ss)+aa*ss*zk7*mp*s2
 sik(k2,m17)=bb*ss*3.*s1*x1*lp
 sik(k2,m21)=s1*x1*lp*(bb*tt-4.*aa*tt+16.*aa*ss-4.*bb*ss)
```

If boundary points do not match with node points then,

```
sik(k2,m20)=aa*s1*x1*lp*(3.*ss-4.*tt)
sik(k2,m22)=ss*s1*x1*lp*(3.*aa-4.*bb)
sik(k2,m25)=aa*s1*x1*lp*(tt-4.*ss)
sik(k2,m26)=ss*s1*x1*lp*(bb-4.*aa)
sik(k2,m28)=aa*ss*s1*x1*lp
```

All coefficient from* sik(k2,1) to sik(k2,nog) *except the above coefficients are zero.
Where, **nog** = number of grid points on the whole body.

```
ck(k2) = ut(jj)*h*k
```

ANNEXURE B1
Coefficient Matrix Generation

FOR BOUNDARY VALUE OF σ_n

```
sai1=((5.+6.*zk10)*s4*x4*mp*mp+(6.*zk9-5.)*s3*x3*lp*lp+
      (3.-10.*zk9)*2.*r*x0*s3*x3*lp*mp)
sai2=(3.*zk9*s3*x3*(2.*lp*mp*r*x0-lp*lp)-3.*zk10*s4*x4*mp*mp)
sai3=1.5*(s3*x3*lp*lp-s4*x4*mp*mp-2.*r*x0*s3*x3*lp*mp)
sai4=((6.-8.*zk9)*s3*x3*lp*lp-1.5*r*x0*s3*x3*2.*lp*
        mp-(6.+8.*zk10)*s4*x4*mp*mp)
sai6=4.*(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai5=2.*r*x0*s3*x3*lp*mp*(12.*zk9-4.)-3.*(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai7=r*x0*s3*x3*4.*lp*mp
sai8=-(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai9=-r*x0*s3*x3*lp*mp
sai10=((3.+2.*zk10)*s4*x4*mp*mp+(2.*zk9-3.)*s3*x3*lp*lp)
sai11=4.*(zk10*s4*x4*mp*mp+zk9*s3*x3*lp*lp+r*x0*s3*x3*lp*mp)
sai12=2.*(1.-6.*zk9)*r*x0*s3*x3*lp*mp
sai14=0.5*(s3*x3*lp*lp-s4*x4*mp*mp)
sai15=-(zk9*s3*x3*lp*lp+zk10*s4*x4*mp*mp)
sai16=-r*x0*s3*x3*lp*mp
sai17=2.*lp*mp*zk9*r*x0*s3*x3
sik(k3,m1)=bb*ss*sai2+bb*tt*sai1+aa*tt*sai3
sik(k3,m2)=bb*tt*sai2
sik(k3,m3)=bb*tt*sai3
sik(k3,m4)=aa*ss*sai2+bb*ss*sai6+bb*tt*sai4+aa*tt*sai1
sik(k3,m5)=bb*ss*sai1+aa*tt*sai7+bb*tt*sai5+aa*ss*sai3
sik(k3,m6)=bb*tt*sai6+aa*tt*sai2
sik(k3,m7)=bb*ss*sai3+bb*tt*sai7
sik(k3,m8)=bb*tt*sai8+aa*tt*sai6
sik(k3,m9)=bb*ss*sai7+bb*tt*sai9
sik(k3,m10)=bb*ss*sai8+aa*ss*sai6+bb*tt*sai10+aa*tt*sai4
sik(k3,m11)=bb*ss*sai4+bb*tt*sai11+aa*tt*sai5+aa*ss*sai1
ik(k3,m12)=bb*ss*sai5+bb*tt*sai12+aa*tt*sai9+aa*ss*sai7
sik(k3,m14)=aa*ss*sai8+bb*tt*sai14+aa*tt*sai10
sik(k3,m15)=bb*ss*sai10+bb*tt*sai15+aa*tt*sai11+aa*ss*sai4
sik(k3,m16)=bb*ss*sai11+bb*tt*sai16+aa*tt*sai12+aa*ss*sai5
sik(k3,m17)=bb*ss*sai12+bb*tt*sai17+aa*ss*sai9
```

If boundary points do not match with node points then,

```
sik(k3,m13)=aa*tt*sai8
sik(k3,m18)=bb*ss*sai9
sik(k3,m19)=aa*tt*sai14
sik(k3,m20)=aa*ss*sai10+bb*ss*sai14+aa*tt*sai15
sik(k3,m21)=aa*tt*sai16+bb*ss*sai15+aa*ss*sai11
sik(k3,m22)=bb*ss*sai16+aa*tt*sai17+aa*ss*sai12
sik(k3,m23)=bb*ss*sai17
sik(k3,m24)=aa*ss*sai14
sik(k3,m25)=aa*ss*sai15
sik(k3,m26)=aa*ss*sai16
sik(k3,m27)=aa*ss*sai17
```

All coefficient from* sik(k3,1) to sik(k3,nog) *except the above coefficients are zero.

Where, **nog** = number of grid points on the whole body.

```
ck(k3)=signn(jj)*h*k
```

ANNEXURE B1
Coefficient Matrix Generation

FOR BOUNDARY VALUE OF σ_t

```
udd1 = lp*mp*(s3*x3*(6.*zk9-5.)-s4*x4*(5.+6.*zk10))+r*x0*s3*x3*
       (mp*mp-lp*lp)*(3.-10.*zk9)
udd2 = 3.*lp*mp*(zk10*s4*x4-zk9*s3*x3)+3.*zk9*r*x0*s3*x3*(mp*mp-lp*lp)
udd3 = 1.5*(lp*mp*(s4*x4+s3*x3)-(r*x0*s3*x3*(mp*mp-lp*lp)))
udd4 = lp*mp*(s4*x4*(6.+8.*zk10)+s3*x3*(6.-8.*zk9))-1.5*r*x0*s3*x3*(mp*mp-lp*lp)
udd5 = 3.*lp*mp*(s4*x4*zk10-s3*x3*zk9)+r*x0*s3*x3*(mp*mp-lp*lp)*(12.*zk9-4.)
udd6=4.*lp*mp*(zk9*s3*x3-zk10*s4*x4)
udd7=2.*r*x0*s3*x3*(mp*mp-lp*lp)
udd8=lp*mp*(zk10*s4*x4-zk9*s3*x3)
udd9=-0.5*r*x0*s3*x3*(mp*mp-lp*lp)
udd10=lp*mp*(s3*x3*(2.*zk9-3.)-s4*x4*(3.+2.*zk10))
udd11=4.*lp*mp*(s3*x3*zk9-s4*x4*zk10)+2.*r*x0*s3*x3*(mp*mp-lp*lp)
udd12=r*x0*s3*x3*(1.-6.*zk9)*(mp*mp-lp*lp)
udd14=0.5*lp*mp*(s4*x4+s3*x3)
udd15=lp*mp*(s4*x4*zk10-s3*x3*zk9)
udd16=-0.5*r*x0*s3*x3*(mp*mp-lp*lp)
udd17=r*x0*s3*x3*zk9*(mp*mp-lp*lp)
sik(k2,m1)=bb*ss*udd2+aa*tt*udd3+bb*tt*udd1
sik(k2,m2)=bb*tt*udd2
sik(k2,m3)=bb*tt*udd3
sik(k2,m4)=aa*ss*udd2+bb*ss*udd6+bb*tt*udd4+aa*tt*udd1
sik(k2,m5)=aa*ss*udd3+bb*ss*udd1+bb*tt*udd5+aa*tt*udd7
sik(k2,m6)=bb*tt*udd6+aa*tt*udd2
sik(k2,m7)=bb*tt*udd7+bb*ss*udd3
sik(k2,m8)=bb*tt*udd8+aa*tt*udd6
sik(k2,m9)=bb*ss*udd7+bb*tt*udd9
sik(k2,m10)=aa*ss*udd6+bb*ss*udd8+bb*tt*udd10+aa*tt*udd4
sik(k2,m11)=aa*ss*udd1+bb*ss*udd4+bb*tt*udd11+aa*tt*udd5
sik(k2,m12)=aa*ss*udd7+bb*ss*udd5+bb*tt*udd12+aa*tt*udd9
sik(k2,m14)=aa*ss*udd8+bb*tt*udd14+aa*tt*udd10
sik(k2,m15)=aa*ss*udd4+bb*ss*udd10+bb*tt*udd15+aa*tt*udd11
sik(k2,m16)=bb*ss*udd11+bb*tt*udd16+aa*tt*udd12+aa*ss*udd5
sik(k2,m17)=bb*ss*udd12+bb*tt*udd17+aa*ss*udd9
```

If boundary points do not match with node points then,

```
sik(k2,m13)=aa*tt*udd8
sik(k2,m18)=bb*ss*udd9
sik(k2,m19)=aa*tt*udd14
sik(k2,m20)=aa*ss*udd10+bb*ss*udd14+aa*tt*udd15
sik(k2,m21)=bb*ss*udd15+aa*tt*udd16+aa*ss*udd11
sik(k2,m22)=bb*ss*udd16+aa*tt*udd17+aa*ss*udd12
sik(k2,m23)=bb*ss*udd17
sik(k2,m24)=aa*ss*udd14
sik(k2,m25)=aa*ss*udd15
sik(k2,m26)=aa*ss*udd16
sik(k2,m27)=aa*ss*udd17
```

All coefficient from* sik(k3,1) to sik(k3,nog) *except the above coefficients are zero.

Where, **nog** = number of grid points on the whole body.

```
ck(k2) = -sigt(jj)*h*k
```

ANNEXURE B1
Coefficient Matrix Generation

FOR THE POINTS INSIDE THE BODY

This coefficient are generating from the GOVERNING EQUATIONS

m1 = a1(i,j)
m2 = a1(i,j+1)
m3 = a1(i,j-1)
m4 = a1(i+1,j)
m5 = a1(i-1,j)
m6 = a1(i+1,j+1)
m7 = a1(i-1,j+1)
m8 = a1(i-1,j-1)
m9 = a1(i+1,j-1)
m10 = a1(i,j+2)
m11 = a1(i-2,j)
m12 = a1(i,j-2)
m13 = a1(i+2,j)

sik(m1,m1) =zk4
sik(m1,m2) =-zk3
sik(m1,m3) =-zk3
sik(m1,m4) =-zk2
sik(m1,m5) =-zk2
sik(m1,m6) =zk5
sik(m1,m7) =zk5
sik(m1,m8) =zk5
sik(m1,m9) =zk5
sik(m1,m10) =1.0
sik(m1,m11) =zk1
sik(m1,m12) =1.0
sik(m1,m13) =zk1

All coefficient from* sik(m1,1) to sik(m1,nog) *except the above coefficients are zero.
Where, **nog** = number of grid points on the whole body.

ck(m1) =0.0

Input Data File 1:

```
[INFILE1=BOUND?.INP] {{input data 1 for sbs}}
  CONSTANTS FOR THE GRAPHICS OF THE DOMAIN CONCERNED
    mm   lmn   bmn   rmn    c    itoplf  ibotlf  ibotrg  itoprg  ihole
   192   49    97   145   0.3     33      65      129     161     1

  range of field weight symboles & no. of additional b.c.
        xmin          xmax        icess
    0.5000E-04    0.1050E-02        4

  transfer & position controllers of the b.c.
j   i     ktb    kpc    kdc    ktci),i=1,mm) boundary value controlers
1    32     4      1      0      0
33   33     4      1     -1      0
34   64     1      4      0      0
65   65     4      1      1      0
66   128    4      1      0      0
129  129    4      1     -1      0
130  160    4      4      0      0
161  161    4      1      1      0
162  192    4      1      0      0

non-zero  boundary values (in pair)
(un(j) ut(j) or signn(j) sigt(j),j=1,mm) pair of boundary value
130   3.00E-04   0.00E+00
131   3.00E-04   0.00E+00
132   3.00E-04   0.00E+00
133   3.00E-04   0.00E+00
134   3.00E-04   0.00E+00
135   3.00E-04   0.00E+00
136   3.00E-04   0.00E+00
137   3.00E-04   0.00E+00
138   3.00E-04   0.00E+00
139   3.00E-04   0.00E+00
140   3.00E-04   0.00E+00
141   3.00E-04   0.00E+00
142   3.00E-04   0.00E+00
143   3.00E-04   0.00E+00
144   3.00E-04   0.00E+00
145   3.00E-04   0.00E+00
146   3.00E-04   0.00E+00
147   3.00E-04   0.00E+00
148   3.00E-04   0.00E+00
149   3.00E-04   0.00E+00
150   3.00E-04   0.00E+00
151   3.00E-04   0.00E+00
152   3.00E-04   0.00E+00
153   3.00E-04   0.00E+00
154   3.00E-04   0.00E+00
155   3.00E-04   0.00E+00
```

```
156   3.00E-04   0.00E+00
157   3.00E-04   0.00E+00
158   3.00E-04   0.00E+00
159   3.00E-04   0.00E+00
160   3.00E-04   0.00E+00
193   3.00E-04   0.00E+00

TRANSFER & POSITION CONTROLLERS FOR ADDITIONAL B.C.
     it(i)      ipc(i)  idc(i)  ivs(i),i=1,icess)
       33         3       1       2
       65         3      -1       2
      129         3       1       3
      161         3      -1       3

boundary values for additional points
(ut(j) un(j) signn(j) or sigt(j),j=1,icess)
          0.0
          0.0
     .3000E-03
     .3000E-03
*******************************************************
Data for the body with internal hole(arbitrary shape)
        mmo      nlm      nbm      nrm    icessh
        44        12       23       34       0

Transfer & Position controllers of the b.c.
j      i      ktbh    kpch    kdch    ktch(i),i=1,mm)  boundary  value
controlers
1     6      4       3       0       0
7     7      4       3       0       1
8    16      4       4       0       0
17   17      4       3       0       1
18   28      4       3       0       0
29   29      4       3       0       1
30   38      4       4       0       0
39   39      4       3       0       1
40   44      4       3       0       0

non-zero  boundary values (in pair)
(un(j) ut(j) or signn(j) sigt(j),j=1,mm) pair of boundary value
   43   0.0000000  0.0000E+00
   44   0.0000000  0.0000E+00
   45   0.0000000  0.0000E+00

  TRANSFER & POSITION CONTROLLERS FOR ADDITIONAL B.C for hole.
     ith(i)    ipch(i) idch(i) ivsh(i),i=1,icess)

boundary values for additional points
(ut(j) un(j) signn(j) or sigt(j),j=1,icess)
```

ANNEXURE C
INPUT DATA FILE

DESCRIPTION OF INFILE 1:

The input file has been developed as per the READ FORMAT.

mm is the no of points on the boundary

lmn bmn rmn are respectively the leftmost, bottom most and rightmost boundary points

c

itoplf ibotlf ibotrg itoprg respectively the topleft, bottom left, bottom right and top right boundary points

ihole tells the presence of hole or not.

For '0' there is no hole inside the body and for '1' there is hole inside the body.

`range of field weight symboles & no. of additional b.c.`

xmin and **xmax** are respectively the highest and lowest value of the symbols for representing results in output file plot.out

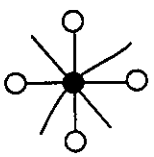
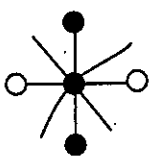
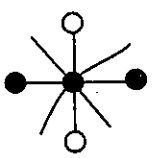
icess number of excess points presents on the boundary as described in art 4.3

j i are the starting and ending boundary points for assigning a set of boundary conditions

ktb kpc kdc ktc boundary value controllers as described in Table B1 in annexure B

The transfer and positional controllers for the additional boundary points **it(i)**, **ipc(i)**, **idc(i)** and **ivs(i)** are selected according to the following table.

Table C1

it/ith	**ipc/ipch**		**idc/idch**		**ivs/ivsh**	
	Value	Control	Value	Control	Value	Control
Serial number of the boundary point corresponding to the additional or previously omitted boundary or for given ψ values	1	k2=m1 	0	lp = l(ij+0) mp = m(ij+0)	1 2	u_t u_n
	2	k2=m2 	1	lp = l(ij+1) mp = m(ij+1)	3 4	σ_n σ_t
	3	k2=m3 	-1	lp = l(ij-1) mp = m(ij-1)	5	Declaration of ψ values

mmo is the no of points on the boundary for hole

nlm nbm nrm are respectively the leftmost, bottom most and rightmost boundary points of the hole

icessh number of excess points presents on the boundary as described in art 4.3

ktbh, kpch, kdch and **ktch** boundary value controllers for hole boundary, similar to **ktb, kpc, kdc** and **ktc** as described in Table B1 in annexure B

ith(i), ipch(i), idch(i) and **ivsh(i)** The transfer and positional controllers for the additional boundary points for the hole boundary as described in Table C1.

Input Data File 2:

```
**********************************
      CONSTANTS FOR THE GRAPHICS OF THE STRESSED BODY
      xl      xr      yt      yb    mn    nn   irc
    2.00   34.00   22.50    6.50   64    32    0

    boundary coordinates
bx(i)    by(i)          i=1, mm+2 boundary coordinates
18.00      22.50
.          .
.          ..
.          .
21.00      22.50
20.50      22.50
20.00      22.50
19.50      22.50
19.00      22.50
18.50      22.50
18.00      22.50
17.50      22.50

boundary   value for internal hole
bx(i),           by(i), i=1, mmo+2
18.0000    18.5000
.          .
.          .
17.5000    18.4686
18.5000    18.4686
18.0000    18.5000
17.5000    18.4686
```

DESCRIPTION OF INFILE 2:

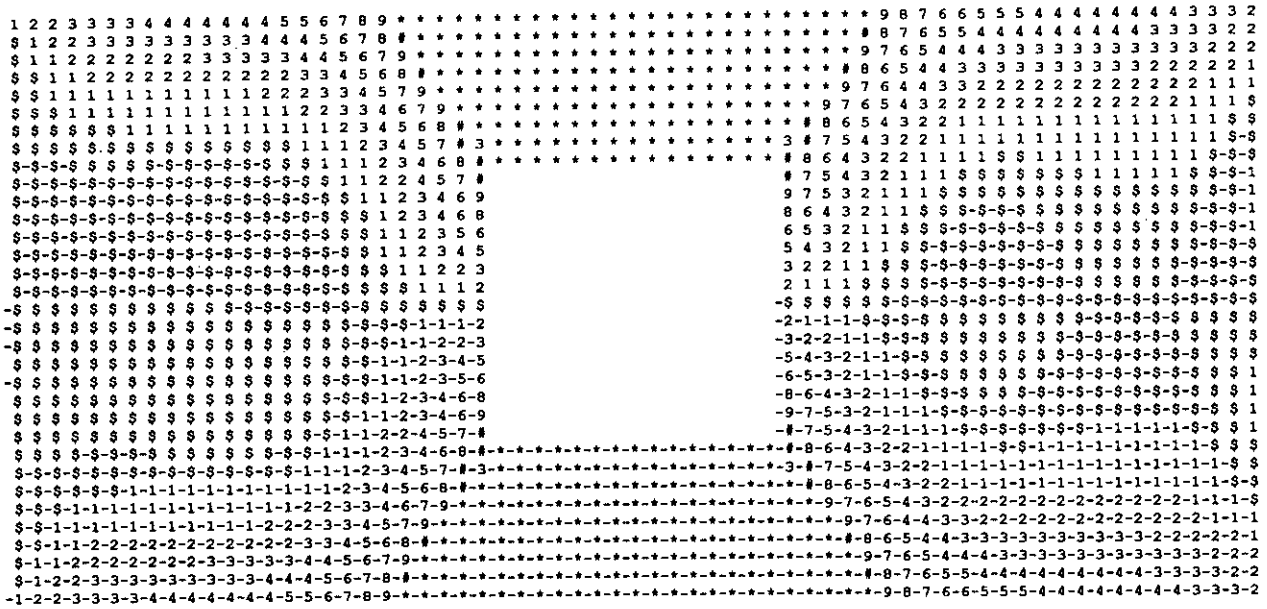
The input file has been developed as per the READ FORMAT.
The boundary coordinates of each boundary points are given in this file

xl xr	The minimum and maximum *x* value of the boundary coordinates
yt yb	The maximum and minimum *y* value of the boundary coordinates
mn nn	Number of Horizontal and vertical mesh
irc	Tells whether the body is rectangular shaped or arbitrary shaped.

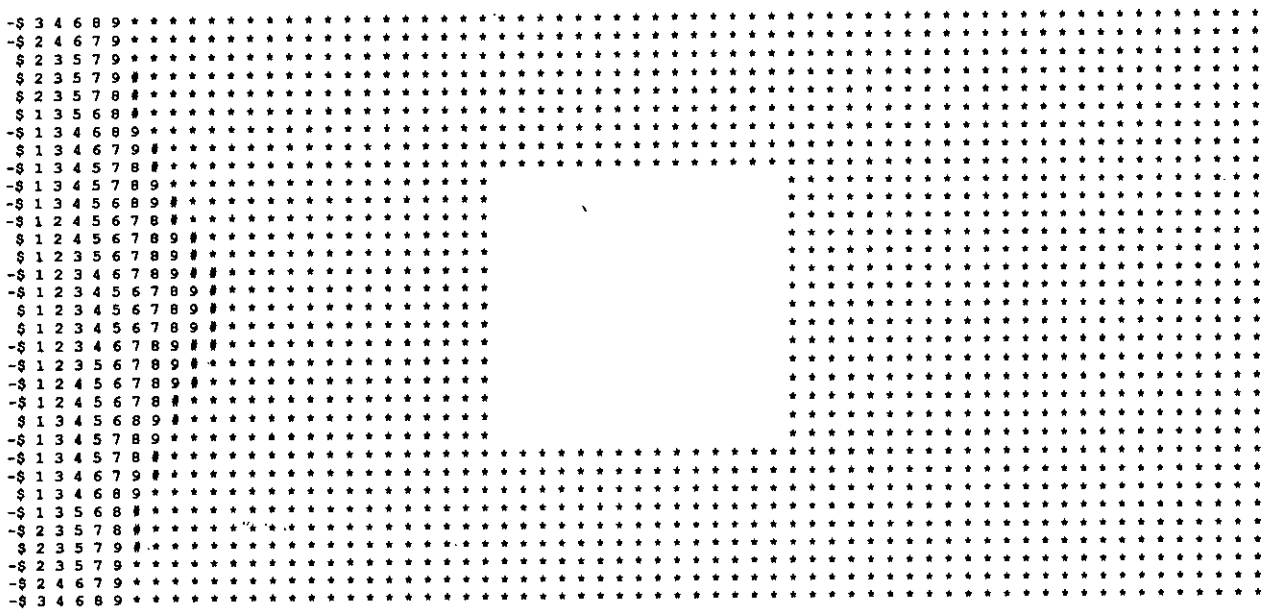
ANNEXURE D1

Solution of the same problem of with rectangular hole

Distribution of *u*



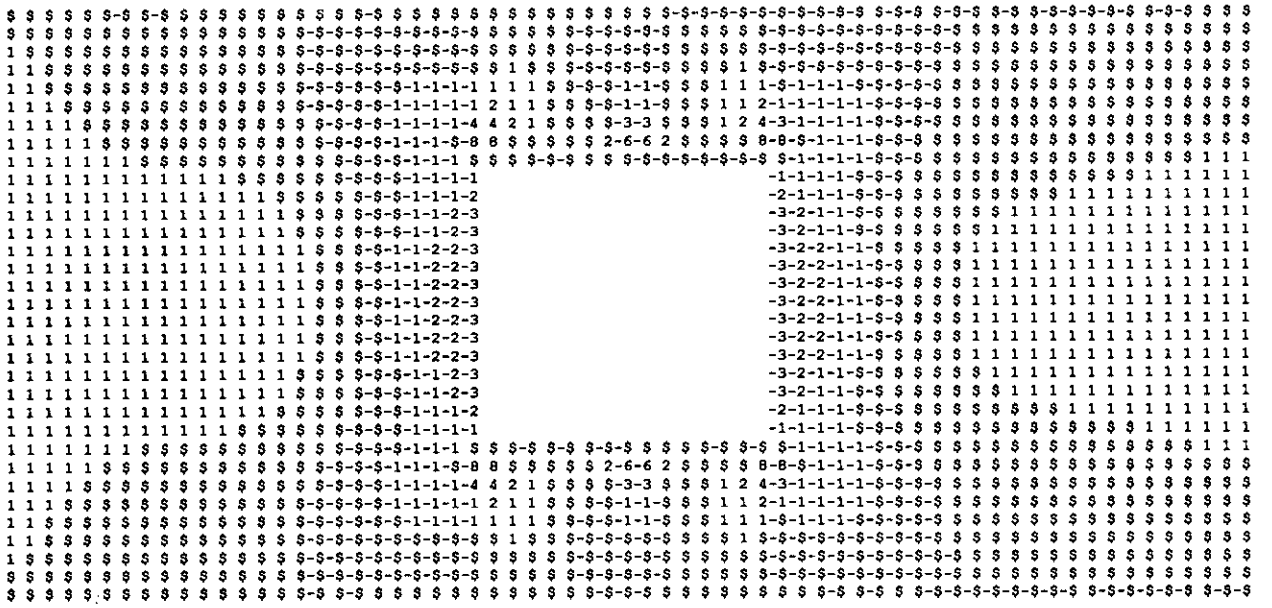
Distribution of *v*



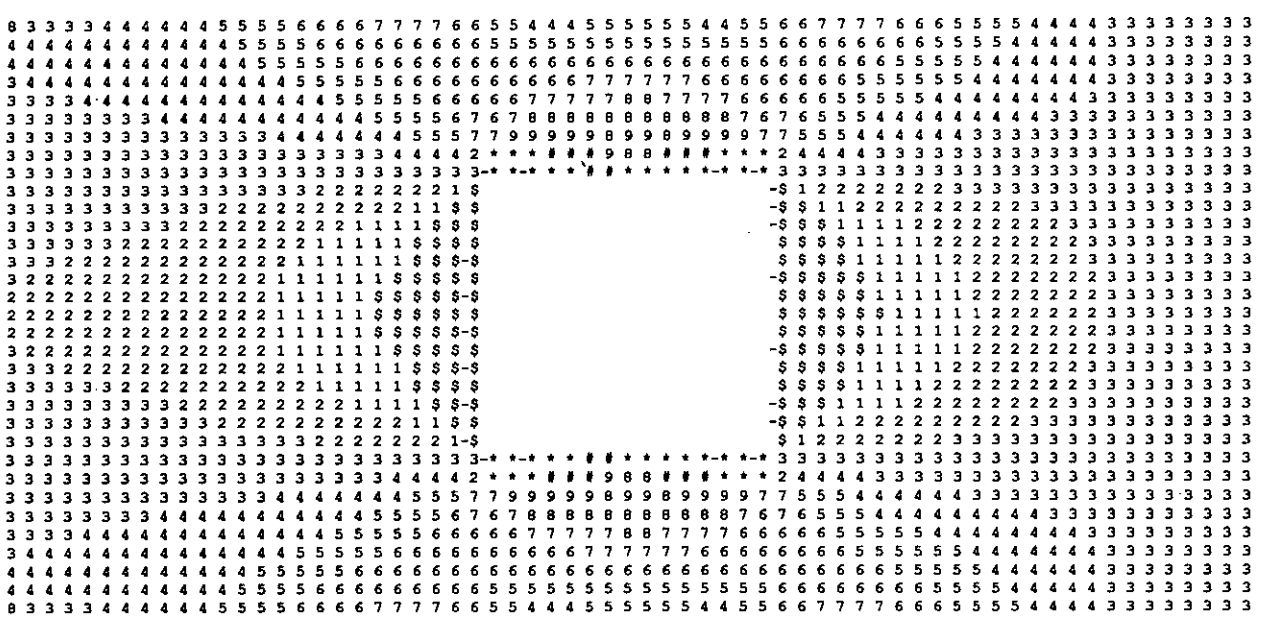
ANNEXURE D1

Solution of the same problem of with rectangular hole

Distribution of σ_x

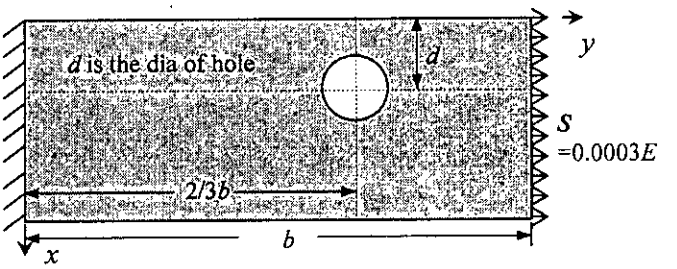


Distribution of σ_y

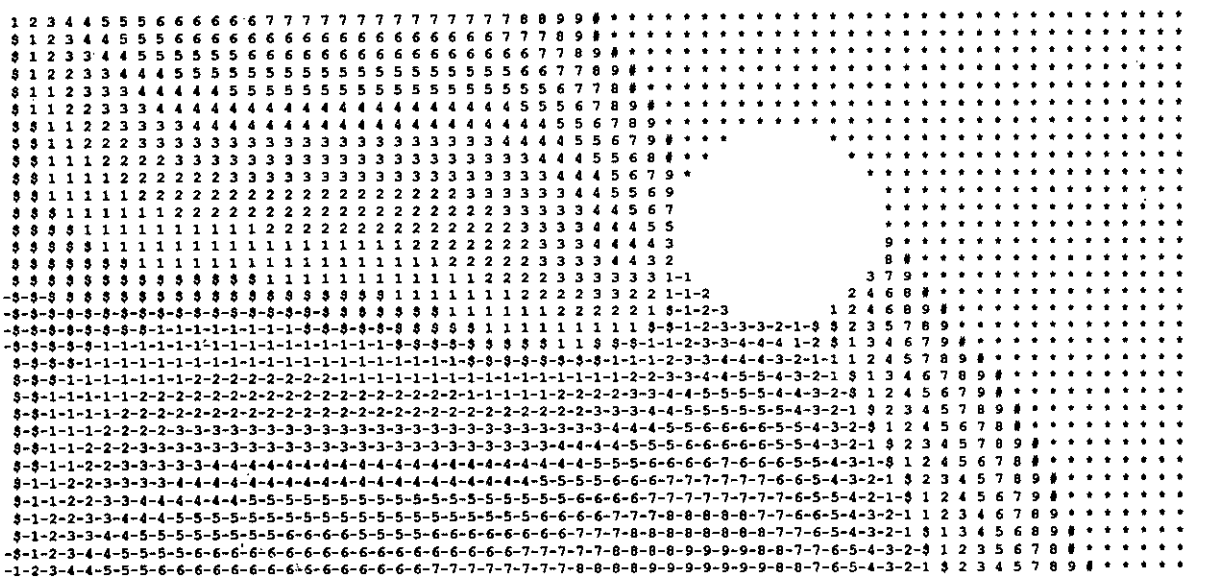


ANNEXURE D2

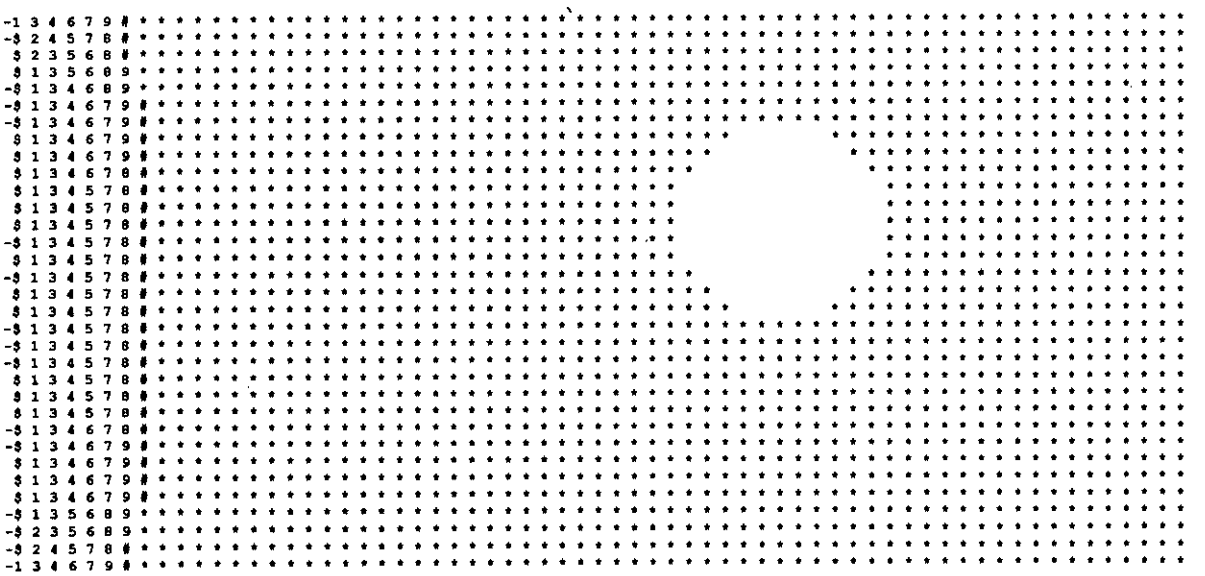
Solution of the same plate changing the position of the hole as shown



Distribution of *u*

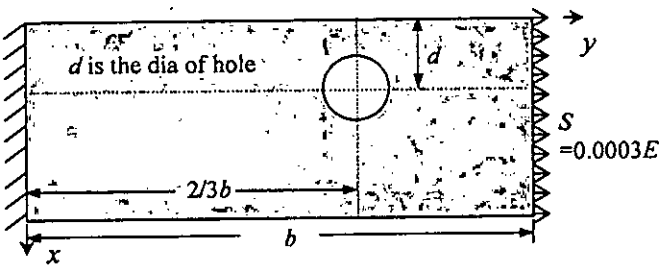


Distribution of *v*

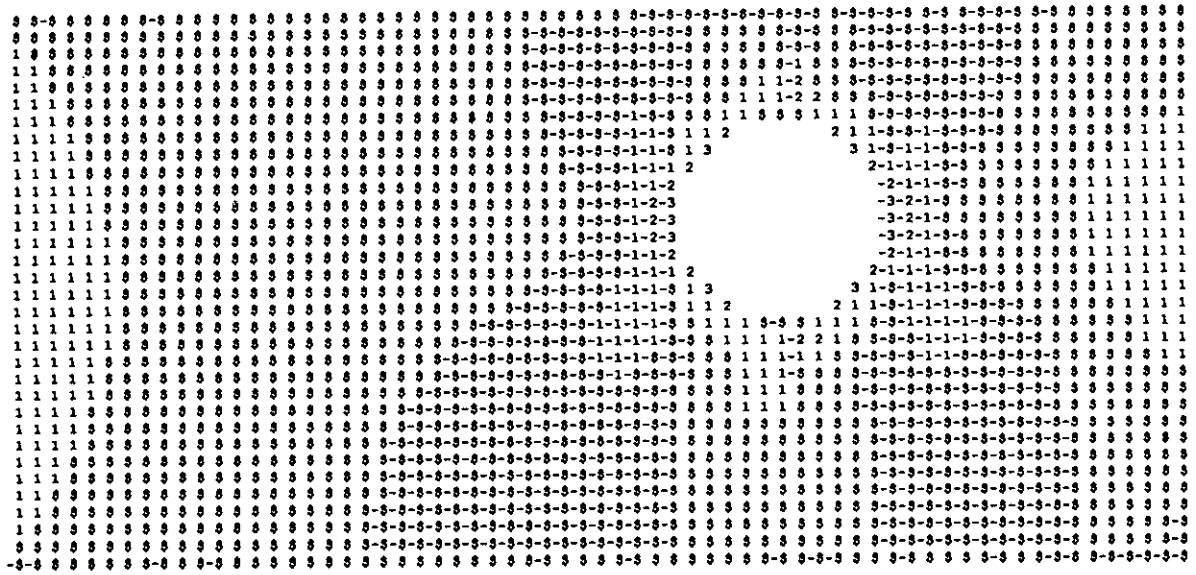


ANNEXURE D2

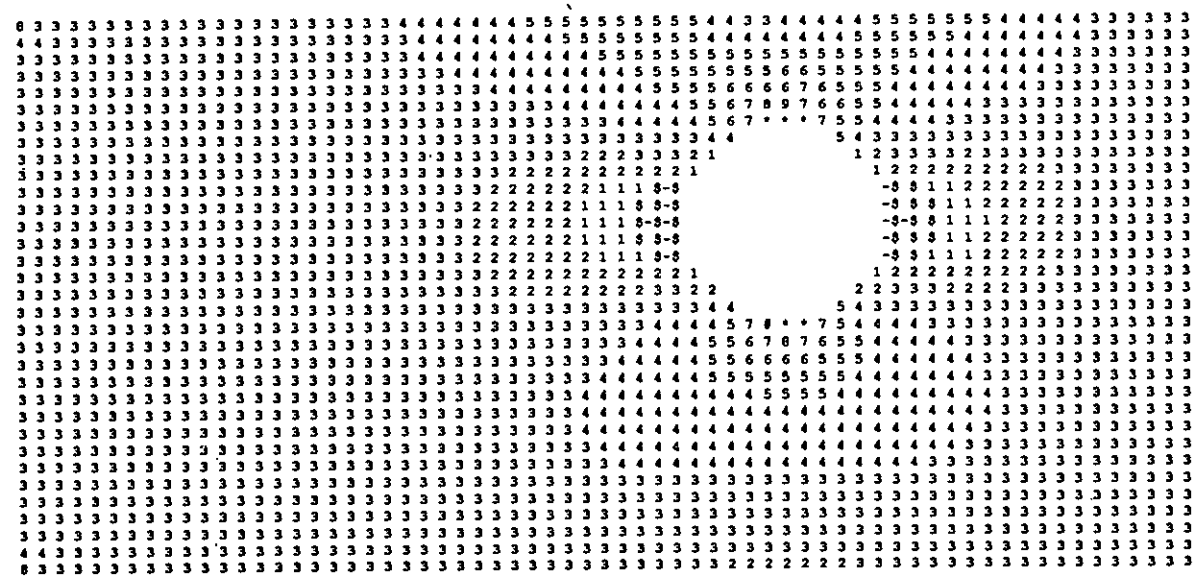
Solution of the same plate changing the position of the hole as shown



Distribution of σ_x

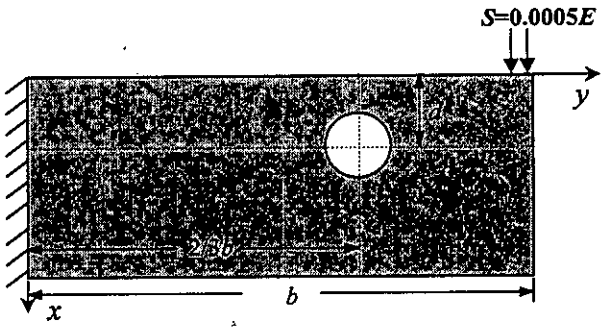


Distribution of σ_y

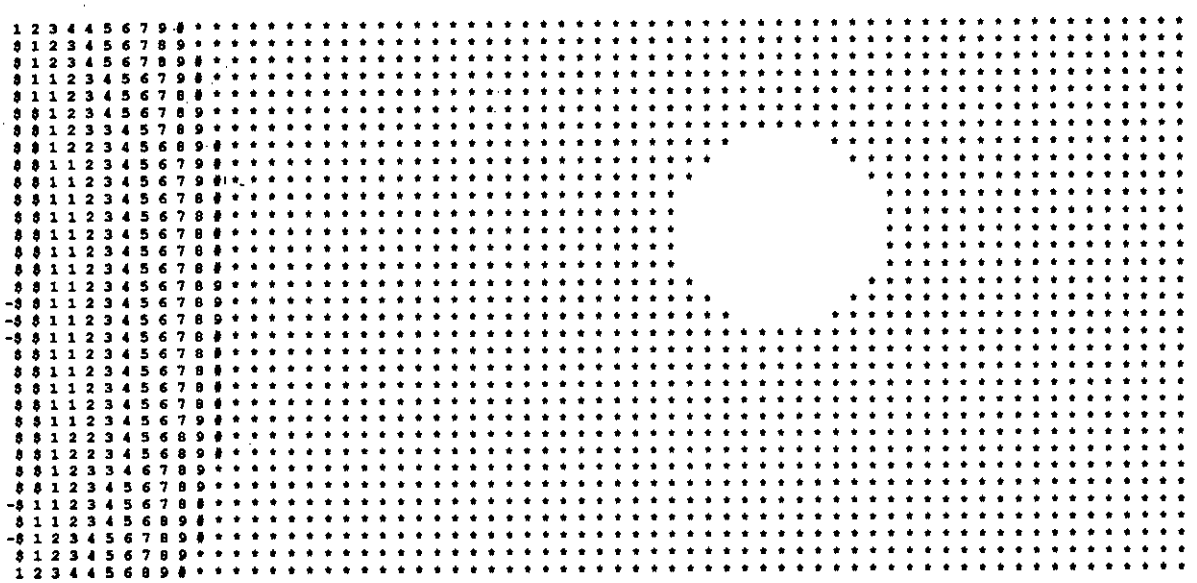


ANNEXURE D3

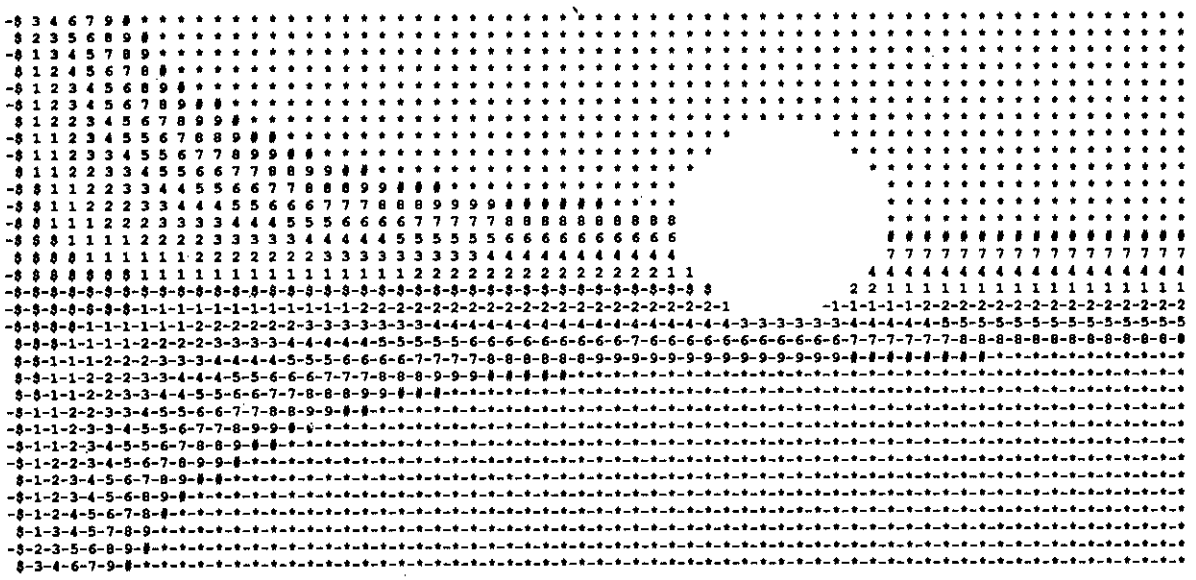
Solution of a rectangular plate with loading at the top right end



Distribution of *u*

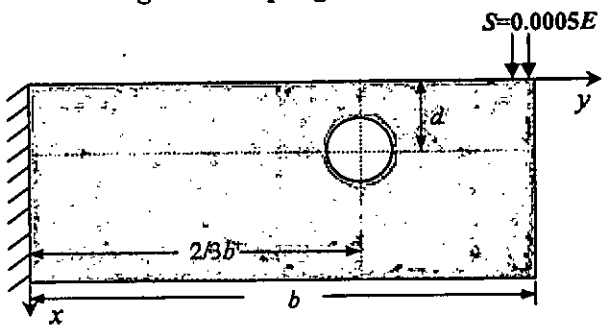


Distribution of *v*

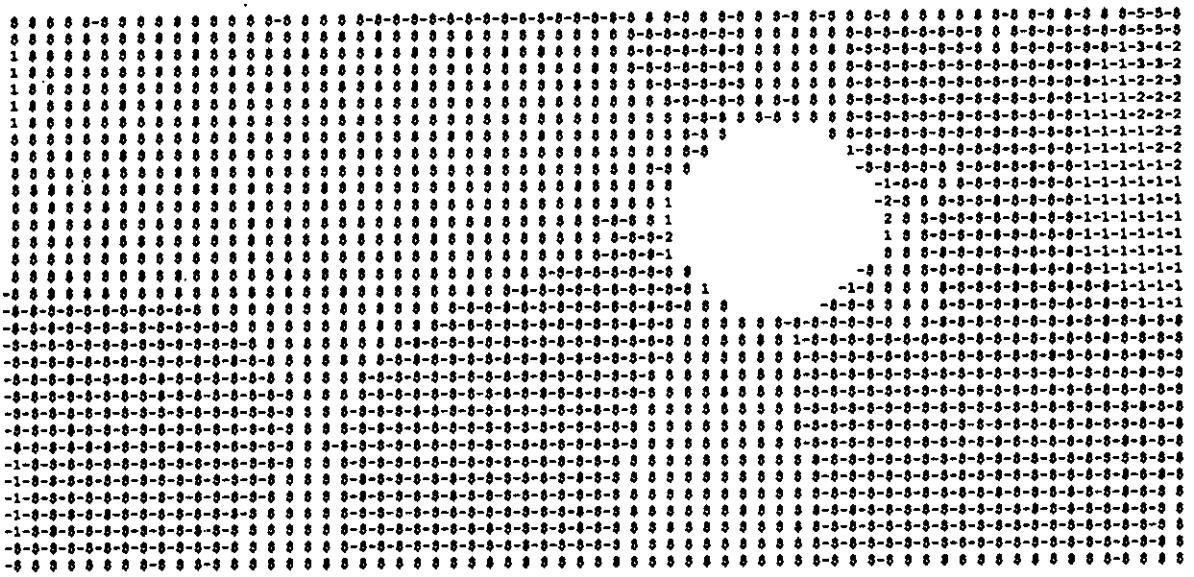


ANNEXURE D3

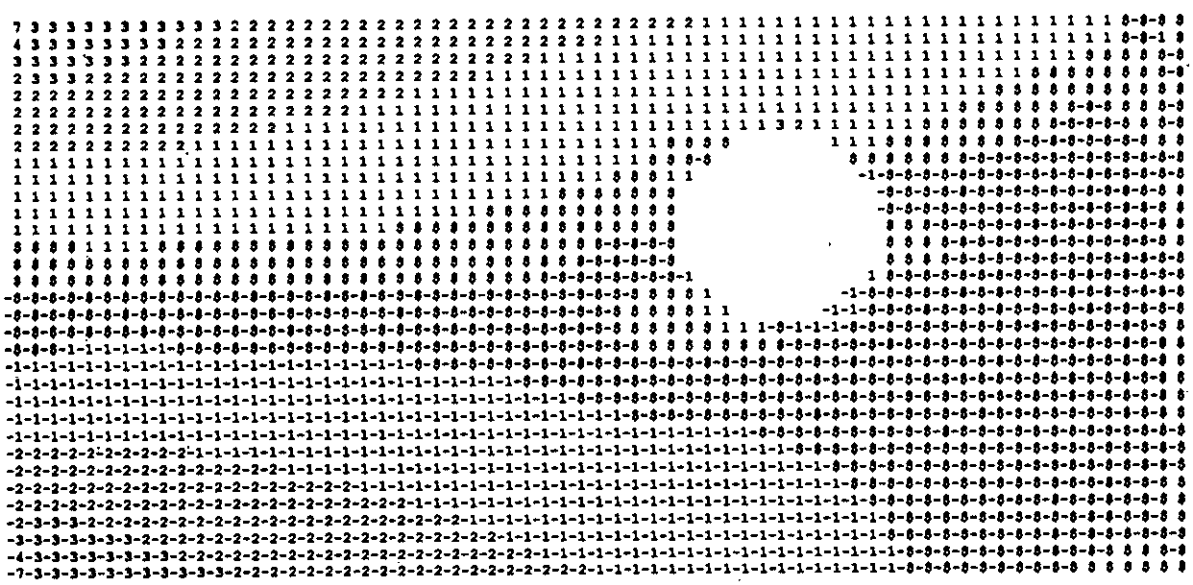
Solution of a rectangular plate with loading at the top right end



Distribution of σ_x

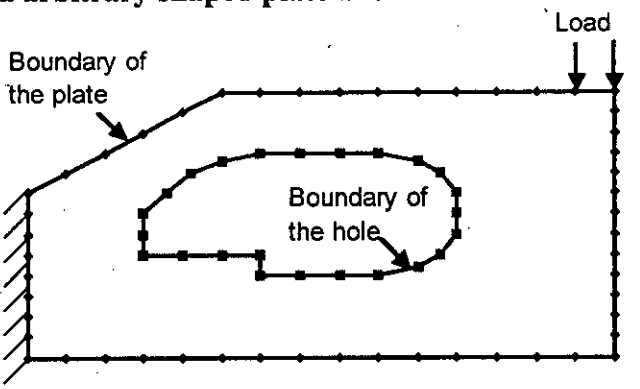


Distribution of σ_y



ANNEXURE D4

Solution of a arbitrary shaped plate and hole



Distribution of *u*

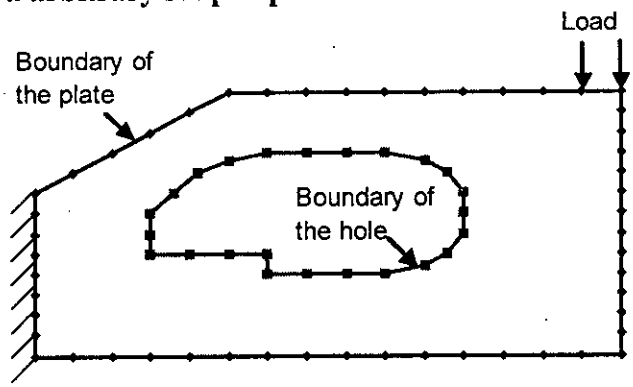
```
                  #  *  *  *  *  *  *  *  *  *  *
               7  #  *  *  *  *  *  *  *  *  *  *
            2  8  *  *  *  *  *  *  *  *  *  *  *
         -6  *-*  *  *  *  *  *  *  *  *  *  *  *
      -*-#-*  9                    *  *  *  *  *
   $-*  *  3  *                    *  *  *  *  *
   $-*-*-3                         *  *  *  *  *
   $-$-4-$                         *  *  *  *  *
-$  4  3  6  *-*  $                *  *  *  *  *
-$  2-6  5  4-*-#  *  *  *  *  *  *  *  *  *  *
-$  2  5  2-*-*  1  *  *  *  *  *  *  *  *  *  *
-$  2  2-5-#-*  4  *  *  *  *  *  *  *  *  *  *
 $  2  $-5-8-6  5  *  *  *  *  *  *  *  *  *  *
-$  2  $-4-7-5  5  *  *  *  *  *  *  *  *  *  *
```

Distribution of *v*

```
                  *  *  *  *  *  *  *  *  *  *  *
               *  *  *  *  *  *  *  *  *  *  *  *
            *  *  *  *  *  *  *  *  *  *  *  *  *
         *  *  *  *  *  *  *  *  9  9  8  8  8  8
      *  8  5  5                    2  2  3  3  3
-$-*-*-*-*                        -3-3-3-2-2
-$-8-*-*                          -8-7-7-7-7
 $-9-*-*                          -*-*-*-*-*-#
-$-5-9-9-*  *  *                  -*-*-*-*-*-*
-$-3-6-9-7  7  *  *-5-2-*-*-*-*-*-*-*-*
 $-2-3-3  1  $  1-1-2-6-9-*-*-*-*-*-*
-$-2-2-2-2-4-7-8-8-9-#-*-*-*-*-*-*
-2-2-3-3-5-9-*-*-*-*-*-*-*-*-*-*-*
-4-3-2-1-6-*-*-*-*-*-*-*-*-*-*-*-*
```

ANNEXURE D4

Solution of a arbitrary shaped plate and hole



Distribution of σ_x

```
                $ $-$-$-$ $ $-$ $-3-3
             -1-$-$ 1 $-$-$ $-$-$-3-3
          -2-$-$-1 2 $-1-1 $-$-1-2-3
       -4 $ 2-1-$-$-$-$-1-$-$-1-2-3
    -*-5 3 6              -1-1-1-2-3
  * *-*-#-*               -1-1-1-2-3
-6 9 * 1                  -2-$-1-2-4
-4 # #-2                   3 1-1-3-5
-3-2-$-$-$ $ *             6 $-1-3-4
-2-1 5-5-* * $-$-$-$-2 5 2-$-2-3
-2-2-3-* * * 7-*-* 7-2 2 3 1-1-2
-1-1-3-7 8 # 3-9-9 1-1 1 1 1-$-1
-$-$-1-2 2 3 1-2-2 $-$ $ $ $-$-$
 $-$-$ $ $-$ $ $ $ $ $ $ $ $ $ $
```

Distribution of σ_y

```
             -$-1-4-1 3 3 2 2 1-$ $
          -1-1-1-1-1 $ $ $ $ $ $-$
       -2-1-1-1-$-1-2-2-1-1-$ $-$
    -4-3-$ 4-1-3-$-*-2-*-*-* * *
  -*-*-3 4                1 * *-*-*
-6 * *-9-#               -$ * *-*-*
-*-*-8 $                  $ * *-*-*
-*-*-*-*                 -$ * *-*-*
-*-*-8 4-* *-*            # * *-*-*
-7-7-8-6-2 * *-* *-*-6 * * *-*-*
-5-5-4 6 4 8-1-8-#-6-7-3-2-1-$ $
-4-3-1 2-3-4-3-$-$-4-2-2-2-1-$ $
 $ $-$-2-7-9-5 $ 2 1 1-$-1-1-$ $
-$ 5 4-4-*-*-8 5 * 8 4 1-1-1 $-$
```


বাংলাদেশ প্রকৌশল বিশ্ববিদ্যালয়
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লাইব্রেরী, ঢাকা