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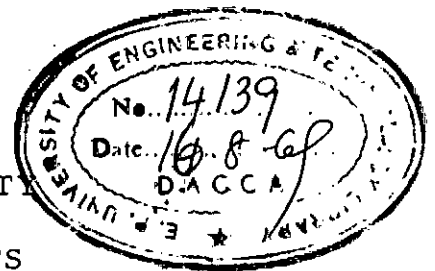
THEORY OF
DIELECTRIC COATED LINEAR ANTENNAS

by
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ABSTRACT

The problem of a dielectric coated linear antenna has been investigated in several steps. In chapter 1, the scattering of an obliquely incident electromagnetic wave by a composite cylinder has been studied and the general result reduced to simpler forms for certain special cases. In chapter 2, the simplest case of an electric dipole within a dielectric sphere and a cylinder has been discussed. In chapter 3, an integral equation for the current distribution on a dielectric coated antenna has been formulated. When the antenna is infinitely long, the integral equation yields an explicit expression for the current distribution in the form of an integral. The asymptotic behavior of this integral has been investigated and the driving point admittance plotted for a set of different parameters. An approach to the solution of a coated antenna of finite length is indicated.

A brief review of the problems related to plasma diagnostics has been presented in appendix 1. The problem of inhomogeneous coating has been discussed in appendix 2 and 7. Several other appendices show the various mathematical details.

CHAPTER 1

SCATTERING OF ELECTROMAGNETIC WAVES

BY A COMPOSITE CYLINDER*

In many problems related to space exploration and guided missiles, one is interested to know the scattering cross-section of a re-entry vehicle, covered with a plasma sheath, or the kind of dielectric coating one should have in order that the scattering cross-section may be appreciably reduced.

The problem of scattering of electromagnetic waves by obstacles have received attention of various authors.¹⁻⁹ Many of them, however, treated the case of normally incident electromagnetic waves only. A consistent approach to the problem of scattering by cylindrical structures has been described by J. R. Wait.¹⁰

The problem to be considered in this chapter is that of scattering of an obliquely incident plane electromagnetic wave to a composite dielectric cylinder. The approach is similar to that of Wait,¹⁰ who treated the problem of a single dielectric cylinder.

1.1 STATEMENT OF THE PROBLEM AND FORMULATION

The geometry of the composite dielectric structure is shown in figure 1.

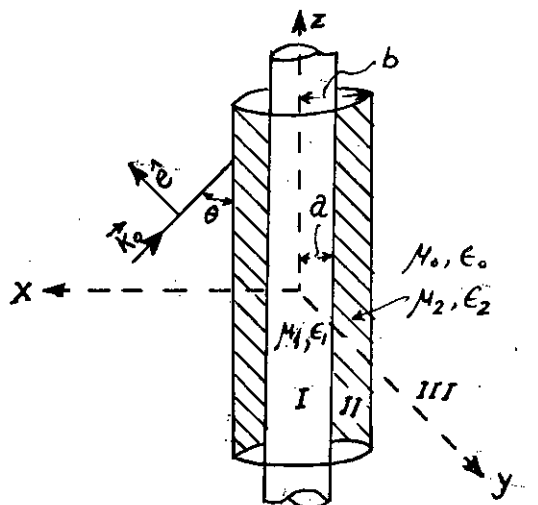


Figure 1 - Composite Cylinder and the Coordinate System

*This article, with slight modification appeared in reference 14.

A plane electromagnetic wave is incident on a composite infinitely long cylinder making an angle θ with the axis of the cylinder.

Let

$\vec{E}^i$ = incident electric field

$\hat{e}$ = unit vector along the direction of $\vec{E}^i$

$\vec{k}_0$ = propagation vector

θ = angle of incidence

$\hat{e}$ will be assumed to be parallel to the plane $\phi = 0$

i.e., = in the x-z plane.

Since the $\vec{H}$ field is assumed to be perpendicular to the axis of cylinder, the wave will be called TM wave. When the $\vec{E}$ field is perpendicular to the axis, the wave will be termed a TE wave. The equation of a plane wave is

$$\vec{E}^i = \hat{e} E_0 e^{i\vec{k}_0 \cdot \vec{R}}$$

In circular cylindrical coordinate system

$$\vec{x} = r \cos \phi; y = r \sin \phi; z = z$$

$$\therefore \vec{E}^i = \hat{e} E_0 \exp [i k_0 z \cos \theta - i \lambda_0 r \cos \phi] \quad (1)$$

where $\lambda_0 = k_0 \sin \theta$

$$\lambda_0 = k_0 \sin \theta$$

Using the expansion of

$$e^{-i z \sin \theta} \text{ (see Ref. 12)}$$

$$e^{-i z \sin \theta} = \sum_{n=-\infty}^{\infty} e^{-i n \theta} J_n(z) \quad (2)$$

one writes

$$e^{-i \lambda_0 r \cos \phi} = e^{-i \lambda_0 r \sin [\frac{\pi}{2} - \phi]} = \sum_{n=-\infty}^{\infty} (-i)^n e^{i n \phi} J_n(\lambda_0 r) \quad (3)$$

Substituting (3) in (1), one obtains

$$\vec{E}^i = \hat{e} E_0 \sum_{n=-\infty}^{\infty} (-i)^n J_n(\lambda_0 r) F_n \quad (4)$$

where

$$F_n = \exp[ik_0 z \cos \theta + in\phi]$$

Therefore, the z-component of $\vec{E}^i$ will be

$$E_z^i = \vec{E}^i \cdot \hat{z} = E_0 \sin \theta \sum_{n=-\infty}^{\infty} (-i)^n J_n(\lambda_0 r) F_n \quad (5)$$

Since the cylinder is of infinite length, there is no discontinuity along the z-direction. As such, the z-variation of all field components must be the same as that of the incident field, i.e., according to the factor $e^{ik_0 z \cos \theta}$. The z-component of the scattered field will then be

$$E_z^s = \sum_{n=-\infty}^{\infty} a_n^s H_n^{(1)}(\lambda_0 r) F_n \quad (6)$$

a_n^s is a coefficient to be determined from suitable boundary condition and $H_n^{(1)}(\lambda_0 r)$ is the Hankel function of the first kind, defined as $H_n^{(1)}(\lambda_0 r) = J_n(\lambda_0 r) + iN_n(\lambda_0 r)$, where $J_n(\lambda_0 r)$ and $N_n(\lambda_0 r)$ are two independent solutions of the Bessel equation. This function is chosen to ensure proper behavior of the wave at infinity. Henceforth, the superscript (1) from $H_n^{(1)}(\lambda_0 r)$ will be omitted.

Because of the assumed polarization, the z-component of the $\vec{H}^i$ field is zero. The z-component of the scattered field, however, will not in general be zero except in the case of normally incident wave or of a perfectly conducting cylinder.¹³ Thus, even though, the incident wave is purely TM, the resultant field have to be constructed as a superposition of a set TE and TM waves.

Therefore, (the summation in the sequel runs from $n = -\infty$ to $+\infty$)

$$H_z^i = 0$$

$$H_z^s = \sum b_n^s H_n(\lambda_0 r) F_n \quad (7)$$

It is now necessary to relate the other field components with E_z^i , E_z^s , H_z^i and H_z^s . To do this, one starts with Maxwell's equations and obtains the following relationships.

$$\begin{aligned} E_\phi &= \frac{1}{k^2 - a^2} \left[i \frac{a}{r} \frac{\partial E_z}{\partial \phi} - i \omega \mu \frac{\partial H_z}{\partial r} \right] \\ H_\phi &= \frac{1}{k^2 - a^2} \left[i \frac{a}{r} \frac{\partial H_z}{\partial \phi} + i \omega \epsilon \frac{\partial E_z}{\partial r} \right] \end{aligned} \quad (8)$$

In the above k , μ , ϵ are the properties of the medium and $\frac{\partial}{\partial z} = i a$. For the present case, the z -dependence is as $e^{i k_0 z \cos \theta}$. Then, $a = k_0 \cos \theta$. It is now easy to write the ϕ component of the incident and the scattered field.

Thus, from (5), (6), (7), and (8)

$$E_\phi^i = \sum \frac{-a n E_0 \sin \theta (-i)^n J_n(\lambda_0 r) F_n}{r \lambda_0^2} \quad (9)$$

$$E_\phi^s = \sum \left[\frac{-a n a_n^s}{r \lambda_0^2} H_n(\lambda_0 r) - \frac{i \omega \mu_0 b_n^s}{\lambda_0} H_n'(\lambda_0 r) \right] F_n \quad (10)$$

$$H_\phi^i = \sum \frac{i k_0^2 E_0 \sin \theta}{\omega \mu_0 \lambda_0} (-i)^n J_n'(\lambda_0 r) F_n \quad (11)$$

$$H_\phi^s = \sum \left[\frac{-a n a_n^s}{r \lambda_0^2} b_n^s H_n(\lambda_0 r) + \frac{i k_0^2}{\omega \mu_0 \lambda_0} a_n^s H_n'(\lambda_0 r) \right] F_n \quad (12)$$

The prime sign with J_n and H_n indicates differentiation and as indicated previously, the summation runs over n from $-\infty$ to ∞ . Thus (5), (6), (7), (9) - (12) are the field representations in Region III. Since the radial components are not required for matching boundary conditions, these have not been set up. It is now possible to set up the field components in Regions II and I following the same principles and observing that in Region II both J_n 's and H_n 's are permissible solutions and in Region I, only the J_n 's are permissible solutions. These are

Region II

$$E_z^{(2)} = \Sigma [b_n J_n(\lambda_2 r) + c_n H_n(\lambda_2 r)] F_n \quad (13)$$

$$H_z^{(2)} = \Sigma [a_n J_n(\lambda_2 r) + e_n H_n(\lambda_2 r)] F_n \quad (14)$$

$$E_\phi^{(2)} = \Sigma \left\{ -\frac{\omega}{r \lambda_2^2} [b_n J_n(\lambda_2 r) + c_n H_n(\lambda_2 r)] - \frac{i \omega \mu_2}{\lambda_2} [a_n J_n'(\lambda_2 r) + e_n H_n'(\lambda_2 r)] \right\} F_n \quad (15)$$

$$H_\phi^{(2)} = \Sigma \left\{ \frac{i k_2^2}{\omega \mu_2 \lambda_2} [b_n J_n'(\lambda_2 r) + c_n H_n'(\lambda_2 r)] - \frac{\omega}{r \lambda_2^2} [a_n J_n(\lambda_2 r) + e_n H_n(\lambda_2 r)] \right\} F_n \quad (16)$$

Region I

$$E_z^{(1)} = \Sigma d_n J_n(\lambda_1 r) F_n \quad (17)$$

$$H_z^{(1)} = \Sigma f_n J_n(\lambda_1 r) F_n \quad (18)$$

$$E_\phi^{(1)} = \Sigma \left[-\frac{\omega}{r \lambda_1^2} d_n J_n(\lambda_1 r) - \frac{i \omega \mu_1}{\lambda_1} f_n J_n'(\lambda_1 r) \right] F_n \quad (19)$$

$$H_\phi^{(1)} = \Sigma \left[\frac{i k_1^2}{\omega \mu_1 \lambda_1} d_n J_n'(\lambda_1 r) - \frac{\omega}{r \lambda_1^2} f_n J_n(\lambda_1 r) \right] F_n \quad (20)$$

All summations run over n from $-\infty$ to ∞ , and the superscripts (1) and (2) indicate the regions in which they are valid.

As indicated previously

$$\alpha = k_0 \cos \theta, \quad \lambda_0 = k_0 \sin \theta$$

$$\lambda_1 = (k_1^2 - k_0^2 \cos^2 \theta)^{1/2}, \quad \lambda_2 = (k_2^2 - k_0^2 \cos^2 \theta)^{1/2}$$

$$F_n = \exp [in\phi + ik_0 z \cos \theta]$$

a_n, b_n etc are suitable coefficients to be determined from the boundary conditions.

These are

$$\begin{aligned} H_z^i + H_z^s &= H_z^{(2)}; H_\phi^i + H_\phi^s = H_\phi^{(2)} \\ E_z^i + E_z^s &= E_z^{(2)}; E_\phi^i + E_\phi^s = E_\phi^{(2)} \end{aligned} \quad \text{at } r = b \quad (21)$$

$$\begin{aligned} H_z^{(2)} &= H_z^{(1)}; H_\phi^{(2)} = H_\phi^{(1)} \\ E_z^{(2)} &= E_z^{(1)}; E_\phi^{(2)} = E_\phi^{(1)} \end{aligned} \quad \text{at } r = a \quad (22)$$

1.2 DETERMINATION OF THE COEFFICIENTS

Since at $r = a$, $E_z^{(1)} = E_z^{(2)}$, one must have

$$\sum d_n J_n(\lambda_1 a) \exp [in\phi + ik_0 z \cos \theta] = \sum [b_n J_n(\lambda_2 a) + c_n H_n(\lambda_2 a)] \exp [in\phi + ik_0 z \cos \theta]$$

Multiplying both sides of above by $e^{-ip\phi}$ and integrating with ϕ between the limits 0 to 2π , the above reduces to

$$d_n J_n(\lambda_1 a) = b_n J_n(\lambda_2 a) + c_n H_n(\lambda_2 a) \quad (23)$$

In a similar manner, the boundary condition (21) and (22) applied to the proper field expressions yield the following:

$$f_n J_n(\lambda_1 a) = a_n J_n(\lambda_2 a) + e_n H_n(\lambda_2 a) \quad (24)$$

$$\frac{an}{a \lambda_1^2} d_n J_n(\lambda_1 a) + \frac{i \omega \mu_1}{\lambda_1} f_n J_n'(\lambda_1 a) =$$

$$= \frac{an}{a \lambda_2^2} [b_n J_n(\lambda_2 a) + c_n H_n(\lambda_2 a)] - \frac{i \omega \mu_2}{\lambda_2} [a_n J_n'(\lambda_2 a) + e_n H_n'(\lambda_2 a)] \quad (25)$$

$$\begin{aligned} \frac{i k_1^2}{\omega \mu_1 \lambda_1} d_n J_n'(\lambda_1 a) - \frac{an}{a \lambda_1^2} f_n J_n(\lambda_1 a) \\ = \frac{i k_2^2}{\omega \mu_2 \lambda_2} [b_n J_n'(\lambda_2 a) + c_n H_n'(\lambda_2 a)] - \frac{an}{a \lambda_2^2} [a_n J_n(\lambda_2 a) + e_n H_n(\lambda_2 a)] \end{aligned} \quad (26)$$

$$E_o \sin \theta (-i)^n J_n(\lambda_o b) + a_n^s H_n(\lambda_o b) = b_n J_n(\lambda_2 b) + c_n H_n(\lambda_2 b) \quad (27)$$

$$b_n^s H_n(\lambda_o b) = a_n J_n(\lambda_2 b) + e_n H_n(\lambda_2 b) \quad (28)$$

$$\begin{aligned} \frac{an}{b \lambda_o^2} [E_o \sin \theta (-i)^n J_n(\lambda_o b) + a_n^s H_n(\lambda_o b)] + \frac{i \omega \mu_o}{\lambda_o} b_n^s H_n'(\lambda_o b) \\ = \frac{an}{b \lambda_2^2} [b_n J_n(\lambda_2 b) + c_n H_n(\lambda_2 b)] + \frac{i \omega \mu_2}{\lambda_2} [a_n J_n'(\lambda_2 b) + e_n H_n'(\lambda_2 b)] \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{i k_o^2}{\omega \mu_o \lambda_o} [E_o \sin \theta (-i)^n J_n'(\lambda_o b) + a_n^s H_n'(\lambda_o b)] - \frac{an}{b \lambda_o^2} b_n^s H_n(\lambda_o b) \\ = \frac{i k_2^2}{\omega \mu_2 \lambda_2} [b_n J_n'(\lambda_2 b) + c_n H_n'(\lambda_2 b)] - \frac{an}{b \lambda_2^2} [a_n J_n(\lambda_2 b) + e_n H_n(\lambda_2 b)] \end{aligned} \quad (30)$$

There are thus eight equations (23) to 30) and eight unknowns a_n , b_n , c_n , d_n , e_n , f_n , a_n^s and b_n^s . By manipulating the equations and using the value of the Wronskian,

$$J_n(\lambda_o b) H_n'(\lambda_o b) - H_n(\lambda_o b) J_n'(\lambda_o b) = \frac{2i}{\pi \lambda_o b}$$

the above eight equations may be reduced to the following set:

$$A_n b_n + B_n c_n = D_n a_n + E_n e_n$$

$$H_n b_n + G_n c_n = A_n a_n + B_n e_n$$

(31)

$$L_n b_n + M_n c_n = P_n a_n + Q_n e_n$$

$$R_n b_n + S_n c_n = L_n a_n + M_n e_n - T_n$$

where a_n , b_n , c_n and e_n are the unknowns and the quantities A_n , B_n etc are defined as follows:

$$\begin{aligned} A_n &= \frac{\omega n}{a} J_n(\lambda_2 a) \left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} \right] \\ B_n &= \frac{\omega n}{a} H_n(\lambda_2 a) \left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} \right] \\ D_n &= \frac{i \omega \mu_2}{\lambda_2} J_n'(\lambda_2 a) - \frac{i \omega \mu_1}{\lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} J_n(\lambda_2 a) \\ E_n &= \frac{i \omega \mu_2}{\lambda_2} H_n'(\lambda_2 a) - \frac{i \omega \mu_1}{\lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} H_n(\lambda_2 a) \\ H_n &= \frac{i k_1^2}{\omega \mu_1 \lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} J_n(\lambda_2 a) - \frac{i k_2^2}{\omega \mu_2 \lambda_2} J_n'(\lambda_2 a) \\ G_n &= \frac{i k_1^2}{\omega \mu_1 \lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} H_n(\lambda_2 a) - \frac{i k_2^2}{\omega \mu_2 \lambda_2} H_n'(\lambda_2 a) \\ L_n &= \frac{\omega n}{b} J_n(\lambda_2 b) \left[\frac{1}{\lambda_0^2} - \frac{1}{\lambda_2^2} \right] \\ M_n &= \frac{\omega n}{b} H_n(\lambda_2 b) \left[\frac{1}{\lambda_0^2} - \frac{1}{\lambda_2^2} \right] \\ P_n &= \frac{i \omega \mu_2}{\lambda_2} J_n'(\lambda_2 b) - \frac{i \omega \mu_0}{\lambda_0} \frac{H_n'(\lambda_0 b)}{H_n(\lambda_0 b)} J_n(\lambda_2 b) \\ Q_n &= \frac{i \omega \mu_2}{\lambda_2} H_n'(\lambda_2 b) - \frac{i \omega \mu_0}{\lambda_0} \frac{H_n'(\lambda_0 b)}{H_n(\lambda_0 b)} H_n(\lambda_2 b) \\ R_n &= \frac{i k_0^2}{\omega \mu_0 \lambda_0} \frac{H_n'(\lambda_0 b)}{H_n(\lambda_0 b)} J_n(\lambda_2 b) - \frac{i k_2^2}{\omega \mu_2 \lambda_2} J_n'(\lambda_2 b) \end{aligned} \quad (32)$$

$$S_n = \frac{i k_o^2}{\omega \mu_o \lambda_o} \frac{H_n'(\lambda_o b)}{H_n(\lambda_o b)} H_n(\lambda_2 b) - \frac{i k_2^2}{\omega \mu_2 \lambda_2} H_n'(\lambda_2 b)$$

$$T_n = \frac{2 k_o^2 (-i)^n}{\pi \omega \mu_o b} \frac{1}{\lambda_o^2} \frac{E_o \sin \theta}{H_n(\lambda_o b)}$$

The solutions for the unknowns may now be formally written down.

$$\begin{aligned} b_n &= \frac{\begin{vmatrix} 0 & B_n & -D_n & -E_n \\ 0 & G_n & -A_n & -B_n \\ 0 & M_n & -P_n & -Q_n \\ -T_n & S_n & -L_n & -M_n \end{vmatrix}}{\Delta} \\ c_n &= \frac{\begin{vmatrix} A_n & 0 & -D_n & -E_n \\ H_n & 0 & -A_n & -B_n \\ L_n & 0 & -P_n & -Q_n \\ R_n & -T_n & -L_n & -M_n \end{vmatrix}}{\Delta} \\ \hat{a}_n &= \frac{\begin{vmatrix} A_n & B_n & 0 & -E_n \\ H_n & G_n & 0 & -B_n \\ L_n & M_n & 0 & -Q_n \\ R_n & S_n & -T_n & -M_n \end{vmatrix}}{\Delta} \\ \Delta &= \begin{vmatrix} A_n & B_n & -D_n & -E_n \\ H_n & G_n & -A_n & -B_n \\ L_n & M_n & -P_n & -Q_n \\ R_n & S_n & -L_n & -M_n \end{vmatrix} \end{aligned} \quad \begin{aligned} e_n &= \frac{\begin{vmatrix} A_n & B_n & -D_n & 0 \\ H_n & G_n & -A_n & 0 \\ L_n & M_n & -P_n & 0 \\ R_n & S_n & -L_n & -T_n \end{vmatrix}}{\Delta} \end{aligned} \quad (33)$$

and || sign represents the determinant. Once a_n , b_n , c_n and e_n are known, the other unknowns may be easily found. These known coefficients may then be substituted in (5), (6), (7), and (9) to (20) to obtain the different field components.

This then is the complete formal solution of the general problem. The result for the case of the magnetic vector in the $\phi = 0$ plane and the electric vector along the y-direction may be obtained using the analogy between the magnetic and electric quantities, namely, by substituting in the above results H for E; -E for H, and μ and ϵ interchanged throughout.

1.3 SPECIAL CASES

It is interesting to see how these complicated results behave in certain very special cases:

Case 1 - If one lets $\mu_2 = \mu_0$, $\epsilon_2 = \epsilon_0$, the above problem should reduce to the problem of a single dielectric cylinder. With the above substitution, (32) becomes

$$\begin{aligned}
 A_n &= \frac{\omega n}{a} J_n(\lambda_0 a) \left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_0^2} \right] \\
 B_n &= \frac{\omega n}{a} H_n(\lambda_0 a) \left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_0^2} \right] \\
 D_n &= \frac{i \omega \mu_0}{\lambda_0} J_n'(\lambda_0 a) - \frac{i \omega \mu_1}{\lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} J_n(\lambda_0 a) \\
 E_n &= \frac{i \omega \mu_0}{\lambda_0} H_n'(\lambda_0 a) - \frac{i \omega \mu_1}{\lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} H_n(\lambda_0 a) \\
 F_n &= \frac{i k_1^2}{\omega \mu_1 \lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} J_n(\lambda_0 a) - \frac{i k_0^2}{\omega \mu_0 \lambda_0} J_n'(\lambda_0 a) \\
 G_n &= \frac{i k_1^2}{\omega \mu_1 \lambda_1} \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} H_n(\lambda_0 a) - \frac{i k_0^2}{\omega \mu_0 \lambda_0} H_n'(\lambda_0 a) \\
 L_n &= M_n = Q_n = S_n = 0
 \end{aligned} \tag{34}$$

$$\begin{aligned}
P_n &= \frac{2 \omega \mu_0}{\pi \lambda_0^2 b H_n(\lambda_0 b)} \\
R_n &= - \frac{2 k_0^2}{\pi \omega \mu_0 \lambda_0^2 b H_n(\lambda_0 b)} \\
T_n &= \frac{2 k_0^2 (-i)^n}{\pi \omega \mu_0 \lambda_0^2 b} \frac{E_0 \sin \theta}{H_n(\lambda_0 b)}
\end{aligned}$$

Therefore (31) reduces to

$$A_n b_n + B_n c_n - D_n a_n - E_n e_n = 0$$

$$H_n b_n + G_n c_n - A_n a_n - B_n e_n = 0$$

$$P_n a_n = 0$$

$$R_n b_n = -T_n$$

(35)

These may be solved in the usual way. The results are

$$a_n = 0$$

$$b_n = (-i)^n E_0 \sin \theta$$

$$b_n^s = e_n = \frac{k_0}{\omega \mu_0} E_0 \sin \theta (-i)^n \frac{2}{\pi \nu^2} \left[\frac{1}{u^2} - \frac{1}{\nu^2} \right] \frac{n \cos \theta}{[H_n(\nu)]^2 D}$$

$$f_n = b_n^s \frac{H_n(\nu)}{J_n(u)}$$

(36)

$$a_n^s = c_n = (-i)^n E_0 \sin \theta \left[- \frac{J_n(\nu)}{H_n(\nu)} + 2i \frac{\frac{H_n'(\nu)}{\nu H_n(\nu)} - \frac{K}{u} \frac{J_n'(u)}{J_n(u)}}{\pi \nu^2 [H_n(\nu)]^2 D} \right]$$

$$d_n = [E_0 \sin \theta (-i)^n J_n(\nu) + a_n^s H_n(\nu)] \cdot \frac{1}{J_n(u)}$$

Where

$$D = \left[\frac{H_n'(\nu)}{\nu H_n(\nu)} - \frac{K}{u} \frac{J_n'(u)}{J_n(u)} \right] \left[\frac{H_n'(\nu)}{\nu H_n(\nu)} - \frac{N^2}{uK} \frac{J_n'(u)}{J_n(u)} \right] - \left[\frac{1}{u^2} - \frac{1}{\nu^2} \right]^2 n^2 \cos^2 \theta$$

$$N = k_1^2/k_0^2, \quad K = \mu_1/\mu_0$$

$$u = \lambda_1 a, \quad \nu = \lambda_0 a$$

These results are, exactly what one expected and agreed with Wait's,¹³ the result for a single dielectric cylinder. The corresponding results for normal incidence may easily be obtained by letting $\theta = 90^\circ$ in (36). The back-scattering cross section for this infinite dielectric cylinder may be defined as

$$\sigma_B \text{ per unit length} = \lim_{r \rightarrow \infty} 2\pi r \left| \frac{E^s}{E^i} \right|^2 = \frac{P_{\text{omni}}^s}{S^i} \quad (37)$$

where P_{omni}^s is the total power re-radiated per unit length of an ideal omnidirectional scatterer that maintains the same field E^s at a radial distance r for all values of ϕ as that maintained by the actual scattering cylinder specifically in the direction toward the source ($\phi = 0$). Thus, for the infinite cylinder under consideration

$$\begin{aligned} \sigma_B \text{ per unit length} &= \lim_{\substack{r \rightarrow \infty \\ \theta = \pi/2 \\ \phi = 0}} 2\pi r \left| \frac{\sum_{n=-\infty}^{\infty} a_n^s H_n(\lambda_0 r) \exp[i n \phi + i k_0 z \cos \theta]}{E_0 \sin \theta \exp[i k_0 z \cos \theta - i k_0 r \sin \theta \cos \phi]} \right|^2 \\ &= 2\pi r \left| \frac{\sum_{n=-\infty}^{\infty} a_n^s \left[\frac{2}{\pi k_0 r} \right]^{1/2} \exp[i k_0 r - i \left(\frac{\pi}{4} + \frac{n\pi}{2} \right)]}{E_0 \exp[-i k_0 r]} \right|^2 \end{aligned}$$

where the asymptotic expression for $H_n(k_0 r)$ has been used. Since only the absolute value is of interest, the above reduces to

$$\sigma_B \text{ per unit length} = \frac{4}{E_0^2 k_0^2} \left| \sum_{n=-\infty}^{\infty} a_n^s \right|^2$$

$$\begin{aligned}
&= \frac{4}{E_0 k_0} \left| \sum_{n=0}^{\infty} \epsilon_n a_n^s \right|^2 \\
&= \frac{4}{k_0} \left| \sum_{n=0}^{\infty} \epsilon_n (-i)^n w_n \right|^2
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
\epsilon_n &= 2 \text{ for } n \neq 0 \\
&= 1 \text{ for } n = 0
\end{aligned}$$

and

$$w_n = a_n^s |E_0 (-i)^n|$$

With $\theta = \frac{\pi}{2}$ and $\mu_1 = \mu_0$ (assumed), from (36)

$$\begin{aligned}
w_n &= \frac{a_n^s}{E_0 (-i)^n} = -\frac{J_n(\nu)}{H_n(\nu)} + 2i \left[\frac{1}{\pi \nu^2 [H_n(\nu)]^2} \left[\frac{H_n'(\nu)}{\nu H_n(\nu)} - \frac{N^2}{uK} \frac{J_n'(u)}{J_n(u)} \right] \right] \\
&= \frac{-k_1 J_n'(k_1 a) J_n(k_0 a) + k_0 J_n(k_1 a) J_n'(k_0 a)}{k_1 J_n'(k_1 a) H_n(k_0 a) - k_0 J_n(k_1 a) H_n'(k_0 a)}
\end{aligned} \tag{39}$$

This result agrees with King and Wu's result in ref. 9 p. 69.

Case 2 - Highly conducting cylinder with a dielectric coaxial cylinder outside.

This is the most important case. Here in Region 1, one assumes $\frac{\sigma_1}{\omega \epsilon_1} \gg 1$.

$$\therefore k_1^2 = -\mu_1 \frac{\partial}{\partial t} (\epsilon_1 \frac{\partial}{\partial t} + \sigma_1) \simeq i \omega \mu_1 \sigma_1$$

$$k_1^2/k_0^2 \simeq \frac{i \omega \mu_1 \sigma_1}{\omega^2 \mu_0 \epsilon_0} \rightarrow \text{very large}$$

$$\lambda_1^2 = k_1^2 - k_0^2 \cos^2 \theta \simeq k_1^2 \simeq i \omega \mu_1 \sigma_1$$

With these restriction, (32) reduces to the following

$$A_n \simeq -\frac{an}{a} \frac{J_n(\lambda_2 a)}{\lambda_2^2}$$

$$\begin{aligned}
B_n &\approx - \frac{an}{a} \frac{H_n(\lambda_2 a)}{\lambda_2^2} \\
D_n &\approx \frac{i \omega \mu_2}{\lambda_2} J_n'(\lambda_2 a) \\
E_n &\approx \frac{i \omega \mu_2}{\lambda_2} H_n'(\lambda_2 a) \\
H_n &\approx \frac{i k_1^2}{\omega \mu_1 \lambda_1} J_n(\lambda_2 a) \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} \\
G_n &\approx \frac{i k_1^2}{\omega \mu_1 \lambda_1} H_n(\lambda_2 a) \frac{J_n'(\lambda_1 a)}{J_n(\lambda_1 a)} \quad (40)
\end{aligned}$$

$L_n, M_n, P_n, Q_n, R_n, S_n$ and T_n are the same as in (32).

One may substitute (40) into (31) and solve for the unknowns. From (40) it is apparent

$$\begin{aligned}
\frac{G_n}{H_n} &\approx \frac{H_n(\lambda_2 a)}{J_n(\lambda_2 a)} \approx \frac{B_n}{A_n} \\
\frac{A_n}{H_n} &\approx \frac{B_n}{H_n} \approx 0
\end{aligned}$$

$$c_n = - \frac{J_n(\lambda_2 a)}{H_n(\lambda_2 a)} b_n; \quad e_n \approx - \frac{J_n'(\lambda_2 a)}{H_n'(\lambda_2 a)} a_n$$

Before solving for the unknowns, some comments regarding the boundary conditions at the surface of a perfect conductor are in order. At the surface of a perfect conductor, the tangential components of the electric field and the normal component of the $\vec{B}$ field must be zero. The normal component of $\vec{D}$ is zero inside the perfect conductor, but is not zero, in general, just outside the conductor, because of the presence of the surface charge density. Likewise, the tangential magnetic field inside a perfect conductor is zero, but is not zero just outside it. This discontinuity gives rise to the surface current density.

Thus, the boundary conditions required to solve a boundary value problem

involving a perfect conductor surface are either

$$E_\phi = 0; \quad E_z = 0 \quad \text{at the surface}$$

(41)

$$\text{or } \frac{\partial}{\partial r} (r H_\phi) = 0; \quad \frac{\partial}{\partial r} H_z = 0 \quad \text{at the surface.}$$

One has a choice now to continue with the solution of (31) with (40) or reformulate the problem with the new set up. If the formulation is correct, both must yield the same result. If instead of solving (31) with (40), one prefers to work directly with the perfect conductor and the outer dielectric cylinder, the procedure will be to start with (5) to (16) and apply boundary conditions (21) and the condition that at $r = a$

$$E_z^{(2)} = E_\phi^{(2)} = 0$$

(42)

This will yield the following set of relationships:

$$E_o \sin \theta (-i)^n J_n(\lambda_o b) + a_n^s H_n(\lambda_o b) = b_n J_n(\lambda_2 b) + c_n H_n(\lambda_2 b)$$

$$b_n^s H_n(\lambda_o b) = a_n J_n(\lambda_2 b) + e_n H_n(\lambda_2 b)$$

$$\begin{aligned} -\frac{an}{b \lambda_o^2} [E_o \sin \theta (-i)^n J_n(\lambda_o b) + a_n^s H_n(\lambda_o b)] - \frac{i \omega \mu_o}{\lambda_o} b_n^s H_n'(\lambda_o b) \\ = -\frac{an}{b \lambda_2^2} [b_n J_n(\lambda_2 b) + c_n H_n(\lambda_2 b)] - \frac{i \omega \mu_2}{\lambda_2} [a_n J_n'(\lambda_2 b) + e_n H_n'(\lambda_2 b)] \end{aligned}$$

$$\begin{aligned} \frac{ik_o^2}{\omega \mu_o \lambda_o} [E_o \sin \theta (-i)^n J_n'(\lambda_o b) + a_n^s H_n'(\lambda_o b)] - \frac{an}{b \lambda_o^2} b_n^s H_n(\lambda_o b) \\ = \frac{ik_2^2}{\omega \mu_2 \lambda_2} [b_n J_n'(\lambda_2 b) + c_n H_n'(\lambda_2 b)] - \frac{an}{b \lambda_2^2} [a_n J_n(\lambda_2 b) + e_n H_n(\lambda_2 b)] \end{aligned}$$

$$b_n J_n(\lambda_2 a) + c_n H_n(\lambda_2 a) = 0$$

$$a_n J_n'(\lambda_2 a) + e_n H_n'(\lambda_2 a) = 0$$

(43)

Here again one has a set of six equations and six unknowns. As pointed out

earlier, solving for the unknowns in (43) will be the same as in (31) with (40).

By either or both methods, one obtains the following results.

$$b_n \left[\frac{1}{\lambda_o^2} - \frac{1}{\lambda_2^2} \right] \frac{an}{b} X = a_n \left[\frac{-i\omega\mu_o}{\lambda_o} \frac{H_n'(\lambda_o b)}{H_n(\lambda_o b)} Y + \frac{i\omega\mu_2}{\lambda_2} W \right] \quad (44)$$

and

$$a_n \left[\frac{1}{\lambda_o^2} - \frac{1}{\lambda_2^2} \right] \frac{an}{b} Y = b_n \left[\frac{-ik_2^2}{\omega\mu_2\lambda_2} V + \frac{ik_o^2}{\omega\mu_o\lambda_o} \frac{H_n'(\lambda_o b)}{H_n(\lambda_o b)} X \right] \quad (45)$$

From (44) and (45), one obtains

$$a_n = \frac{2k_o^2(-i)^n E_o \sin \theta}{\pi \omega \mu_o \lambda_o^2 b H_n(\lambda_o b)} \left[\frac{\omega\mu_2}{\lambda_2} W - \frac{\omega\mu_o}{\lambda_o} \frac{H_n'(\lambda_o b)}{H_n(\lambda_o b)} Y \right] \quad (46)$$

$$+ \frac{2k_o^2(-i)^n E_o \sin \theta}{\pi \omega \mu_o \lambda_o^2 b H_n(\lambda_o b)} \left[\frac{1}{\lambda_o^2} - \frac{1}{\lambda_2^2} \right] \frac{an}{b} X + \left[\frac{k_o^2}{\omega\mu_o\lambda_o} \frac{H_n'(\lambda_o b)}{H_n(\lambda_o b)} X - \frac{k_2^2}{\omega\mu_2\lambda_2} V \right]$$

In (44), (45), and (46) above, the functions X, Y, W, V are defined as follows:

$$X = \frac{J_n(\lambda_2 b) H_n(\lambda_2 a) - J_n(\lambda_2 a) H_n(\lambda_2 b)}{H_n(\lambda_2 a)}$$

$$Y = \frac{J_n'(\lambda_2 b) H_n'(\lambda_2 a) - J_n'(\lambda_2 a) H_n'(\lambda_2 b)}{H_n'(\lambda_2 a)}$$

$$W = \frac{J_n'(\lambda_2 b) H_n'(\lambda_2 a) - J_n'(\lambda_2 a) H_n'(\lambda_2 b)}{H_n'(\lambda_2 a)}$$

$$V = \frac{J_n'(\lambda_2 b) H_n(\lambda_2 a) - J_n(\lambda_2 a) H_n'(\lambda_2 b)}{H_n(\lambda_2 a)} \quad (47)$$

Since a_n is now known from (46), b_n is obtained using (44) and (46) and hence

c_n and e_n also are obtained from the last two relationships of (43). From the

second equation of (43)

$$b_n^s = a_n \frac{J_n(\lambda_2 b)}{H_n(\lambda_o b)} + e_n \frac{H_n(\lambda_2 b)}{H_n(\lambda_o b)}$$

a_n and e_n both being known, b_n^s can be found. From the first equation of (43)

$$a_n^s = \frac{1}{H_n(\lambda_o b)} [b_n X - E_o \sin \theta (-i)^n J_n(\lambda_o b)]$$

Thus

$$a_n^s = - \frac{E_o \sin \theta (-i)^n J_n(\lambda_o b)}{H_n(\lambda_o b)} + \frac{\frac{-2i k_o^2 (-i)^n E_o \sin \theta}{\pi \omega \mu_o \lambda_o^2 b [H_n(\lambda_o b)]^2} \left[\frac{\omega \mu_2}{\lambda_2} W - \frac{\omega \mu_o}{\lambda_o} \frac{H_n'(\lambda_o b)}{H_n(\lambda_o b)} Y \right]}{\left[\frac{1}{\lambda_o^2} - \frac{1}{\lambda_2^2} \right]^2 \left[\frac{an}{b} \right]^2 Y + \left[\frac{k_o^2 H_n'(\lambda_o b)}{\omega \mu_o \lambda_o H_n(\lambda_o b)} - \frac{k_2^2}{\omega \mu_2 \lambda_2} \frac{V}{X} \right] \left[\frac{\omega \mu_2}{\lambda_2} W - \frac{\omega \mu_o}{\lambda_o} \frac{H_n'(\lambda_o b)}{H_n(\lambda_o b)} Y \right]}$$

Thus one obtains the complete solution of the problem. It is interesting to see how a_n^s behaves for the case of normal incidence. For this case,

$$a = k_o \cos \theta = 0$$

$$\lambda_o = k_o \sin \theta = k_o$$

$$\lambda_2 = [k_2^2 - k_o^2 \cos^2 \theta]^{1/2} = k_2$$

also assume

$$\mu_2 = \mu_o$$

Then

$$a_n^s = - \frac{E_o (-i)^n}{H_n(\lambda_o b)} \left[J_n(\lambda_o b) - \frac{\frac{2i}{\pi b \omega \mu_o H_n(\lambda_o b)}}{\frac{k_o^2 H_n'(\lambda_o b)}{\omega \mu_o \lambda_o H_n(\lambda_o b)} - \frac{k_2^2}{\omega \mu_2 \lambda_2} \frac{V}{X}} \right]$$

Note:

$$\frac{2i}{\pi b} = \lambda_o [J_n(\lambda_o b) H_n'(\lambda_o b) - J_n'(\lambda_o b) H_n(\lambda_o b)]$$

also

$$\begin{aligned}
\frac{V}{X} &= \frac{J_n'(\lambda_2 b) H_n(\lambda_2 a) - J_n(\lambda_2 a) H_n'(\lambda_2 b)}{J_n(\lambda_2 b) H_n(\lambda_2 a) - J_n(\lambda_2 a) H_n(\lambda_2 b)} \\
&= \frac{J_n(\lambda_2 a) N_n'(\lambda_2 b) - J_n'(\lambda_2 b) N_n(\lambda_2 a)}{J_n(\lambda_2 a) N_n(\lambda_2 b) - J_n(\lambda_2 b) N_n(\lambda_2 a)} \\
\therefore a_n^s &= - \frac{E_o (-i)^n}{H_n(\lambda_o b)} \left[J_n(\lambda_o b) - \frac{\frac{2i}{\pi b} [J_n(\lambda_2 b) N_n(\lambda_2 a) - J_n(\lambda_2 a) N_n(\lambda_2 b)]}{\{k_o H_n'(\lambda_o b) [J_n(\lambda_2 b) N_n(\lambda_2 a) - J_n(\lambda_2 a) N_n(\lambda_2 b)] - k_2 H_n(\lambda_o b) [J_n'(\lambda_2 b) N_n(\lambda_2 a) - J_n(\lambda_2 a) N_n'(\lambda_2 b)]\}} \right] \\
&= E_o (-i)^n \left\{ \frac{k_2 J_n(k_o b) [J_n(k_2 a) N_n'(k_2 b) - J_n'(k_2 b) N_n(k_2 a)] - k_o J_n'(k_o b) [J_n(k_2 a) N_n(k_2 b) - J_n(k_2 b) N_n(k_2 a)]}{-k_2 H_n(k_o b) [J_n(k_2 a) N_n'(k_2 b) - J_n'(k_2 b) N_n(k_2 a)] + k_o H_n'(k_o b) [J_n(k_2 a) N_n(k_2 b) - J_n(k_2 b) N_n(k_2 a)]} \right\} \quad (48)
\end{aligned}$$

Obviously, the coefficient of the backscattering cross section $W_n = \frac{a_n^s}{E_o (-i)^n}$
= the quantity in { } above.

This result agrees with Eq. 18.3 on p. 69, ref. 9.

The solution for the problem of a single conducting cylinder may simply be obtained by letting $a = b$, $\lambda_2 = \lambda_o$, $\mu_2 = \mu_o$ in the result for the above case.

For this case, from (47)

$$\begin{aligned}
X &= W = 0 \\
Y &= \frac{2i}{\pi \lambda_o a H_n'(\lambda_o a)} \\
V &= - \frac{2i}{\pi \lambda_o a H_n(\lambda_o a)} \quad (49)
\end{aligned}$$

Then from (46) and (43)

$$a_n = e_n = 0 \quad (50)$$

From (45) and (49)

$$b_n = (-i)^n E_o \sin \theta \quad (51)$$

$$\therefore c_n = - \frac{J_n(\lambda_0 a)}{H_n(\lambda_0 a)} (-i)^n E_0 \sin \theta \quad (52)$$

$$a_n^s = - \frac{J_n(\lambda_0 a)}{H_n(\lambda_0 a)} (-i)^n E_0 \sin \theta = c_n \quad (53)$$

From (43) and (50)

$$b_n^s = 0 \quad (54)$$

Looking back into the field components in the various regions, it is observed that for the case of a single conducting cylinder there is no z-component of the H-field. One may thus make an important observation that for the case of a single conducting cylinder, if the incident wave is TE, only TE modes are excited. Similarly, it can be shown that for this case, if the incident wave is TM, only TM modes are excited. However, for a dielectric cylinder with an obliquely incident wave, a combination of TE and TM modes will be required to satisfy the boundary conditions even though the incident wave may be just TE or TM.

The results of various simplified cases may thus be obtained from the general formulation.

CHAPTER 2

THE ELECTROMAGNETIC FIELD OF AN ELECTRIC DIPOLE

SURROUNDED BY A SPHERE AND A CYLINDER

The effect of material bodies on the behavior of antennas has been the subject of numerous investigations. The emphasis, however, had been only on loop and biconical antennas.¹⁻⁶ Galejs⁷, Katzin¹⁰ and Adler¹¹ treated certain aspects of an electric dipole inside dielectric sphere and cylinder. In this paper, the general case of an electric dipole inside dielectric and permeable bodies in the form of spheres and infinitely long cylinders will be studied. The effects of the variation of the material properties on the power outflow will be discussed, for the special cases of spheres and cylinders having diameters very small and very large compared to the free space wave length. The importance of large diameter sphere and cylinder arises when a dipole is embedded in a material medium such as the ionosphere which is large, yet of finite dimensions.

2.1 Electric Dipole Surrounded by a Sphere

The method involves the study of the primary field (the field of an electric dipole in an unbounded medium) in a spherical geometry and then construct the secondary field (the field created by the presence of the material sphere) using similar wave functions.

2.1.1. The Primary Field Due to an Electric Dipole

A short electric dipole of dipole moment ' Ih ' is located at the origin of the coordinate system and is oriented along the z -direction. (Figure 1).

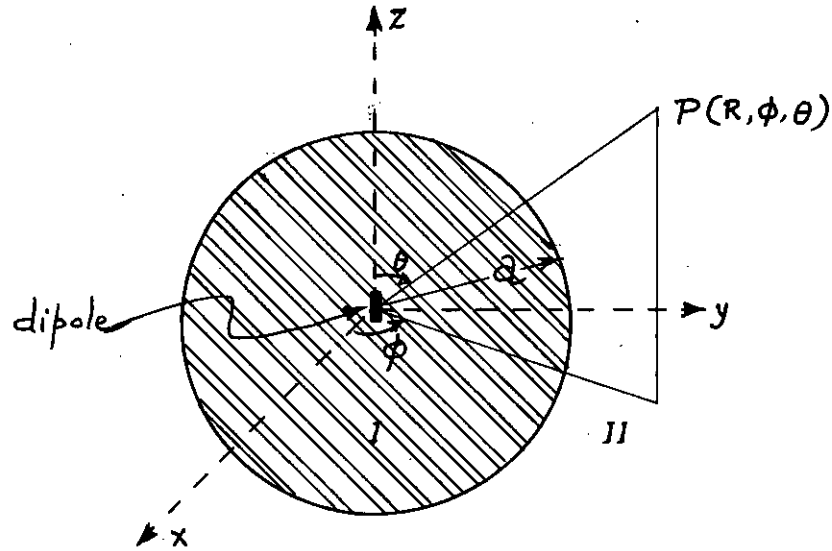


Figure 1 - An Electric Dipole Surrounded by a Sphere.

For a z-oriented electric dipole, the vector potential is given by

$$\vec{A} = \hat{z} \cdot \frac{\mu}{4\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} I \frac{e^{ikR(z,z')}}{R(z,z')} dz' \approx \hat{z} \frac{\mu Ih}{4\pi} \frac{e^{ikR}}{R} \quad (1)$$

where

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

x', y', z' is the location of the dipole and

x, y, z is the point in space where A_z is desired.

The vector potential in (1) is to be expressed in terms of some spherical functions. A suitable function is the spherical Hankel function, $h_0(kR)$, since this function has a singularity at the origin, exactly where the source is located.

Using the following identities

$$h_0(z) = j_0(z) + in_0(z) = -\frac{i}{z} e^{iz}$$

$$P_1(\cos \theta) = \cos \theta$$

$$\frac{d}{d\theta} P_1(\cos \theta) = -\sin \theta$$

(2)

where $P_1(\cos \theta)$ is the Legendre function, equation (1) reduces to

$$\begin{aligned}
 \vec{A} &= \hat{z} \cdot \frac{i\mu I h k}{4\pi} h_0(kR) \\
 &= \hat{R} \frac{i\mu I h k}{4\pi} h_0(kR) \cos \theta - \hat{\theta} \frac{i\mu I h k}{4\pi} h_0(kR) \sin \theta \\
 &= \hat{R} \frac{i\mu I h k}{4\pi} h_0(kR) P_1(\cos \theta) + \hat{\theta} \frac{i\mu I h k}{4\pi} h_0(kR) \frac{d}{d\theta} P_1(\cos \theta)
 \end{aligned} \tag{3}$$

where $\hat{R}, \hat{z}, \hat{\theta}$ are the unit vectors.

The electric and magnetic fields are derived from the vector potential using the following relationships:

$$\begin{aligned}
 \vec{E} &= \frac{i}{\omega\mu\epsilon} (\nabla \nabla \cdot \vec{A} + k^2 \vec{A}) \\
 \vec{H} &= \frac{1}{\mu} \nabla \times \vec{A}
 \end{aligned} \tag{4}$$

These are

$$\begin{aligned}
 \vec{E} &= -\hat{R} \frac{\omega\mu I h}{2\pi R} h_1(kR) P_1(\cos \theta) \\
 &\quad - \hat{\theta} \frac{\omega\mu I h}{4\pi R} \frac{d}{dR} [R h_1(kR)] \frac{d}{d\theta} P_1(\cos \theta) \\
 \vec{H} &= \hat{\phi} \frac{I h k^2}{i4\pi} h_1(kR) \frac{d}{d\theta} P_1(\cos \theta)
 \end{aligned} \tag{5}$$

It is observed that $\vec{E}$ has the E_R and E_θ components and $\vec{H}$ has only the H_ϕ component.

2.1.2 The Secondary Field Due to the Presence of the Sphere

The secondary field must be of the same form as the primary field given in (5). The boundary conditions on the surface of the sphere are

$$\begin{aligned}
 E_\theta^{(1)} &= E_\theta^{(2)} \\
 &\quad \text{at } R = a \\
 H_\phi^{(1)} &= H_\phi^{(2)}
 \end{aligned} \tag{6}$$

The superscripts of E and H represent the region in which they are valid. To obtain unique solutions one must also have an outgoing wave at infinity. It is now very simple to write the field quantities for the two regions, noting that in

Region I, one requires the homogeneous and the inhomogeneous solutions of the wave equations, and in Region II, just the homogeneous solution.

Region I

$$E_{\theta}^{(1)} = - \frac{\omega \mu I h}{4 \pi R} \frac{d}{dR} [R h_1(kR)] \frac{d}{d\theta} P_1(\cos \theta) - e_1 \frac{1}{kR} \frac{d}{dR} [R j_1(kR)] \frac{d}{d\theta} P_1(\cos \theta) \quad (7)$$

$$H_{\phi}^{(1)} = \frac{I h k^2}{i 4 \pi} h_1(kR) \frac{d}{d\theta} P_1(\cos \theta) + \frac{e_1 k}{i \omega \mu} j_1(kR) \frac{d}{d\theta} P_1(\cos \theta)$$

Region II

$$E_{\theta}^{(2)} = - f_1 \frac{1}{k_o R} \frac{d}{dR} [R h_1(k_o R)] \frac{d}{d\theta} P_1(\cos \theta) H_{\phi}^{(2)} = \frac{f_1 k_o}{i \omega \mu_o} h_1(k_o R) \frac{d}{d\theta} P_1(\cos \theta) \quad (8)$$

In (7) and (8), e_1 , and f_1 are suitable constants to be determined from conditions in (6). Thus at $R = a$

$$\frac{\omega \mu I h}{4 \pi R} \frac{d}{dR} [R h_1(kR)] + \frac{e_1}{kR} \frac{d}{dR} [R j_1(kR)] = \frac{f_1}{k_o R} \frac{d}{dR} [R h_1(k_o R)] \frac{I h k^2}{i 4 \pi} h_1(kR) + \frac{e_1 k}{i \omega \mu} j_1(kR) = \frac{f_1 k_o}{i \omega \mu_o} h_1(k_o R) \quad (9)$$

It is now necessary to solve for e_1 and f_1 from the two equations. For the present case, since only the field outside the sphere is of interest, only f_1 need be evaluated. The result is

$$f_1 = \frac{i h \omega \mu}{4 \pi k a^2} \frac{1}{j_1(ka) h_1'(k_0 a) + j_1'(ka) h_1(k_0 a) \left[\frac{1}{k_0 a} - \frac{K_m}{Nka} \right] - \frac{K_m}{N} h_1(k_0 a) j_1'(ka)} \quad (10)$$

where

$$K_m = \mu / \mu_0$$

$$N = k / k_0$$

Combining (8) and (10), one obtains the formal solution for the radiated field outside the sphere.

2.1.3 Special Cases

The value of f_1 in (10) may be reduced to simpler forms for the special cases; namely, the case of a small sphere ($k_0 a, ka \ll 1$), and the case of a large sphere ($k_0 a, ka \gg 1$).

Case 1 $k_0 a, ka \ll 1$ (small sphere, low frequency case.)

For this case the following approximations are possible.

$$j_1(k_0 a) \simeq \frac{1}{3} k_0 a$$

$$h_1(k_0 a) \simeq \frac{-i}{k_0^2 a^2}$$

$$j_1'(k_0 a) \simeq \frac{1}{3}$$

with these, (10) reduces to

$$f_1 = \frac{i h \omega \mu_0 k_0}{4 \pi} \frac{3}{\epsilon_r + 2} \quad (11)$$

where

$$\epsilon_r = \epsilon / \epsilon_0$$

The magnetic field is then, from (8) and (11)

$$\begin{aligned} H_{\phi}^{(2)} &= - \frac{i h \omega \mu_0 k_0}{4 \pi} \cdot \frac{3}{\epsilon_r + 2} \cdot \frac{k_0}{i \omega \mu_0} h_1(k_0 R) \sin \theta \\ &= \frac{i h k_0^2}{4 \pi} h_1(k_0 R) \sin \theta \frac{3}{\epsilon_r + 2} \end{aligned} \quad (12)$$

with $\epsilon_r = 1$, the above reduces to $H_{\phi 0}^{(2)} = \frac{i j h k_o^2}{4\pi} h_1(k_o R) \sin \theta$ = magnetic field due to a dipole alone in free space. In the far zone, the electric field is

$$E_{\theta}^{(2)} = \sqrt{\frac{\mu_o}{\epsilon_o}} H_{\phi}^{(2)} = 30 i j h k_o^2 h_1(k_o R) \sin \theta \frac{3}{\epsilon_r + 2} \quad (13)$$

The radiated power is

$$W = \frac{1}{2} \int_0^\pi H_{\phi}^{(2)} E_{\theta}^{(2)*} 2\pi R^2 \sin \theta d\theta \quad (14)$$

where * indicates complex conjugate. Using the asymptotic expression for $h_1(k_o R)$, the following is obtained.

$$h_1(k_o R) h_1^*(k_o R) \approx \frac{1}{k_o^2 R^2}$$

and

$$W = 10 I^2 h^2 k_o^2 \left| \frac{3}{\epsilon_r + 2} \right|^2$$

The radiation resistance is

$$R_r = \frac{2W}{I^2} = 80 \pi^2 \left[\frac{h}{\lambda_o} \right]^2 \left| \frac{3}{\epsilon_r + 2} \right|^2 = R_o \left| \frac{3}{\epsilon_r + 2} \right|^2 \quad (15)$$

where R_o is the radiation resistance of a dipole alone in free space. The above result is not valid for ϵ_r equal to minus 2, since then R_r is not defined. It is obvious that as ϵ_r becomes less than unity, the radiation resistance for the case of a small sphere increases. As ϵ_r increases, R_r decreases rapidly. The quantity $\frac{3h}{\epsilon_r + 2}$ may be called the effective length of the dipole. It is also seen that when the sphere is small, the radiation resistance does not depend on the relative permeability K_m .

Case 2 $k_0 a$, $ka \gg 1$ (large sphere, high frequency case). For this case

$$j_1(k_0 a) \approx - \frac{\cos(k_0 a)}{k_0 a}$$

$$h_1(k_0 a) \approx - \frac{e^{ik_0 a}}{k_0 a}$$

$$j_1'(k_0 a) \approx \frac{\sin(k_0 a)}{k_0 a}$$

$$h_1'(k_0 a) \approx -i \frac{e^{ik_0 a}}{k_0 a}$$

with these, (10) reduces to (neglecting all terms of order higher than $(\frac{1}{ka})^2$ and $(\frac{1}{k_0 a})^2$)

$$f_1 \approx \frac{Ih\omega\mu k_0}{4\pi} \frac{e^{-ik_0 a}}{\cos(ka) - i \frac{K_m}{N} \sin(ka)} \quad (16)$$

$$H_{\phi}^{(2)} \approx - \frac{iIhk_0 K_m}{4\pi R} \frac{e^{ik_0(R-a)} \sin \theta}{\cos(ka) - i \frac{K_m}{N} \sin(ka)} \quad (17)$$

If one lets $K_m = 1$, $N = 1$, in above,

$$H_{\phi}^{(2)} \approx - \frac{ik_0 Ih}{4\pi R} e^{ik_0 R} \sin \theta \quad (18)$$

which is the usual expression for the magnetic field of a dipole in free space.

Thus

$$H_{\phi}^{(2)} \approx H_{\phi}^{(2)} \frac{K_m e^{-ik_0 a}}{\cos(ka) - i \frac{K_m}{N} \sin(ka)} \quad (19)$$

The far zone electric field is $120 \pi H_{\phi}^{(2)}$ and the radiated power is

$$W = \frac{10k_0^2 I^2 h^2 K_m^2}{\left| \cos(ka) - i \frac{K_m}{N} \sin(ka) \right|^2} \quad (20)$$

The radiation resistance is

$$R_r = \frac{2W}{I^2} = R_o \frac{K_m^2}{\left| \cos(ka) - i \frac{K_m}{N} \sin(ka) \right|^2} \quad (21)$$

where $R_o = 80\pi^2 (h/\lambda_o)^2$ = the free space rad. resistance. Comparing (21) with (15), it is seen that whereas for the low frequency case, is independent of K_m , the radiation resistance R_r for the high frequency case depends both on K_m and ϵ_r .

It is interesting to see the behavior of R_r / R_o for various values of K_m and N . For simplicity both K_m and N will be assumed to be real.

Case 1

$$K_m = \epsilon_r = 1$$

$$R_r/R_o = 1$$

Case 2

$$K_m = 1, k_o a = \text{constant}, \epsilon_r \text{ varies.}$$

$$\frac{R_r}{R_o} \simeq \frac{1}{1 - \left(1 - \frac{1}{\epsilon_r}\right) \sin^2(\sqrt{\epsilon_r} k_o a)} = \text{say } \frac{1}{D}$$

To investigate the extremum points of R_o / R_r differentiate D with respect to ϵ_r , keeping $k_o a$ constant and equate the result to zero.

$$\frac{dD}{d\epsilon_r} = 0$$

Then

$$\sin(\sqrt{\epsilon_r} k_o a) = 0$$

$$\tan(\sqrt{\epsilon_r} k_o a) = (1 - \epsilon_r) \sqrt{\epsilon_r} k_o a \quad (22)$$

The first equation of (22) requires

$$\sqrt{\epsilon_r} k_o a = n\pi, n = \pm 1, \pm 2, \pm 3, \dots$$

At these points,

$$R_r/R_o = 1$$

The second equation of (22) has many solutions. If ϵ_r is greater than unity, $\tan(\sqrt{\epsilon_r} k_o a)$ is negative and therefore the solutions must be in the second and the fourth quadrants.* Without explicitly evaluating these points, one may proceed to find the values of R_r / R_o at these points. The result is

$$\frac{R_r}{R_o} \simeq \frac{1}{1 - \left(1 - \frac{1}{\epsilon_r}\right) \frac{(1 - \epsilon_r)^2 \epsilon_r (k_o a)^2}{1 + (1 - \epsilon_r)^2 \epsilon_r (k_o a)^2}} \quad (23)$$

If $\sqrt{\epsilon_r}$ is sufficiently large, $R_r / R_o \simeq \epsilon_r$

The plot of R_r / R_o is shown in Figure 2.

Case 3 $K_m = 1$, $\epsilon_r = \text{constant}$, $k_o a$ increasing

$$\frac{R_r}{R_o} \simeq \frac{1}{\cos^2(\sqrt{\epsilon_r} k_o a) + \frac{1}{\epsilon_r} \sin^2(\sqrt{\epsilon_r} k_o a)} \quad (24)$$

The plot of R_r / R_o for this case is shown in Figure 3.

Case 4 $\epsilon_n = 1$, K_m varies, $k_o a = \text{constant}$

$$\frac{R_r}{R_o} = \frac{K_m^2}{1 + (K_m - 1) \sin^2(\sqrt{K_m} k_o a)} \quad (25)$$

Apparently, the maximum points are near $\sqrt{K_m} k_o a = n\pi$, $n = \pm 1, \pm 2, \dots$ and the minimum points are near $\sqrt{K_m} k_o a = (2m + 1)\frac{\pi}{2}$, $n = 0, 1, 2, \dots$. The maximum values of R_r / R_o are about K_m^2 and the minimum values are about K_m . The plot of R_r / R_o vs. K_m is shown in Figure 4.

*If, however, ϵ_r is less than unity, the solutions are at $(\sqrt{\epsilon_r} k_o a)$ in the first and the third quadrants.

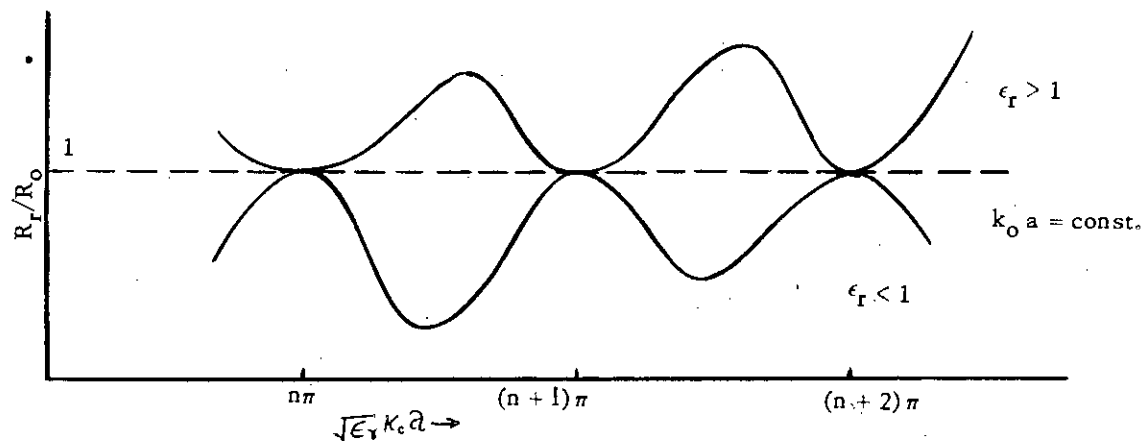


Figure 2 - Relative Radiation Resistance vs $\sqrt{\epsilon_r}$ (not to scale)

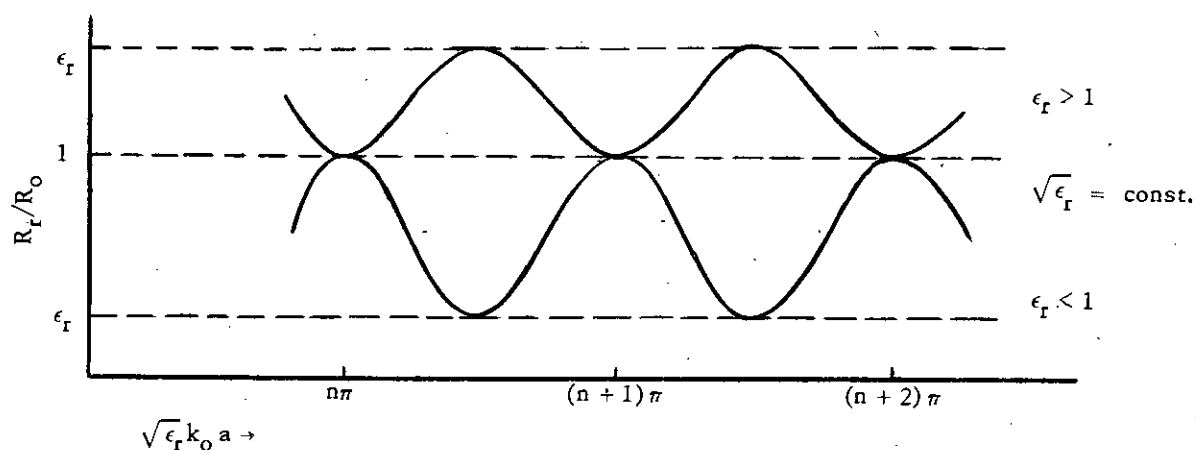


Figure 3 - Relative Radiation Resistance vs $k_0 a$ (not to scale)

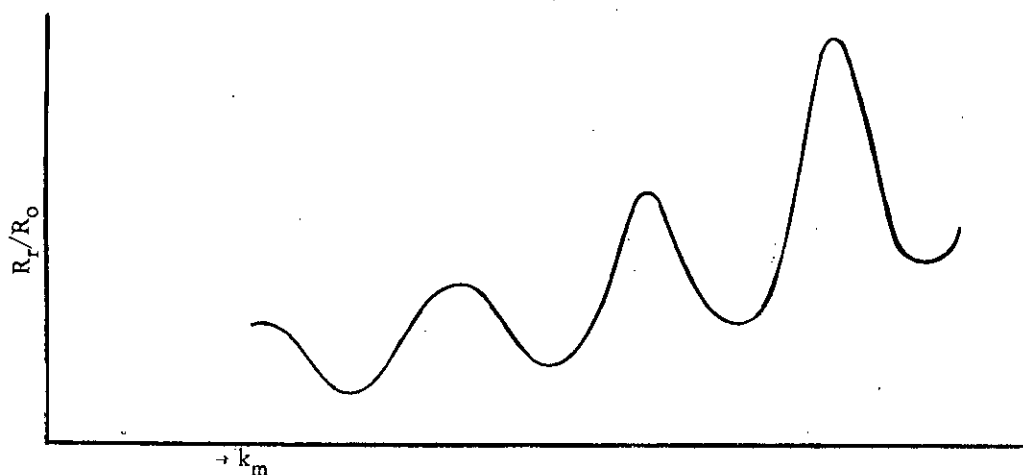


Figure 4 - Relative Radiation Resistance vs k_m (not to scale)

2.2 Electric Dipole Surrounded by an Infinitely Long Cylinder

The electric dipole of moment Ih is assumed to be z-oriented and located at the center of the coordinate system as shown in figure 5.

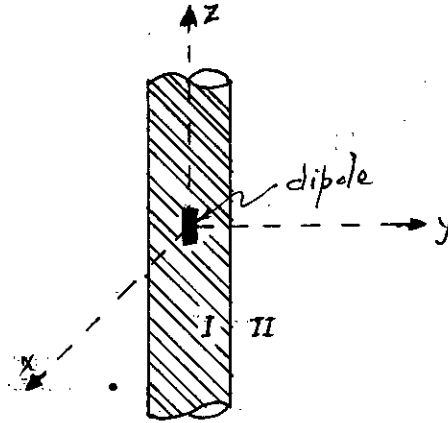


Figure 5 - Electric dipole and an infinitely long cylinder.

As in the spherical case, here also one writes

$$A_z \approx \frac{\mu Ih}{4\pi} \frac{e^{ikR}}{R} \quad (26)$$

and attempts to express e^{ikR}/R as a combination of the functions involving the cylindrical Hankel function of zeroth order. There being no restriction on the eigen-value, it will be continuous, and therefore e^{ikR}/R should be expressed as an integral over all a 's. One may then use Weyrich's formula which is

$$\frac{e^{ikR}}{R} = \frac{i}{2} \int_{-\infty}^{\infty} e^{iaz} H_0^{(1)}(\sqrt{k^2 - a^2} r) da \quad (27)$$

subject to the conditions

$$(a) \text{ } r \text{ and } z \text{ real; } (b) 0 \leq \arg \sqrt{k^2 - a^2} < \pi; \text{ } (c) 0 \leq \arg k < \pi$$

In the present case, condition (a) is obviously satisfied, and so is (c) since $k = \omega \sqrt{\mu} \sqrt{\pm \epsilon + i \frac{\sigma}{\omega}}$. Condition (b) will be used to choose the contour of integration. Using (27), (26) reduces to

$$A_z = \frac{i \mu I h}{8 \pi} \int_{-\infty}^{\infty} H_0(\sqrt{k^2 - a^2} r) e^{i a z} da \quad (28)$$

The superscript (1) from H_0 has been omitted for simplicity in writing.

Also

$$\begin{aligned} \vec{H} &= \frac{1}{\mu} \nabla \times \vec{A} = -\hat{\phi} \frac{1}{\mu} \frac{\partial A_z}{\partial r} \\ \vec{E} &= \frac{i \omega}{k^2} \nabla (\nabla \cdot \vec{A}) + i \omega \vec{A} \end{aligned} \quad (29)$$

Therefore

$$\begin{aligned} E_r &= \frac{i \omega}{k^2} \frac{\partial^2 A_z}{\partial r \partial z} \\ E_{\phi} &= 0 \\ E_z &= \frac{i \omega}{k^2} \left[k^2 + \frac{\partial^2}{\partial z^2} \right] A_z \end{aligned} \quad (30)$$

The boundary conditions are

$$\begin{aligned} H_{\phi}^{(1)} &= H_{\phi}^{(2)} \\ &\text{at } r = a \\ E_z^{(1)} &= E_z^{(2)} \end{aligned} \quad (31)$$

It is observed that in region I, the vector potential will consist of the homogeneous solution of the wave equation and that due to the source. In region II, however, it will consist only of the homogeneous solution of the wave equation. In the two regions, the vector potentials are, therefore

$$\begin{aligned}
A_z(1) &= \frac{i\mu Ih}{8\pi} \int_{-\infty}^{\infty} H_0(\sqrt{k^2 - a^2} r) e^{iaz} da + \int_{-\infty}^{\infty} a(a) J_0(\sqrt{k^2 - a^2} r) e^{iaz} da \\
A_z(2) &= \int_{-\infty}^{\infty} b(a) H_0(\sqrt{k_0^2 - a^2} r) e^{iaz} da
\end{aligned} \tag{32}$$

where $a(a)$ and $b(a)$ are to be determined from the boundary conditions.

Applying now the conditions in (31) and noting that since the exponential e^{iaz} is appearing in all the integrals, the integrands alone may be equated, the following is obtained:

$$a(a) = \frac{\frac{i\mu Ih}{8\pi} \left[H_0(l_0 a) H'_0(l a) - \frac{K_m}{N^2} \frac{l}{l_0} H_0(l a) H'_0(l_0 a) \right]}{\frac{K_m}{N^2} \frac{l}{l_0} J_0(l a) H'_0(l_0 a) - J'_0(l a) H_0(l_0 a)} \tag{33}$$

$$b(a) = \frac{l\mu Ih}{4N^2 l_0^2 \pi^2 a} \left[J'_0(l a) H_0(l_0 a) - \frac{K_m}{N^2} \frac{l}{l_0} J_0(l a) H'_0(l_0 a) \right]$$

where

$$l = \sqrt{k^2 - a^2}, \quad l_0 = \sqrt{k_0^2 - a^2}$$

$$K_m = \mu / \mu_0, \quad N = k/k_0$$

The prime sign with J_0 and H_0 indicates differentiation with respect to the argument.

Thus, the solution of the vector potential is formally given by (32) and (33).

With $K_m = 1$, $N = 1$, $a(a)$ in (33) vanishes and

$$b(a) = \frac{i \mu_0 I h}{4 \ell_0^2 \pi^2 a [J_0'(\ell_0 a) H_0(\ell_0 a) - J_0(\ell_0 a) H_0'(\ell_0 a)]}$$

But

$$J_0(z) H_0'(z) - J_0'(z) H_0(z) = \frac{2i}{\pi z}$$

Then

$$b(a) = \frac{i \mu_0 I h}{8\pi}$$

This is exactly what was expected because in absence of the cylinder, the solution for A_z must be given by (28).

2.2. Radiated Power

The poynting vector is given by

$$\vec{P} = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} [\hat{z} E_r H_\phi^* - \hat{r} E_z H_\phi^*]$$

In order to find the radiated energy, one may integrate the poynting vector along an infinitely long cylinder. Thus, the energy will be $W = \int_{-\infty}^{\infty} P_r 2\pi r dz +$ contributions of ends of the cylinder located at $\pm \infty$.

This energy flow across the ends of the cylinder at $\pm \infty$ does not contribute to the field radiated away from the cylinder and will not be considered here.

Thus

$$W \simeq \int_{-\infty}^{\infty} P_r 2\pi r dz$$

$$= \pi r \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \frac{i\omega}{k_0^2} b(a) (k_0^2 - a^2) H_0(\sqrt{k_0^2 - a^2} r) e^{iaz} da$$

$$\int_{-\infty}^{\infty} \frac{1}{\mu_0} [b(a) \sqrt{k_0^2 - a^2} H_0(\sqrt{k_0^2 - a^2} r)]^* e^{-iaz} da$$

By interchanging the order of integration and using the following properties of the Dirac delta function,

$$2\pi \delta(a_1 - a_2) = \int_{-\infty}^{\infty} e^{i(a_1 - a_2)z} dz$$

$$\int_{-\infty}^{\infty} g(a_1) \delta(a_1 - a_2) da_1 = g(a_2)$$

the above reduces to

$$W = \frac{2i\omega\pi^2 r}{\mu_0 k_0^2} \int_{-\infty}^{\infty} b(a) b(a)^* \frac{k_0}{k_0} \frac{k_0}{k_0} H_0^{(2)}(k_0 r) H_0^{(1)}(k_0 r) da \quad (34)$$

Equation (34) is a complete expression for the radiated energy both in the near and the far field. Evaluation of the expression appears to be difficult without resorting to numerical techniques. Instead of going into any further detail on this, an approximate method will be discussed in the next section for evaluating the far field.

2.2.2 Asymptotic evaluation of the electromagnetic fields by the method of steepest descent.

An asymptotic evaluation of integral of the form (32) is required.

$$A_z^{(2)} = \int_{-\infty}^{\infty} b(a) H_0(\sqrt{k_0^2 - a^2} r) e^{iaz} da$$

The variable of integration a runs along the real axis from $-\infty$ to $+\infty$.

However, there are two branch points at $a = \pm k_0$ which lie on the real axis.

This difficulty can be avoided if k_0 is first assumed to have a small imaginary

part corresponding to a small but finite conductivity of air and then let this imaginary part approach zero. With sufficiently large r , it is then admissible to assume:

$$\sqrt{k_0^2 - a^2} r \gg 1.$$

Then

$$A_z^{(2)} \underset{r \rightarrow \infty}{\approx} \int_{-\infty}^{\infty} b(a) \sqrt{\frac{2}{\pi \sqrt{k_0^2 - a^2} r}} e^{-i \frac{\pi}{4}} e^{i [az + \sqrt{k_0^2 - a^2} r]} da \quad (35)$$

In order to integrate (35) by the method of steepest descent, the following substitution will be made:

$$\begin{aligned} a &= k_0 \sin \phi \\ \phi &= \phi_1 + i \phi_2 \\ z &= R \sin \theta \\ \rho &= R \cos \theta \end{aligned} \quad (36)$$

It may be noted that in (36) the angle θ is measured from the r -coordinate (and not from the z -axis, as is usually done).

The detail of the integration is shown in appendix 3. The result of the integration is

$$A_z^{(2)} \underset{\substack{r \rightarrow \infty \\ R \rightarrow \infty}}{\approx} 2b(k_0 \sin \theta) e^{-i \frac{\pi}{2}} \frac{e^{i k_0 R}}{R} \quad (37)$$

The electromagnetic far fields are then

$$\begin{aligned}
H_{\phi}^{(2)} &\underset{\substack{r \rightarrow \infty \\ R \rightarrow \infty}}{\simeq} - \frac{2}{\mu_0} b(k_0 \sin \theta) k_0 \cos \theta \frac{e^{ik_0 R}}{R} \\
E_{\theta}^{(2)} &\underset{\substack{R \rightarrow \infty \\ r \rightarrow \infty}}{\simeq} \sqrt{\mu_0 / \epsilon_0} H_{\phi}^{(2)}
\end{aligned} \tag{38}$$

where $b(k_0 \sin \theta)$ is the value of $b(a)$ in (33) with $a = k_0 \sin \theta$.

Equation (38) is the formal solution for the far field.

The expression in (38) may be reduced to a simple form if it is assumed that the radius of the cylinder is small such that $\ell_0 a \ll 1$ and $\ell_a \ll 1$ are valid assumptions. There is, however, one restriction. Since a runs from $-\infty$ to $+\infty$, ℓ_0 and ℓ_a also run from $-\infty$ to $+\infty$. If z is assumed to be large, then the major contribution to the integral (35) will be from regions where a is not too large. Thus for large z , $\ell_0 a$ and ℓ_a may assume values small compared to unity.

For this special case, $b(a)$ in (33) reduces to

$$b(a) \simeq \frac{\mu_0 I h}{4\pi^2} \frac{1}{\frac{\ell_0^2 a^2}{2} (1 - \epsilon_r) - \frac{2i}{\pi}} \tag{39}$$

and

$$H_{\phi}^{(2)} \simeq H_{\phi_0} \left[\frac{1}{1 - \frac{i\pi k_0^2 a^2 \cos^2 \theta (\epsilon_r - 1)}{4}} \right] ; H_{\phi_0} = - \frac{i I h k_0 \cos \theta}{4\pi} \frac{e^{ik_0 R}}{R} \tag{40}$$

where H_{ϕ_0} is the magnetic field in free space due to a dipole alone. If $\epsilon_r = 1$, the quantity in the bracket, $[\]$, becomes unity and as expected, and (40) reduces to the far zone magnetic field due to a dipole alone. The electric field is obtained simply by multiplying (40) with $\sqrt{\mu_0 / \epsilon_0}$. Unlike in the case of the

sphere, here $H_{\phi}^{(2)}$ does not increase if ϵ_r is less than unity. As is evident from (40), for all values of ϵ_r other than unity, $H_{\phi}^{(2)}$ decreases compared to H_{ϕ_0} .

This is not surprising because in this case, the infinitely long dielectric rod behaves as a wave guide and some energy will be flowing along the rod.

If the diameter of the cylinder is assumed to be large such that l_a and $l_o a$ are both much greater than unity (the validity of this assumption has already been discussed), then it is possible to use the large argument approximation of the Bessel and Hankel functions and reduce (33) to

$$b(a) \approx \frac{i \mu_0 I h}{8 \pi} \left[\frac{l}{l_o} \right]^{1/2} \frac{e^{-i s_0}}{\cos s - i \frac{\epsilon_r l_o}{l} \sin s}$$

where

$$s = l_a - \pi/4, \quad s_0 = l_o a - \pi/4 \quad (41)$$

Then from (38)

$$H_{\phi}^{(2)} \underset{\substack{R \rightarrow \infty \\ r \rightarrow \infty}}{\approx} H_{\phi_0} \frac{(l/l_o)^{1/2} e^{-i s_0}}{\cos s - i \frac{\epsilon_r l_o}{l} \sin s}$$

where H_{ϕ_0} is given by (40).

Because of the difficulty in carrying out the integration about an infinite cylinder to obtain the total radiated power, only the poynting vector will be considered for the purpose of comparison. The ratio of the magnitudes of the Poynting vector for this case to that for the case of dipole in free space is then:

$$\frac{P_r}{P_o} \approx \left| \frac{(l/l_o)^{1/2}}{\cos s - i \frac{\epsilon_r l_o}{l} \sin s} \right|^2 \quad (42)$$

So far there was no restriction on the value of ϵ and μ . To simplify the matter, in the following discussion, both ϵ and μ will be assumed to be real.

Note:

$$\frac{l}{l_0} = \frac{\sqrt{k^2 - k_0^2 \sin^2 \theta}}{k_0 \cos \theta} = \frac{\sqrt{N^2 - \sin^2 \theta}}{\cos \theta}$$

Therefore, with ϵ and μ real,

$$\frac{P_r}{P_o} \simeq \frac{\frac{\sqrt{N^2 - \sin^2 \theta}}{\cos \theta}}{1 + \left[\frac{\epsilon_r^2 \cos^2 \theta}{N^2 - \sin^2 \theta} - 1 \right] \sin^2 \theta} \quad (43)$$

If N and ϵ_r are both unity, P_r/P_o reduces to unity, as expected. It is interesting to study the behavior of P_r/P_o as K_m and ϵ_r varies, for any fixed θ . Consider the case when $\theta = 0$. Then

$$\frac{P_r}{P_o} \simeq \frac{\sqrt{\epsilon_r K_m}}{1 + (\epsilon_r/K_m - 1) \sin^2 \left[\sqrt{\epsilon_r K_m} k_0 a - \frac{\pi}{4} \right]} \quad (44)$$

A few simple cases may now be considered.

Case 1. $K_m = 1$, $(\sqrt{\epsilon_r} k_0 a)$ varies

$$P_r/P_o \simeq \frac{\sqrt{\epsilon_r}}{1 + (\epsilon_r - 1) \sin^2 \left[\sqrt{\epsilon_r} k_0 a - \frac{\pi}{4} \right]} \quad (45)$$

The maximum value of P_r/P_o is about

$$\sqrt{\epsilon_r} k_0 a - \frac{\pi}{4} = n\pi, n = 0, 1, 2, \dots$$

At these points

$$P_r/P_o \simeq \sqrt{\epsilon_r}$$

The minimum value of P_r/P_o is about

$$\sqrt{\epsilon_r} k_0 a - \frac{\pi}{4} = (2m+1) \frac{\pi}{2}, m = 0, 1, 2, \dots$$

At these points

$$P_r/P_o \approx \sqrt{\epsilon_r}$$

The nature of variation of P_r/P_o with ϵ_r & $k_o a$ is shown in figure 6.

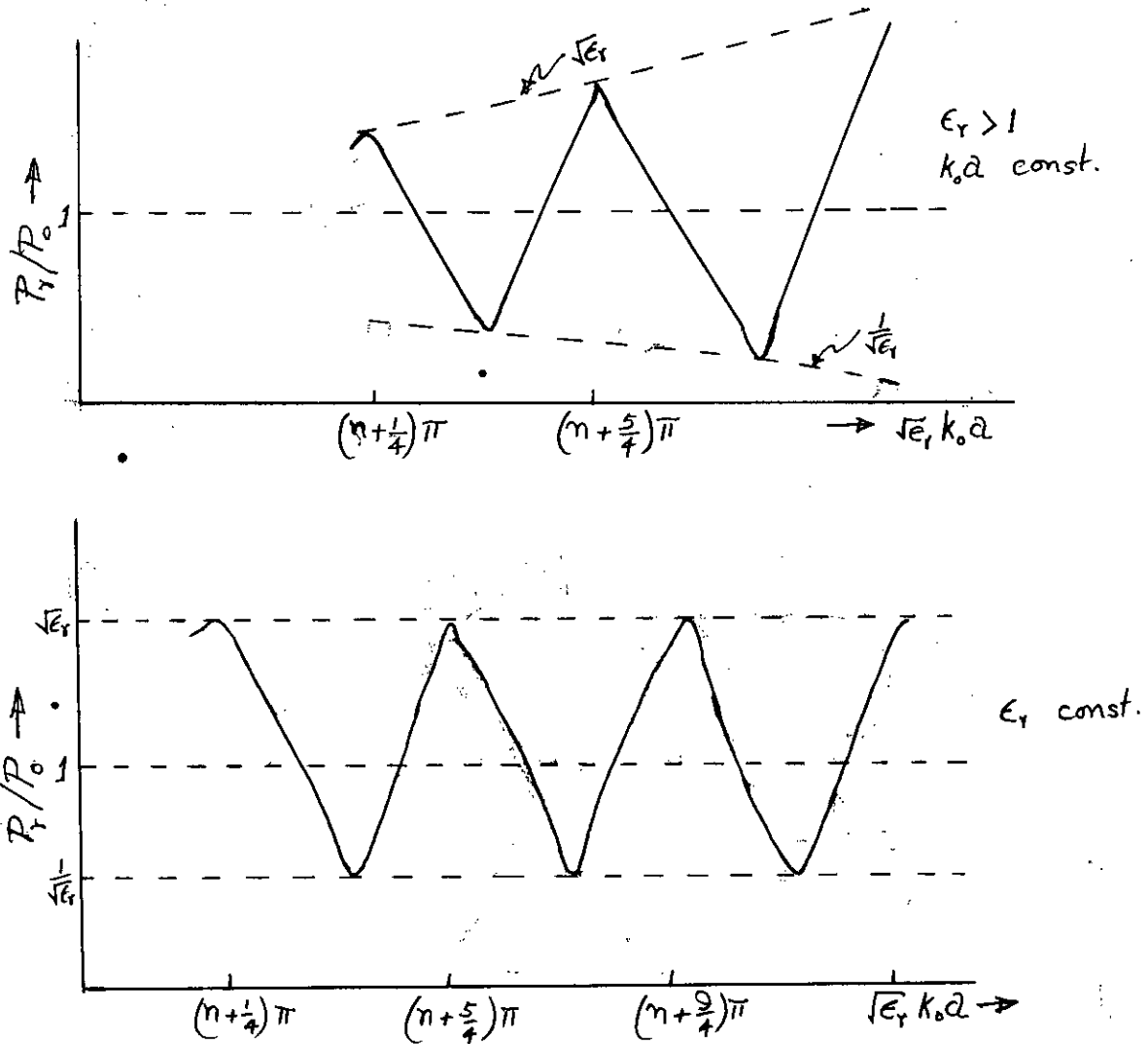


Figure 6. - Variation of radiated power with ϵ_r and $k_o a$ (not to scale).

Case II

K_m varies, $\epsilon_r = 1$, $k_o a = \text{constant}$

$$P_r/P_o \approx \sqrt{K_m} / \left[1 + \left(\frac{1}{K_m} - 1 \right) \sin^2 \left(\sqrt{K_m} k_o a - \frac{\pi}{4} \right) \right]$$

The minimums are about

$$\sqrt{K_m} k_o a - \frac{\pi}{4} = n\pi; n = 0, 1, 2, \dots$$

$$[P_r/P_o]_{\min} \approx \sqrt{K_m}$$

The maximum points are about

$$\sqrt{K_m} k_0 a = \pi/4 + (2m+1)\pi/2; m = 0, 1, 2, \dots$$

$$[P_r/P_0]_{\max} \simeq (K_m)^{3/2}$$

This is shown in figure 7.

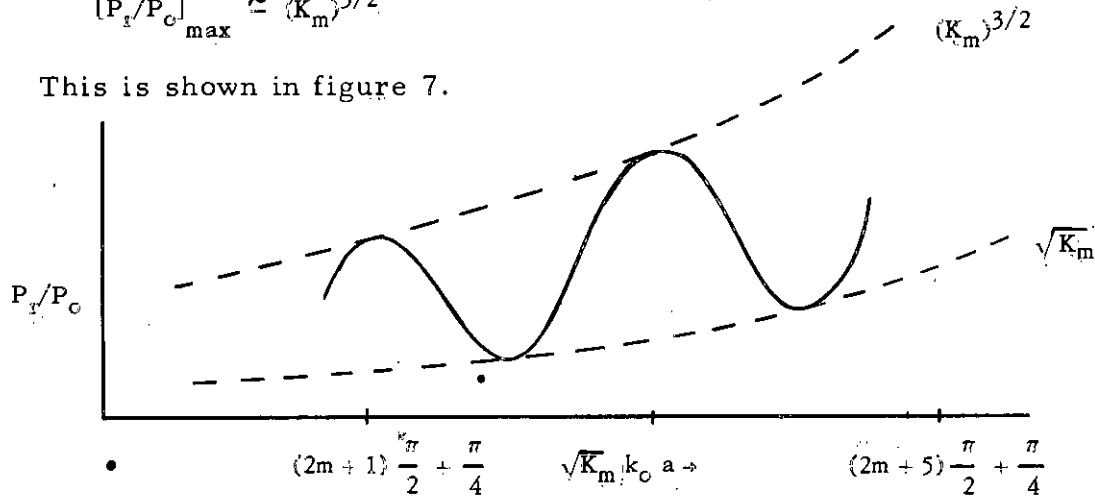


Figure 7. Variation of radiated power with K_m (not to scale).

Discussion

From the plots in figures 2, 3, 4, 6 and 7, it is apparent that in the cases of both the large sphere and the large cylinder, the radiation depends on K_m and ϵ_r , and there are resonant and antiresonant points. The radiation increases with the increase of K_m and ϵ_r , according to the pattern shown in the figures. One possible explanation for the existence of such resonance and antiresonance points is that both the sphere and the cylinder behave like "resonance cavities", with resonances occurring at intervals of $ka = \pi$ radians. A knowledge of the positions of these points would be helpful in designing any antenna structure. The curves also reveal that if maximizing the radiation resistance is the criterion for the design, increasing K_m or ϵ_r is more helpful than increasing the physical size of the structure because this way no benefit beyond a certain range can be obtained no matter how large the sphere or the cylinder is made.

CURRENT DISTRIBUTION ON A CYLINDRICAL ANTENNA
WITHIN A COAXIAL MATERIAL CYLINDER

Figure 1 shows a cylindrical shell with a central longitudinal slot. The shell has an outer radius r and a slot width $2b$. The slot has a depth $2h$ and a thickness 2δ . A vertical z -axis is shown, with the origin $z=0$ at the center of the slot. The diagram illustrates the geometry for a stress analysis problem.

If the current distribution is assumed to be $I(z)$ along the length of the antenna, then the vector potential 'A' at any point (r, ϕ, z) , $r > a$, in an unbounded medium with properties μ and ϵ is given by

$$A_{zp} = \frac{\mu}{4\pi} \int_{-h}^h I(\xi) g(r, z/a, \xi) d\xi \quad (1)$$

where

$$g(r, z/a, \xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{ikR}}{R} d\phi \quad (2)$$

$$R = \sqrt{r^2 + a^2 - 2ar \cos \phi + (z - \xi)^2}$$

The subscript zp of 'A' indicates that the vector potential has only the z-component and Eq. (1) gives the primary excitation only.

One may now use the formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{ikR}}{R} d\phi = \frac{i}{2} \int_{-\infty}^{\infty} J_0(la) H_0^{(1)}(lr) e^{-i\alpha(z-\xi)} d\alpha \quad (3)$$

where $l = \sqrt{k^2 - \alpha^2}$

Using (3), Eq. (1) reduces to

$$A_{zp} = \int_{-\infty}^{\infty} u(\alpha) H_0^{(1)}(lr) e^{-i\alpha z} d\alpha \quad (4)$$

$$\text{where } u(\alpha) = \frac{i\mu}{8\pi} \int_{-h}^h J_0(la) I(\xi) e^{i\alpha\xi} d\xi$$

3.1 Total vector potentials inside and outside the material cylinder.

Formally one may write

$$A_z^{(1)} = \int_{-\infty}^{\infty} a(a) J_0(\ell) e^{-iaz} da + \int_{-\infty}^{\infty} b(a) H_0^{(1)}(\ell) e^{-iaz} da$$

$$A_z^{(2)} = \int_{-\infty}^{\infty} c(a) H_0^{(1)}(\ell_o r) e^{-iaz} da \quad (5)$$

where $\ell_o = \sqrt{k_o^2 - a^2}$ and $a(a)$, $b(a)$ and $c(a)$ are suitable functions of a . The values of these functions will be determined from the boundary and the excitation conditions. The first term $A_z^{(1)}$ arises due to the diffraction from the boundary surface at $r=b$. The second term, which is singular at $r=0$, accounts for the excitation by the current distribution in the enclosed antenna. A comparison between (4) and (5) suggests that $b(a)$ in (5) should be chosen to be equal to $u(a)$ in (4). Equation (5) differs from Equation (32) in Chapter 2 in that the sign in the exponent is different. This is, however, only a question of convenience and does not change the character of the integrals because all functions involved are even in a .

As in the case of the electric dipole in Chapter 2, the continuity of the tangential electric and magnetic fields at $r=b$ gives the following value of $a(a)$.

$$a(a) = u(a) f(a)$$

where

$$f(a) = \frac{H_0'(\ell b) H_0(\ell_o b) - \frac{K_m}{N^2} \frac{\ell}{\ell_o} H_0(\ell b) H_0'(\ell_o b)}{\frac{K_m}{N^2} \frac{\ell}{\ell_o} J_0(\ell b) H_0'(\ell_o b) - J_0'(\ell b) H_0(\ell_o b)} \quad (6)$$

The prime sign indicates differentiation and $H_0 ()$ has been written to mean $H_0^{(1)} ()$ for convenience. This abbreviation will be used throughout. Combining (5) and (6), one obtains the vector potential on the surface of the antenna (at $r = a$). Thus

$$[A_z^{(1)}]_{r=a} = \int_{-\infty}^{\infty} u(a) [f(a) J_0(ka) + H_0(ka)] e^{-iaz} da \quad (7)$$

In what follows, the radius of the antenna will be assumed to be very small compared to the wave length involved, so that only the z-component of $\vec{A}$ and $\vec{E}$ are significant. The condition that E_z must vanish on the surface of the antenna requires then that

$$\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = 0$$

From the usual linear antenna theory, one then writes for a symmetrically fed antenna

$$[A_z^{(1)}]_{r=a} = B_1 \cos(kz) + C_1 e^{ik|z|} \quad (8)$$

where it has been assumed that the gap width is vanishingly small. B_1 and C_1 are suitable constants. B_1 is determined from the condition that I (th) must vanish and C_1 from the condition that there is a discontinuity of the scalar potential at the gap at $z=0$, due to the applied voltage to feed the antenna.

Lorentz condition gives the relationship between the vector and the scalar potential. Thus

$$\phi(z) = -\frac{i\omega}{k^2} \frac{\partial A_z}{\partial z} = \frac{\omega}{k} c_1 e^{ik|z|} + \frac{i\omega B_1}{k} \sin(kz)$$

$$\lim_{\delta \rightarrow 0} [\phi(\delta) - \phi(-\delta)] = 2 \frac{\omega}{k} c_1 = V, \text{ the applied voltage}$$

So that

$$c_1 = \frac{k}{\omega} \frac{1}{2} V \quad (9)$$

Combining (7), (8), and (9), one obtains after some algebraic manipulation, an integral equation for the current distribution on the antenna.

$$\int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} da F(a) e^{-ia(z-\xi)} = -\frac{i8\pi k}{\omega\mu} \left[\frac{1}{2} V e^{ik|z|} + B \cos(kz) \right] \quad (10)$$

where

$$F(a) = J_0(la) [f(a) J_0(la) + H_0(la)]$$

and $f(a)$ is given by (6).

The constant B is different from the previous constant B_1 .

The first part of $F(a)$ arises due to the presence of the material cylinder and the second part due to the antenna itself. This can be seen easily by letting $\mu = \mu_0$ and $\epsilon = \epsilon_0$ in (6). For then $f(a)$ vanishes and $F(a)$ reduces to $J_0(la)H_0(la)$, which is indeed the kernel for an uncoated antenna. In order to check if the first relationship of (5) along with the value of $a(a)$ in (6) and $b(a)$ equal to $u(a)$ is actually the value of the vector potential within the cylinder, one may use

the usual method of formulation of the Green's function involving sources and boundary surfaces. In Appendix 4, it is shown that the vector potential, as given in this chapter is indeed the vector potential for the problem.

Equation (10) has to be solved for $I(z)$ to obtain the current distribution along the length of the antenna. Unfortunately, it is very difficult to solve this equation exactly. Even for the uncoated antenna, only approximate and cumbersome solutions are available in the literature. Before attempting to solve Eqn. (10), the simplified case of an infinitely long antenna will be discussed.

3.2 Infinitely long antenna within a dielectric cylinder

The limiting case of a cylindrical antenna with $h \rightarrow \infty$ is theoretically simpler compared to the finite length antenna. For this case, the current and potential in the antenna form travelling waves starting from the source point at $z = 0$. Hallén² has shown that for an infinitely long antenna, the constant B in (10) is zero because of the absence of any standing wave, so that the integral equation for such an antenna is

$$\int_{-\infty}^{\infty} d\xi I(\xi) \int_{-\infty}^{\infty} d\alpha F(\alpha) e^{-i\alpha(z-\xi)} = -\frac{i4\pi k}{\omega\mu} V e^{ik|z|} \quad (11)$$

The following remarks of Hallén³ regarding the properties of an infinitely long antenna deserve some attention:

"An infinite antenna does not exist. But the behavior of outgoing travelling current waves on an antenna of finite length must be the same as if it were

infinite. If we have correctly determined the current on a finite antenna, which is the more difficult problem, we must necessarily be able to extract expressions for the travelling waves. In other words, the solution for the infinite antenna must be included in the solution for a finite one. This gives a good check on the more difficult problem."

The solution of (11) is simplified by virtue of the existence of the following identity:

$$e^{ik|z|} = -\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{2k e^{-iaz}}{a^2 - k^2} da \quad (12)$$

Combining (11) and (12) and taking the inverse Fourier Transform of both sides, one obtains after some straightforward manipulations

$$\int_{-\infty}^{\infty} I(\xi) e^{ia\xi} d\xi = 4\omega\epsilon V \frac{1}{l^2 F(a)}$$

so that

$$I(z) = \frac{2\omega\epsilon V}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iaz}}{l^2 F(a)} da \quad (13)$$

Equation (13) is the expression for the current distribution along the antenna. It is difficult to evaluate the integral explicitly. Therefore, in the following, an attempt will be made to study only its asymptotic behavior for large and small $k/|z|$.

3.2.1 Asymptotic evaluation for the current distribution for large $k|z|$ by the method of steepest descent.

Various authors have discussed the asymptotic behavior of the current distribution in the infinitely long uncoated antenna.⁴⁻⁶ Their methods being complicated, it is difficult to extend them to the case of coated antenna. It is, however, possible to use the method of steepest descent to evaluate the asymptotic behavior of the current and this method will be used here. The result is usually obtained as an infinite series, but since for large enough $k|z|$ the first term of the series is predominant, only the first term will be evaluated here. The details of the method are shown in Appendix 5.

It will be assumed that the antenna and its coating is thin, compared to the wave length. It is first necessary to simplify the function $F(\alpha)$ as given in (10) and (6). It is observed that $F(\alpha)$ contains terms $\ell (= \sqrt{k^2 - \alpha^2})$ and $\ell_0 (= \sqrt{k_0^2 - \alpha^2})$. Since the integration is along the real axis, difficulty arises due to the fact that k_0 is real, because the points $\alpha = \pm k_0$, are branch points. This difficulty, however, is usually avoided by assuming the existence of a vanishingly small imaginary part of k_0 , corresponding to a slight conductivity of air. Similar trouble arises if k is real, but may be avoided by assuming again a vanishing small imaginary part of k . These points were discussed in Chapter 2. Since only the current distribution for large z is of interest, it is apparent from (13) that the contribution to the integral from large values of α is negligible because of the presence of the factor $e^{-i\alpha z}$, which becomes rapidly oscillatory with increase of α (z being already large). Therefore, if 'a' and 'b' are sufficiently small, ℓ_a , and ℓ_{0b} may be assumed

to be small compared to unity so that the small parameter approximations of the Bessel and Hankel functions may be used to reduce $F(a)$ to

$$F(a) \simeq i \frac{2\ell_o^2 \epsilon_r}{\pi \ell^2} \ln \frac{\Gamma \ell_o b}{2} - i \frac{2}{\pi} \ln \frac{b}{a} \quad (14)$$

With slight rearrangement of the terms, Eq. (14) may be written as

$$F(a) \simeq i \frac{2}{\pi} \frac{\epsilon_r}{\ell^2} \left[\ell_o^2 \ln \frac{\Gamma \ell_o c}{2} - \frac{k^2 - k_o^2}{\epsilon_r} \ln \frac{b}{a} \right] \quad (15)$$

$$\text{where } c = b \left[\frac{a}{b} \right]^{\frac{1}{\epsilon_r}}$$

Therefore (13) becomes

$$I(z) = -i\omega V \epsilon_o \int_{-\infty}^{\infty} \frac{e^{-iaz} d\alpha}{\ell_o^2 \left[\ln \frac{\Gamma \ell_o c}{2} - \frac{k^2 - k_o^2}{\epsilon_r \ell_o^2} \ln \frac{b}{a} \right]} \quad (16)$$

The first term of the series of the asymptotic value of the integral (16) may be obtained directly by using the results shown in Eq. (10) in Appendix 5.

$$\text{Thus } G(a) = \ln \frac{\Gamma \ell_o c}{2} - \frac{k^2 - k_o^2}{\epsilon_r \ell_o^2} \ln \frac{b}{a}$$

$$\text{With } \cos \psi_o \simeq \sqrt{\frac{i}{k_o |z|}}$$

$$[G(a)]_{a=k_o \sin \psi_o} = G(k_o \sin \psi_o) \simeq \frac{1}{2} \ln \frac{\Gamma^2 k_o c^2 i}{4|z|} + \frac{i[k^2 - k_o^2]}{\epsilon_r k_o} |z| \ln \frac{b}{a}$$

Therefore

$$I(z) \simeq 2 \omega \epsilon_0 V \sqrt{e\pi} \frac{e^{ik_0 |z|}}{k_0 \ln \frac{-i4|z|}{\Gamma^2 k_0 c^2} - \frac{2i(k^2 - k_0^2)}{\epsilon_f} |z| \ln \frac{b}{a}} \quad (17)$$

Thus the current distribution far away from the source is given approximately by Eqn. (17).

If in Eqn. (17), one lets $\epsilon_f = 1$, the second term in the denominator vanishes, and $I(z)$ reduces to

$$I(z) \simeq \frac{2\omega\epsilon_0 V \sqrt{e\pi}}{k_0} \frac{e^{ik_0 |z|}}{\ln \frac{-i4|z|}{\Gamma^2 a^2 k_0}} \quad (18)$$

Eqn. (18) obviously is the current distribution of an uncoated antenna far away from the source, as expected. This result compares closely to the result obtained by other authors. For example, Kunz⁵ gives the following result for an uncoated antenna.

$$I(z) \simeq \frac{2\pi V}{\eta_0} \frac{e^{ik_0 |z|}}{\ln \frac{-i4|z|}{\Gamma^2 k_0 a^2}} \cdot \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad (19)$$

Since $k_0 |z|$ is large, $k_0 a$ small and $\sqrt{e\pi} \simeq \pi$, results (18) and (19) are comparable. Chen and Keller⁶ gave a similar result.

If ϵ_f is not too large compared to unity, and the coating is thin, then the second term in the denominator of (17) is negligible compared to the first term, so that

$$I(z) \simeq \frac{2\omega\epsilon_0 V \sqrt{e\pi}}{k_0} \frac{e^{ik_0 |z|}}{\ln \frac{-i4|z|}{\Gamma^2 k_0 c^2}} \quad (20)$$

Comparing (20) with (18), it is clear that the term $c (= b \left[\frac{a}{b} \right]^{\frac{1}{\epsilon_r}})$ may be defined as the effective radius of the coated antenna. An effective radius may be defined as the radius of an uncoated antenna which will have the same current distribution as that of a coated antenna. This observation agrees with that of Wu⁷, who describes the second term in the denominator of (17) as the surface impedance term due to the coating.

3.2.2 Current distribution close to the source and the Driving Point Admittance.

It is desired to obtain an approximate value of the integral in (13) for z close to zero. Chen and Keller⁶ has evaluated a similar integral by a method which is based on the fact that the behavior of the integral for small z is primarily dependent on the behavior of the integrand for large α . A technique very similar to the above will be used here.

It is convenient to define certain quantities β, γ, c, p and c' as follows:

$$\beta = k|z|, \quad \gamma = a|z|, \quad \gamma = \beta \sqrt{1 + \left[\frac{c}{ka} \right]^2} \quad (21)$$

$$p = k_0|z|, \quad \gamma = p \sqrt{1 + \left[\frac{c'}{k_0 b} \right]^2}$$

then

$$la = ic, \quad lb = i \frac{b}{a} c, \quad l_0 b = ic'$$

Chen and Keller⁶ have shown that "no matter how large the constant c is, the absolute values of the argument of Bessel and Hankel functions

exceed c for all γ greater than a certain complex multiple of β which vanishes as β does. Consequently, for all these values of γ , the Bessel and Hankel functions differ by an arbitrarily small amount from their asymptotic forms."

Without going into any further details about the justification of replacing the Bessel and Hankel functions by their asymptotic forms for large argument, (since this is already well explained in Chen and Keller's paper), one may use the method to find the behavior of (13) for z very close to zero. To obtain the desired result, it is not even necessary to utilize the substitutions in (21).

Thus for large a , one may write

$$l = i\sqrt{a^2 - k^2} = iu, \quad l_0 = i\sqrt{a^2 - k_0^2} = iu_0$$

$$J_0(la) = I_0(ua) \simeq \frac{e^{ua}}{\sqrt{2\pi ua}}$$

$$H_0^{(1)}(la) = \frac{2}{\pi i} K_0(ua) \simeq \frac{2}{\pi i} \sqrt{\frac{\pi}{2ua}} e^{-ua} \quad (22)$$

$$J_0'(la) \simeq i I_1(ua) \simeq -i \frac{e^{ua}}{\sqrt{2\pi ua}}, \quad H_0'(la) = \frac{2}{\pi} K_1(ua) \simeq \frac{2}{\pi} \sqrt{\frac{\pi}{2ua}} e^{-ua}$$

In (22), only the first terms of the following series expansions of the modified Bessel and Hankel functions have been retained.⁸

For $|z| \gg 1$, $|z| \gg |\nu|^2$ and $0 \leq \text{phase } z \leq \pi$

$$I_\nu(z) \simeq \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{(4\nu^2 - 1^2)}{1! 8z} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2! (8z)^2} - \dots \right] \\ + e^{(\nu + \frac{1}{2})\pi i} \frac{e^{-z}}{\sqrt{2\pi z}} \left[1 + \frac{(4\nu^2 - 1^2)}{1! 8z} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2! (8z)^2} + \dots \right]$$

For $|z| \gg 1$, $|z| \gg |\nu|^2$, $-\pi < \text{phase } z \leq \pi$

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{(4\nu^2 - 1^2)}{1! 8z} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2! (8z)^2} \right. \\ \left. + \dots + \frac{(4\nu^2 - 1^2) \dots [4\nu^2 - (2r-3)^2]}{(r-1)! (8z)^{r-1}} + \dots \right]$$

With the approximations in (22), the function $F(a)$ defined in (10) and (6) reduces to

$$F(a) \simeq \frac{-i[1 + y_1 e^{-ux}]}{\pi u a}$$

$$\text{where } y_1 = \left[1 - \frac{1}{\epsilon_r} \frac{u}{u_0} \right] / \left[1 + \frac{1}{\epsilon_r} \frac{u}{u_0} \right]; \quad x = 2(b-a)$$

Therefore (13) reduces to

$$I(z) \simeq -2i\omega\epsilon V a \int_{-\infty}^{\infty} \frac{e^{-iaz}}{u[1 + y_1 e^{-ux}]} da \quad (23)$$

The above integral is quite complicated. Some simplifications are possible if one assumes ϵ_r to be not too large, because then $u/u_0 \simeq 1$ (a being large). For this case, y_1 reduces to $y_1 \simeq y = (\epsilon_r - 1)/(\epsilon_r + 1)$ so that

$$I(z) \simeq -2i\omega\epsilon V a \sum_{n=0}^{\infty} (-y)^n \int_{-\infty}^{\infty} [e^{iaz - nx\sqrt{a^2 - k^2}}] \frac{da}{\sqrt{a^2 - k^2}} \quad (24)$$

Apparently the term e^{-iaz} may be changed to e^{iaz} without affecting the result.

The following property of the Fourier Transform (Formula 141, p-199, ref. 8) is useful.

$$\int_{-\infty}^{\infty} \frac{e^{ibu - z\sqrt{u^2 - a^2}}}{\sqrt{u^2 - a^2}} du = \pi i H_0^{(1)} [a\sqrt{z^2 + b^2}]$$

with a, b, z real > 0 , $\sqrt{u^2 - a^2}$ positive for $u > a$, negative imaginary for $-a < u < a$.

Using this formula, Eq. (24) may be reduced to

$$I(z) \simeq 2\omega\epsilon V a \pi \left[H_0^{(1)}(k|z|) + \sum_{n=1}^{\infty} (-y)^n H_0^{(1)}(k\sqrt{n^2 x^2 + |z|^2}) \right] \quad (25)$$

In (25), z and nx are apparently real, k has been assumed to be real although it can be shown that k may also be complex. Eq. (25) may be reduced to simpler forms for certain special cases.

Case I. $x = 2(b - a) \rightarrow$ very large

For this case, the second term in (25) decreases rapidly with the increase of x , so that

$$I(z) \simeq 2\omega\epsilon V a \pi H_0^{(1)}(k|z|) \quad (26)$$

This result is also apparent, if one starts with (23), where the term e^{-ux} vanishes for large x (u being large). The resulting integral may be performed using a Table of Fourier Transforms¹¹. In this integral, ϵ will be assumed to be complex, or having a vanishingly small imaginary part. The result is the same as in (26).

Case 2. $\epsilon_r = 1$, uncoated antenna

For this case, $y = 0$

$$I(z) \simeq 2\omega\epsilon_0 V a \pi H_0^{(1)}(k_0 |z|) \quad (27a)$$

This is the usual result for an uncoated antenna.

Case 3. $|z| \ll x$; ϵ_r large (so that $\epsilon_r \gg 1$) but not too large (so that the previous assumption of $u/u_0 \simeq 1$ is still a valid assumption). The functions ϵ_r and a being independent variables, the above requirement is easily realizable.

For this case, $y = 1$

$$I(z) \simeq 2\omega\epsilon V a \pi \left[H_0^{(1)}(k|z|) + \sum_{n=1}^{\infty} (-1)^n H_0^{(1)}(nkx) \right] \quad (27b)$$

Case 4. $|z| \ll x$; ϵ_r small compared to unity.

For this case, $y = -1$

$$I(z) \simeq 2\omega\epsilon V a \pi \left[H_0^{(1)}(k|z|) + \sum_{n=1}^{\infty} H_0^{(1)}(nkx) \right] \quad (27c)$$

Comparing (26) and (27), it is observed that the current distribution in a coated antenna consists of two parts. The first part is the same as though the antenna was located in an unbounded medium having the same property as the material of the coating. This part depends on the position of the point where the current is to be measured. The second part of the current distribution consists of an infinite series. This part depends on the thickness of the coating.

The driving point admittance is defined as the ratio of the current at $|z| = \delta$ (δ being half the gap size) and the voltage applied at the gap. Thus from (25), the driving point admittance of a coated antenna is (with $|\delta| \ll x$)

$$Y_{dp} = \frac{I(\delta)}{V} \simeq 2\omega\epsilon a \pi \left[H_0^{(1)}(k|\delta|) + \sum_{n=1}^{\infty} (-y)^n [J_0(nkx) + iN_0(nkx)] \right] \quad (28)$$

The first term of Y_{dp} is easy to find once the gap size is given. The second term involves the infinite series and as such is more complicated. A general treatment of this series is quite involved and beyond the scope of this project. Only the special cases in (27b) and (27c) will be discussed.

$$\sum_{n=1}^{\infty} (-1)^n H_0^{(1)}(nkx) = \sum_{n=1}^{\infty} (-1)^n [J_0(nkx) + i N_0(nkx)] \quad (29)$$

The series $\sum_{n=1}^{\infty} (-1)^n J_0(nkx)$ is a special case of Schlömilch series. A detailed discussion about this series has been presented by Watson¹⁶. It is only necessary to mention here that by using Parseval's integral, he has proved the following theorem

$$\sum_{n=1}^{\infty} (-1)^n J_0(nkx) = -\frac{1}{2} \quad (30)$$

provided that $0 < kx < \pi$; the series oscillates when $kx = 0$ and diverges to $+\infty$ when $kx = \pi$. Values of the series for kx larger than π may be obtained using techniques discussed by Infeld,¹⁵ et al. They also treated the series $\sum_{n=1}^{\infty} (-1)^n N_0(nkx)$ in detail. The discussions, being quite involved, will be omitted from here. Only some numerical values of this series will be reproduced from their paper for ready reference (Table 1.)

The other two series $\sum J_0$ and $\sum N_0$ may be evaluated using the following relationships⁹.

$$\begin{aligned} \sum_{n=1}^{\infty} J_0(nx) &= -\frac{1}{2} + \frac{1}{x} \\ \sum_{n=1}^{\infty} N_0(nx) &= -\frac{1}{\pi} \ln \frac{\Gamma x}{4\pi} - \frac{1}{\pi} \sum_{l=1}^{\infty} \frac{1}{l} \left[\frac{1}{\sqrt{1 - \left[\frac{x}{2\pi l}\right]^2}} - 1 \right] \end{aligned} \quad (31)$$

with $0 < x < 2\pi$.

TABLE 1 (taken from ref. 15)

x	$\sum_{n=1}^{\infty} (-1)^n N_0(n\pi x)$	x	$\sum_{n=1}^{\infty} (-1)^n N_0(n\pi x)$
0.0	$+\infty$	1.6	0.25190
0.2	0.31478	1.8	0.19638
0.4	0.04728	2.0	0.13919
0.6	-0.18637	2.2	0.07657
0.8	-0.54814	2.4	0.00177
1.0(-)	$-\infty$	2.6	-0.10178
1.0(+)	0.43521	2.8	-0.29843
1.2	0.36854	3.0(-)	$-\infty$
1.4	0.30845	3.0(+)	0.26222

Eq. (31) is only a special case of the following more general relationships.

$$\begin{aligned}
 \sum_{k=1}^{\infty} J_0(kx) \cos(kxt) &= -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - (2\pi l + tx)^2}} \\
 &\quad + \frac{1}{x\sqrt{1-t^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - (2\pi l - tx)^2}} \\
 \sum_{k=1}^{\infty} N_0(kx) \cos(kxt) &= -\frac{1}{\pi} \ln \frac{\Gamma x}{4\pi} + \frac{1}{2\pi} \left[\sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right] \\
 &\quad - \sum_{l=m+1}^{\infty} \left[\frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right] - \sum_{l=n+1}^{\infty} \left[\frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right]
 \end{aligned}$$

provided that $x > 0$, $0 \leq t < 1$, $2m\pi < x(1-t) < 2(m+1)\pi$, $2n\pi < x(1+t) < 2(n+1)\pi$;

$(m+1)$ and $(n+1)$ are natural numbers and $\Gamma = 1.781072$.

The first summation in (31) (i.e. $\sum J_0$) can, of course, be easily found if x is given. The second summation is slightly complicated. One obvious simplification arises if $x = 2k(b-a)$ is very small compared to unity. Then it is clear

$$\sqrt{1 - \left[\frac{x}{2\pi l} \right]^2} - 1 \approx 0$$

so that

$$\sum_{n=1}^{\infty} N_0(nx) \approx -\frac{1}{\pi} \ln \frac{\Gamma x}{4\pi}, \quad x < 2\pi \quad (32)$$

When this is not the case, then one may use a technique used by Wessel.¹⁰

This uses the following approximation:

$$\sum_{l=1}^{\infty} \frac{1}{l} \left[\frac{1}{\sqrt{1 - \left[\frac{x}{2\pi l} \right]^2}} - 1 \right] \approx \sum_{l=1}^m \frac{1}{l} \left[\frac{1}{\sqrt{1 - \left[\frac{x}{2\pi l} \right]^2}} - 1 \right] + \ln \frac{2}{1 + \sqrt{1 - \left[\frac{x}{2\pi m} \right]^2}} \quad (33)$$

where m is any arbitrary large integer. Now it is not too difficult to calculate the value of the series. Wessel¹⁰ has actually tabulated the values of $\sum J_0$ and $\sum N_0$ over a range of values x from 0.3 to 13.5. The table is reproduced on the following page without modification: For values of x less than 0.3 the series $\sum_{n=1}^{\infty} N_0(nx)$ may be calculated from (32). However, one observes that in order to arrive at (27), it was assumed that

$$k|z| < 2k(b-a)$$

In other words, the thickness of the coating $(b-a)$ must be large compared to one fourth of the total gap width.

TABLE 2 (taken from ref. 10)

x	$\sum_{n=1}^{\infty} J_0(nx)$	$\sum_{n=1}^{\infty} N_0(nx)$	x	$\sum_{n=1}^{\infty} J_0(nx)$	$\sum_{n=1}^{\infty} N_0(nx)$	x	$\sum_{n=1}^{\infty} J_0(nx)$	$\sum_{n=1}^{\infty} N_0(nx)$
0.3	2.8333	1.0047	4.8	-0.2917	-0.0729	9.3	-0.1008	0.1222
0.6	1.1667	0.7828	5.1	-0.3039	-0.1466	9.6	-0.1202	0.0998
0.9	0.6111	0.6515	5.4	-0.3148	-0.2456	9.9	-0.1376	0.0756
1.2	0.3333	0.5568	5.7	-0.3246	-0.4004	10.2	-0.1531	0.0494
1.5	0.1667	0.4815	6.0	-0.3333	-0.7356	10.5	-0.1671	0.0201
1.8	0.0556	0.4183	6.3	4.0029	0.3164	10.8	-0.1797	-0.0132
2.1	-0.0238	0.3627	6.6	0.6416	0.2973	11.1	-0.1913	-0.0533
2.4	-0.0833	0.3124	6.9	0.3462	0.2782	11.4	-0.2021	-0.1039
2.7	-0.1296	0.2655	7.2	0.2078	0.2595	11.7	-0.2119	-0.1739
3.0	-0.1667	0.2207	7.5	0.1220	0.2407	12.0	-0.2211	-0.2851
3.3	-0.1970	0.1769	7.8	0.0609	0.2219	12.3	-0.2296	-0.5380
3.6	-0.2222	0.1331	8.1	0.0147	0.2022	12.6	1.9249	0.2272
3.9	-0.2436	0.0880	8.4	-0.0223	0.1843	12.9	0.4413	0.2152
4.2	-0.2619	0.0401	8.7	-0.0527	0.1639	13.2	0.2431	0.2030
4.5	-0.2778	-0.0125	9.0	-0.0785	0.1435	13.5	0.1470	0.1907

The driving points admittance for an uncoated antenna is obtained from (27) and (28). Thus

$$Y_{dp} = \frac{I(\delta)}{V} \approx 2\omega\epsilon_0 a \pi [J_0(k_0|\delta|) + iN_0(k_0|\delta|)]$$

If $k_0|\delta|$ is assumed to be small compared to unity, which obviously should be the case, then

$$Y_{dp} \approx 4\omega\epsilon_0 a \left[\frac{\pi}{2} + i \ln \frac{\Gamma k_0 |\delta|}{2} \right] \quad (34)$$

Thus the input admittance consists of two parts. The first part is the conductive term and it is independent of the gap size. The second term is a capacitive term and is dependent on the gap size. The result is comparable with the result obtained by Chen and Keller.⁶

3.3 The cylindrical Antenna of finite length within a dielectric cylinder.

The integral equation for the current distribution on a thin cylindrical antenna was given in Eq. (10). The constant B is expected to be determined from the condition that the current at the ends of the antenna is zero.

For an uncoated antenna, the function $F(a)$ in (10) is simply $[J_0(l_a)H_0(l_a)]$. For the coated antenna $F(a)$ becomes more complicated, as shown previously. For short, uncoated antennas, (of the order of a wave length or less), the current distribution has been found quite accurately by many authors. A good collection of many of the works in this topic appears in King's book.¹³ Wu⁷ and Raemer¹² have done some works for coated antennas.

For long antennas (long compared to wave length), a method suggested by Hallén² is very useful. He pictured a long antenna as a transmission line with

current and potential waves emanating from the source at the gap and reflected from the ends of the antenna. The two reflected waves from the two ends travel toward the source and again get reflected at the gap. Thus there is an infinite series of reflected waves. Other authors^{6, 14} have used a similar principle to solve the finite but long antenna problem. Since the coated antenna current distribution is given by an integral equation similar to that of the uncoated antenna, and the physical situation is somewhat the same, a similar approach would be useful for the coated antenna problem.

Following Hallén's² theory, one attempts to expand the constant B in an infinite series of waves:

$$B = 2 \sum_{n=1}^{\infty} (-1)^n V_n e^{i 2 n k h} \quad (35)$$

This substituted in (10) yields

$$\begin{aligned} & \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} d\alpha F(\alpha) e^{-i\alpha(z-\xi)} \\ &= -\frac{i 8 \pi k}{\omega \mu} \left[\frac{1}{2} V e^{i k |z|} + \sum_{n=1}^{\infty} (-1)^n V_n e^{i k h (2n-1)} [e^{i k (h-z)} + e^{i k (h+z)}] \right] \end{aligned} \quad (36)$$

for $-h < z < h$.

The right side of (36) is essentially the vector potential due to the current distribution in the antenna. It is apparent that the current distribution should have the same form as the vector potential, so that one may write for the current,

$$I(z) = I_0(z) + \sum_{n=1}^{\infty} (-1)^n e^{ikh(2n-1)} [I_n(h+z) + I_n(h-z)] \quad (37)$$

In (37), $I_0(z)$ is the outgoing current wave in an infinitely long antenna, the solution of which was discussed in section 3.2. I_n 's are the reflected current waves: $I_n(h+z)$ is travelling in the positive z -direction, and $I_n(h-z)$ in the negative z -direction. In the phase factor, the term $h(2n-1)$ refers to the total distance over which the wave has travelled before the last reflection. The factor $(-1)^n$ indicates that the waves travel back and forth and the direction of the current is reversed at the reflection.

It is now necessary to determine V_n and I_n . By substituting (37) into (36) and using the condition that the total current must be zero at the ends of the antenna, which requires that the successive current waves must satisfy the relations

$$\begin{aligned} I_1(0) &= e^{-ikh} I_0(h) \\ I_n(0) &= e^{-ik2h} I_{n-1}(2h), \quad n > 1 \end{aligned} \quad (38)$$

Hallén was able to formulate one common integral equation for the reflected current waves. Following his steps, one may write for the case of the coated antenna

$$\int_{-\infty}^{\infty} I_n(\zeta) d\zeta \int_{-\infty}^{\infty} F(a) e^{-ia(z-\zeta)} da = \begin{cases} -\frac{i8\pi k}{\omega\mu} V_n e^{ikz} & , z > 0 \\ m(z) & , z < 0 \end{cases} \quad (39)$$

The function $m(z)$ is unknown. There is an infinite set of integral equations corresponding to each n . $I_0(z)$ is the current in an infinitely long antenna. Hallén has shown that the different reflected current waves approach the form of the outgoing wave $I_0(z)$. They differ from $I_0(z)$ mainly at small values of z . In the limit of large h , the incident wave comes from infinity and therefore has a constant amplitude when it approaches the end. The limit value of the reflected wave $I(z)$, when the incident wave comes from infinity is

$$I(z) = I(0) e^{-ikz}, \quad z < 0 \quad (40)$$

Hallén has shown methods for calculating all the reflected waves. In the following only the limiting value of the first reflected wave will be calculated. A convenient way to calculate this reflected current wave is by using the Wiener-Hopf technique. This method has been used to find the reflected wave. The details of the method is shown in Appendix 6. It has been shown there that the first reflected current wave is given by (Eq. (22), Appendix 6)

$$I_1(z) = \frac{4k^2 V \phi_2(k)}{\omega \mu \pi} \int_{-\infty}^{\infty} \frac{\phi_1(\alpha) e^{-i\alpha z}}{l^2 F(\alpha)} d\alpha \quad (41)$$

where ϕ_1 and ϕ_2 are given by Eq. (14) in Appendix 6.

The evaluation of $I_1(z)$ from (41) is quite involved. The complication is due to the fact that even $\phi_1(\alpha)$ and $\phi_2(\alpha)$ are known only as integrals and $F(\alpha)$ is not just a simple function. Numerical techniques have to be adopted to find an answer. No attempt will be made to evaluate (41). Instead, the function $I_1(z)$ will be examined for its behavior for large $k|z|$.

An integral similar to (41) was evaluated asymptotically by the method of steepest descent in Appendix 5. From Equation (10) in Appendix 5, it is apparent that the asymptotic behavior of $I_1(z)$ in (41) is largely dependent on the value of $\phi_1(a)/F(a)$ at $a = k \sin \psi_0$. For large enough $k|z|$, it was also shown in the same appendix that $\sin \psi_0$ is approximately unity. Since $\phi_1(a)$ is analytic and non-vanishing in the entire upper half plane, for $\phi_1(a)$ at $a = k$, one writes simply $\phi_1(k)$, which is a constant. Therefore

$$I_1(z) \underset{k|z| \rightarrow \infty}{\simeq} \frac{4k^2 v \phi_2(k) \phi_1(k)}{\omega \mu \pi} \int_{-\infty}^{\infty} \frac{e^{-i a z}}{l^2 F(a)} da$$

Using (13), this reduces to

$$I_1(z) \underset{k|z| \rightarrow \infty}{\simeq} 2 \phi_1(k) \phi_2(k) I_0(z) \quad (42)$$

where $I_0(z)$ is the current in an infinitely long antenna.

Equation (42) shows that the first reflected current wave in a long antenna is proportional to the outgoing current in an infinitely long antenna.

It is thus possible to evaluate many properties of a coated antenna in a manner exactly analogous to that for an uncoated antenna. Since, once the properties of the infinitely long coated antenna are known, many of the properties of a long but finite coated antenna can be determined by a comparison to the case of an uncoated antenna (except for the added difficulties in calculations due to the more complicated $F(a)$) the discussion for the finite antenna will be concluded here. Attention will instead be focused to certain additional features and some numerical results for the infinitely long antenna.

3.4 Additional Topics on Infinitely long coated antenna

In section 3.2, the integral equation for the current distribution on an infinitely long coated antenna was formulated. An explicit expression for the current distribution function was also obtained in the form of an integral. The integral was evaluated approximately and asymptotic values for $I(z)$ for large $k|z|$ and small $k|z|$ were obtained. In all these discussions, however, the dielectric property of the material of the coating was assumed to be real or with a vanishingly small lossy part. In the following it will be shown that many of the results obtained in section 3.2 are also valid for complex dielectric coating.

3.4.1 Infinitely long antenna within a dielectric cylinder having complex dielectric property.

Since most of the derivations will be exactly similar to those in section 3.2, only some salient points will be discussed here. Whenever necessary, pertinent equations in section 3.2 will be referred to avoid duplication. It must be noted, however, that the conductivity of the dielectric cylinder will be assumed to be small compared to conductivity of the antenna so that the conduction current is predominantly flowing along the antenna. Here ϵ is complex, so that

$$\epsilon = \epsilon' + i \frac{\sigma}{\omega} \quad (43)$$

where σ is the conductivity, ω is the radian frequency and ϵ' is the real part of ϵ .

Therefore

$$k = k_1 + ik_2$$

where

$$\begin{aligned}
 k_1 &= \sqrt{\omega^2 \mu \epsilon'} \sqrt{\frac{1}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega \epsilon'} \right]^2} + 1 \right]} \\
 k_2 &= \sqrt{\omega^2 \mu \epsilon'} \sqrt{\frac{1}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega \epsilon'} \right]^2} - 1 \right]}
 \end{aligned} \tag{44}$$

Without difficulty, it may be shown that the current distribution on the antenna will be given by the integral equation (11) and the explicit solution for the current by (13), with k and ϵ as defined in (43) and (44). The asymptotic values of the current distribution function will, as in the previous case, be given by (17) for large $k|z|$ and by (25) for small $k|z|$. The driving point admittance will be given by (28). In each of these cases, k and ϵ will have to be interpreted in the sense of (43) and (44).

In this connection, it should be mentioned that relationships ³⁰(30) to (33) are not valid for complex k . Therefore it will be much more difficult to obtain the numerical values for the driving point admittance. The difficulty is with the evaluation of the two infinite series $\sum_{n=1}^{\infty} J_0(nx)$ and $\sum_{n=1}^{\infty} N_0(nx)$ for complex x . At the present, no information about these two series appears to be available in the literature. Numerical techniques may, however, be used to evaluate them.

3.4.2 Electromagnetic Field and Radiated Power from an infinitely long coated antenna.

Without explicitly evaluating the current distribution along the antenna, it is possible to derive expressions for the electromagnetic fields and the

radiated power. For this purpose, one may start from (5) with $b(a) = u(a)$ and, applying the boundary condition of continuity of tangential electric and magnetic fields on the surface at $r = b$, obtain the following expression for $c(a)$.

$$c(a) = \frac{\omega \mu \epsilon V}{\pi^2 l_0^2 N^2} \frac{1}{\{H_0(l_0 b) [J_0'(lb) H_0(la) - J_0(la) H_0'(lb)] + \frac{K_m}{N^2} \frac{l}{l_0} [J_0(la) H_0(lb) - J_0(lb) H_0(la)] H_0'(l_0 b)\}} \quad (45)$$

where the value of $u(a)$ as given by (4) and (13) has been used.

The magnetic field is then (outside the cylinder)

$$H_\phi^{(2)} = \frac{1}{\mu_0} \int_{-\infty}^{\infty} l_0 c(a) H_1(l_0 r) e^{-iaz} da \quad (46)$$

where $c(a)$ is given by (45).

E_r and E_z may now be found in a straightforward manner using relationships (30) in chapter 2. The radiated power may also be found following exactly the same steps as in Chapter 2. The asymptotic behavior of $H_\phi^{(2)}$ in (46) may easily be found by the method of steepest descent, described in Appendix 3. It is not intended to go into details about these aspects of the problem and the discussion will be concluded here. Attention will now be focused to a more useful measurable parameter, namely the driving point admittance.

3.4.3 Driving point admittance, Numerical results.

The expressions for the driving point admittances for an uncoated and a coated antenna were given in (34) and (28) respectively. For the purpose of

plotting the driving point admittance, it is convenient to normalize the function and separate into real and imaginary parts in the following manner. Thus

$$G_u = \text{Re} \frac{Y_{dpu}}{2 \omega \epsilon_0 a \pi} = J_0(k_0 |\delta|)$$

$$B_u = \text{Im} \frac{Y_{dpu}}{2 \omega \epsilon_0 a \pi} = N_0(k_0 |\delta|)$$

$$G_c = \text{Re} \frac{Y_{dpc}}{2 \omega \epsilon_0 a \pi} = \epsilon_r \left[J_0(k_0 |\delta| \sqrt{\epsilon_r}) + \sum_{n=1}^{\infty} (\pm 1)^n J_0[2n \sqrt{\epsilon_r} k_0 (b-a)] \right]$$

$$B_c = \text{Im} \frac{Y_{dpc}}{2 \omega \epsilon_0 a \pi} = \epsilon_r \left[N_0(k_0 |\delta| \sqrt{\epsilon_r}) + \sum_{n=1}^{\infty} (\pm 1)^n N_0[2n \sqrt{\epsilon_r} k_0 (b-a)] \right]$$

The subscripts u and c are used to indicate uncoated and coated cases respectively. The variables are δ , $(b-a)$, and ϵ_r . The results of variation of these quantities on the value of G and B are plotted on Figures 2-4.

In figure 2, first keeping the thickness of the coating constant, G_u and G_c were plotted by varying the gap size for different values of the dielectric constant. Then the gap size was kept constant and the thickness of the coating varied. G_c was then plotted for various values of ϵ_r . In this figure, ϵ_r has been assumed to be small compared to unity. It is apparent from figure 2 that G_u and G_c are both independent of the gap size, as long as the gap size is small, and that G_c varies with x (the thickness of the coating) as well as ϵ_r . The curves reveal that the conductance increases directly with ϵ_r and inversely with the thickness of the coating. This is exactly what one would expect from elementary considerations. It is observed that by varying ϵ_r and $k_0 x$, it is possible to obtain conductance less than, equal to, or greater than the uncoated antenna value.

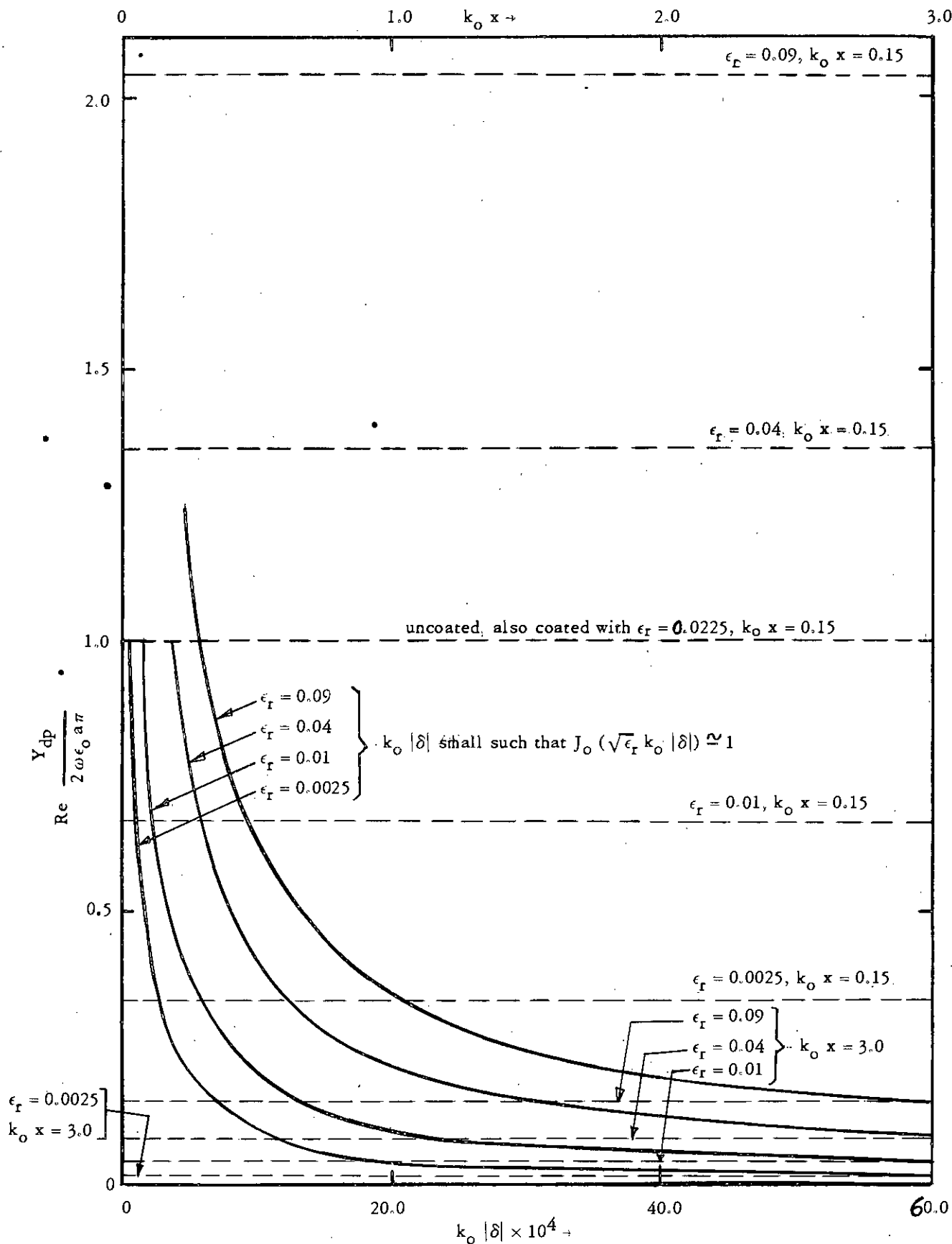


Figure 2 - Driving Point Conductance vs Gap Size (dashed line) and Thickness of Dielectric Coating (solid line)

In figure 3, B_u and B_c are plotted for various values of ϵ_r ($\epsilon_r \ll 1$), first keeping the thickness of the coating constant and varying the gap size, and then, keeping the gap size constant and varying the coating thickness. In both cases, it is seen that the susceptance increases (negatively) directly with the dielectric constant and inversely with the gap size. These results are exactly what one would have expected. Intuitively, the susceptance of an antenna should be comparable to the case of a simple parallel plate capacitor. In such a case, the susceptance (which is directly proportional to the capacitance for a fixed frequency) increases directly with the magnitude of ϵ_r and the area of the plates, but inversely with the distance between the plates.

In figure 4, similar information to that in figure 2 is shown, however, with $\epsilon_r \gg 1$. For this case also the above comments made for figure 2 apply. The driving point conductance for this case (i. e. for $\epsilon_r \gg 1$) will be given simply by $G_c = \frac{1}{2} \epsilon_r$. This means that for large ϵ_r , G_c is directly proportional to ϵ_r and independent of the coating thickness provided that $\sqrt{\epsilon_r} k_0 |\delta| \ll 1$ and $0 < \sqrt{\epsilon_r} k_0 x < \pi$. The plot being just a straight line has not been shown.

From the formulas given in (27), (28) and (34), it is possible to plot many more results without any difficulty. The formulas being in such simple forms, it is considered unnecessary to go into any more details. Figures 2-4 reveal many of the salient features of a dielectric coated antenna. Since plotting more of the same properties would not change the general picture already presented, the discussions will be concluded here.

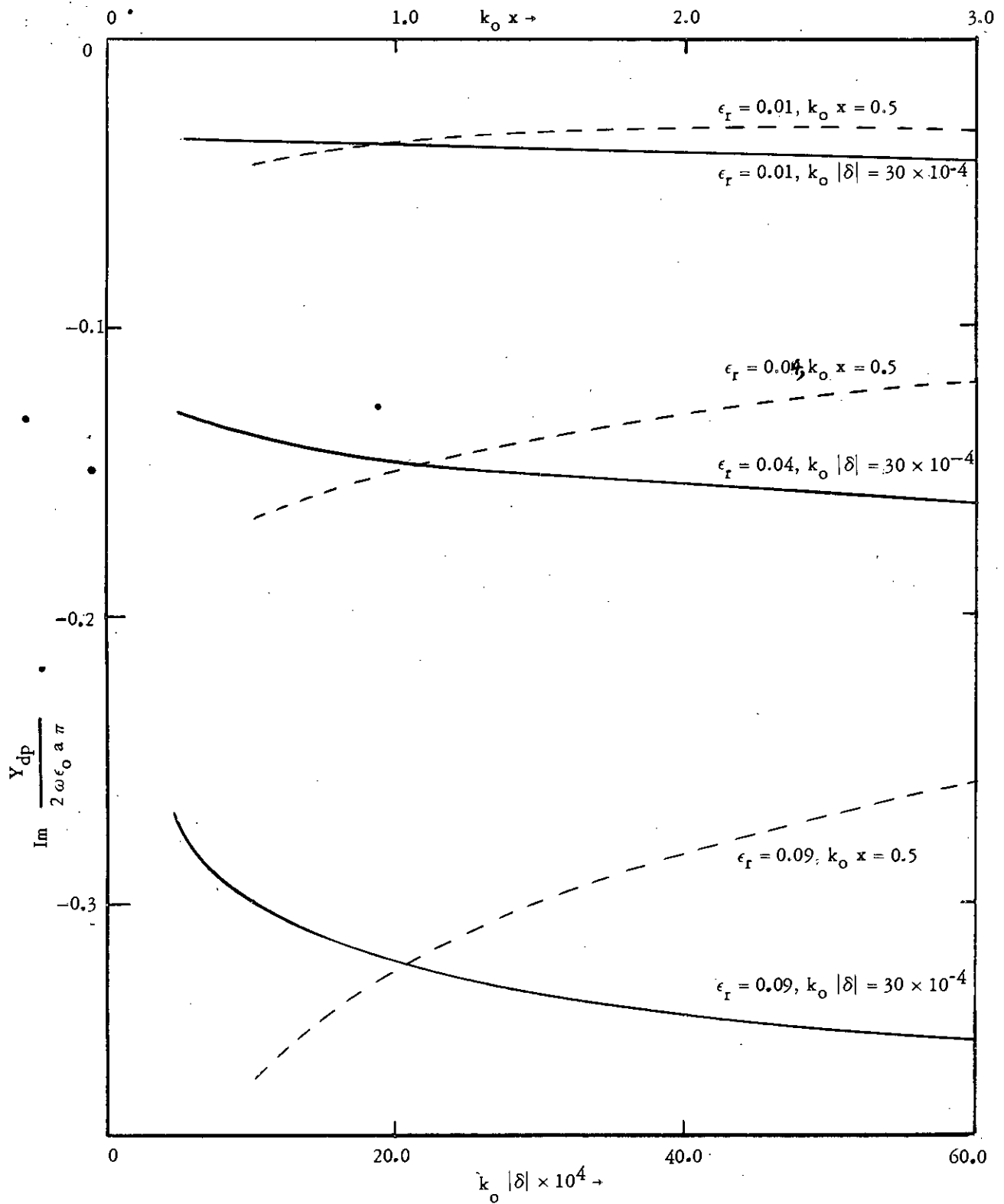


Figure 3 - Driving Point Susceptance vs Gap Size (dashed line)
and Thickness of Dielectric Coating (solid line)

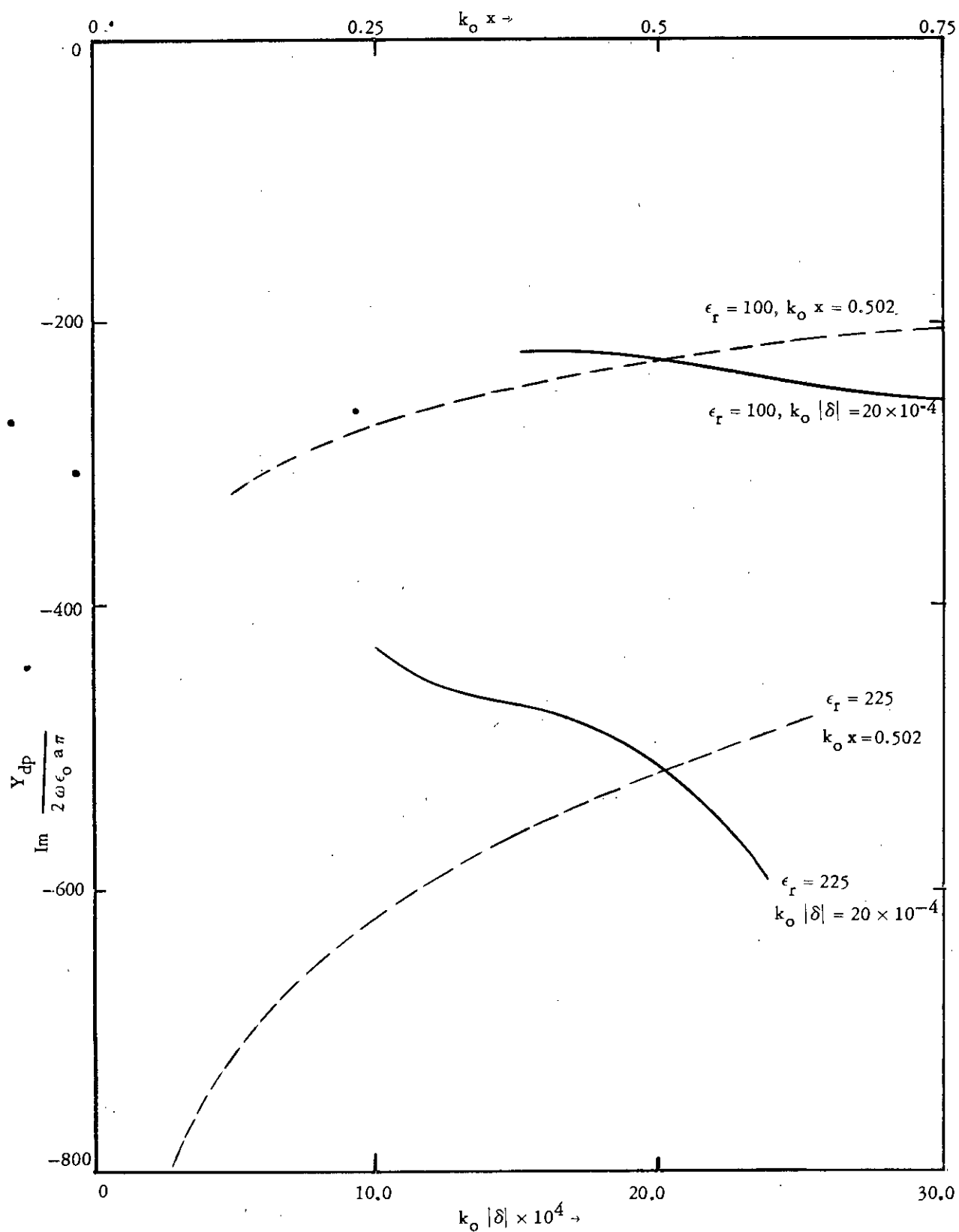


Figure 4 - Driving Point Susceptance vs Gap Size (dashed line)
and Thickness of Dielectric Coating (solid line)

APPENDIX 1

THE PROBLEM OF PLASMA DIAGNOSTICS AND THE PROBE THEORY

In a gaseous plasma, some of the most important characteristics are the temperature and density. Since a plasma consists of electrons, ions and neutral gas particles, the terms temperature and density have to be associated with each species of particles (namely ions, electrons etc.). In many applications such as in the field of thermo-nuclear fusion, it is important to know these properties accurately.

Of the various methods for the measurements of temperature and density in a plasma, the important ones are:

- (1) the probe techniques
- (2) microwave diagnostic techniques
- (3) the spectroscopic techniques.

Each method has advantages and disadvantages and is suitable under different circumstances. A survey of the various methods, along with their advantages and disadvantages have been presented by Dickson¹. Only the probe technique will be considered here.

One important characteristic of a plasma is its tendency to adjust its constituents in such a way as to maintain an overall neutrality and forcefree condition inspite of any external forces applied to it. As a consequence when an electrode or even a neutral conducting body is introduced in a plasma medium which was originally neutral, an envelope (the so-called "plasma sheath") will be formed about the body so as to shield the plasma region from

the stresses that otherwise would be caused by such a body. This shielding is, however, not quite perfect in an actual case, and a small potential drop between the electrode and the plasma may penetrate beyond the edge of the plasma sheath.

When an absorbing body is introduced in a neutral isothermal plasma, the initial flow of electrons will exceed the flow of ions because of the relative weight and mobility of the two kinds of particles. This is due to the fact that the velocity of the particles in thermal equilibrium is inversely proportional to the square root of their masses. Therefore there will be an accumulation of net negative charges until the probe reaches a potential such that an equal number of positive and negative particles reaches this probe in any time. Thus if the electrons and ions have energy corresponding to +1 electron volt, the probe potential cannot exceed -1 volt because if it does, the probe will continue to receive ions and no electrons, which would restore the probe to its original potential. The temperature of the electrons usually is higher than that of the ions and therefore the potential of air insulated probe can never exceed the potential of the plasma.

From 1923 to 1929, Langmuir, Mott-Smith, Blodgett, and Tonks developed the probe theory.⁷⁻¹⁰ Just as in a diode, Langmuir viewed the discharge to be separated from any collecting surface (probe surface) by a space charge sheath. He showed that the thickness of this sheath is, in general, a small multiple of the so-called Debye length. In 1929, Langmuir and Tonks introduced the word "plasma" to distinguish it from the "sheath". In a plasma, the two kinds of charged particles are equal in number. On the other hand, in a sheath, the densities of the two kinds of particles are such that one of them can be neglected at least in the first order. The other type of particles gives rise to a space charge, which produces a large electric field, which in turn drives a current large

compared to the local random current. In plasma, both kinds of particles being of the same number density, space charge effects may be neglected. However, the currents (small compared to the random currents which flow in the plasma), require electric fields to drive them and the divergence of the field determines a small difference in densities of the two kinds of particles. Allis³ summarizes the distinguishing characteristics of the two kinds in the following manner.

$$\text{plasmas: } n_+ - n_- < n_-; J \ll J_{\text{random}}; n_+ - n_- = \frac{\epsilon_0}{e} \nabla \cdot \vec{E}$$

$$\text{sheaths: } n_+ > 2n_-; J > J_{\text{random}}; E = \frac{e}{\epsilon_0} \int (n_+ - n_-) dn$$

In sheaths, the last equation is essential. In the above discussion, the probe surface was assumed to be perfectly absorbing. This means that all the electrons and ions striking the probe surface are absorbed and no secondary emission takes place.⁴ Although the major contribution to the probe current is due to the ion current, there exist many other factors to influence the probe current. Some of these are as follows:⁵

- (1) Ion densities and velocities near the probe
- (2) Electric field set up by the probe
- (3) Collisions between electrons, ions, and gas molecules
- (4) Ultraviolet rays and metastable atoms striking the probe
- (5) Plasma oscillations
- (6) Effect of removal of ions by probe from surrounding plasma

The usual assumptions made in probe theory are:

- (1) Probe small compared to mean free path, which is defined as follows:

If the molecules are regarded as rigid bodies, in absence of any external field, the molecules are quite free through a certain distance between two

successive impacts. The free distance is called the free path and the average free path of a system of molecules is the "mean free path."

- (2) Sheath small compared with probe radius
- (3) Ionization in sheath and secondary emission both negligible
- (4) Plasma oscillations absent

With the above assumptions, the probe current is equal to the sum of the random thermally diffused ion and electron currents. When a probe is introduced into a plasma medium, it collects positive ions at a rate determined by the random thermal currents in the plasma and by the electric fields that guide these currents towards or away from the probe. Since the thermal currents and the electric field affect each other through the space charge effect, the problem is complicated. It may be shown that under actual conditions the positive ion collection is determined not by the temperature of the positive ions but mainly by the electron temperature.

1.1 Fundamentals of Kinetic Theory of Gases

1.1.1 Concept of Velocity Distribution

A simple gas, whose molecules are spherical and possess only energy of translation and are subject to no external forces, will be considered. The molecules of the gas will be moving at random and as such there will be collisions and encounters resulting in the changes of velocity in both magnitude and direction of the colliding molecules. If the collisions are assumed to be elastic, then it is conceivable that some molecules after the encounter will be left with zero velocity and others will have a relatively high velocity. Thus physically it is apparent that there will be a whole range of velocities from zero to some high value, when a whole system of particles are considered.

The function which represents the distribution of velocities in a system of molecules is termed the velocity distribution function. Maxwell and Boltzmann derived independently by different methods the nature of the distribution function and it is often termed as the Maxwell-Boltzmann distribution function. Maxwell's method is relatively simple but mathematically less rigorous. Boltzmann showed the same result by his more elegant H-theorem. Lorentz later improved the argument to the same problem. For simplicity Maxwell's method will be considered here.

Maxwell assumed that as the components of the velocity u_x , u_y and u_z are perpendicular to each other, the distribution of one of these components among the gas molecules will be independent of the other components, and they will also be spherically symmetric. Thus

$$\overline{u_x^2} = \overline{u_y^2} = \overline{u_z^2} = \frac{1}{3} \overline{u^2}$$

$$\overline{u_x u_y} = \overline{u_y u_z} = \overline{u_x u_z} = 0$$

The bar on u_x , u_y etc. represents the average value*. If one assumes that $F(u_x) du_x$ is the probability that a molecule should possess an x-component of velocity between u_x and $u_x + du_x$ and that $F(u_x)$ is independent of u_y and u_z , then the probabilities that its y and z components should have values between u_y and $u_y + du_y$, u_z and $u_z + du_z$ are similarly $F(u_y) du_y$ and $F(u_z) du_z$ respectively. Hence if $f(u_x, u_y, u_z) du_x du_y du_z$ represents the number of molecules per unit volume then $f(u_x, u_y, u_z) du_x du_y du_z = n F(u_x) du_x F(u_y) du_y F(u_z) du_z$ where f is the distribution function.

Since in a gas which is in equilibrium, there is no preferred direction, then $f(u_x, u_y, u_z)$ can depend on u_x, u_y, u_z only through the invariant $u_x^2 + u_y^2 + u_z^2 = u^2$. Thus

*The derivation that follows is based on reference 6.

$$n F(u_x) du_x \cdot F(u_y) du_y \cdot F(u_z) du_z = f(u_x, u_y, u_z) du_x du_y du_z = \phi(u_x^2 + u_y^2 + u_z^2) du_x du_y du_z$$

Therefore

$$n F(u_x) F(u_y) F(u_z) = f(u_x, u_y, u_z) = \phi(u^2) \quad (1)$$

Differentiating (1) with respect to u_x , one obtains

$$n \frac{dF(u_x)}{du_x} F(u_y) F(u_z) = 2u_x \frac{d\phi}{du^2}$$

$$\text{But } n F(u_y) F(u_z) = \phi / F(u_x)$$

Then

$$\frac{1}{F(u_x)} \cdot \frac{dF(u_x)}{du_x} = \frac{2u_x}{\phi} \frac{d\phi}{du^2}$$

or

$$\frac{1}{\phi} \frac{d\phi}{du^2} = \frac{1}{2u_x F(u_x)} \frac{dF(u_x)}{du_x}$$

Similarly

$$\frac{1}{\phi} \frac{d\phi}{du^2} = \frac{1}{2u_y F(u_y)} \frac{dF(u_y)}{du_y} = \frac{1}{2u_z F(u_z)} \frac{dF(u_z)}{du_z}$$

Then

$$\frac{1}{2u_x F(u_x)} \frac{dF(u_x)}{du_x} = \frac{1}{2u_y F(u_y)} \frac{dF(u_y)}{du_y} = \frac{1}{2u_z F(u_z)} \frac{dF(u_z)}{du_z}$$

$$\left| \frac{1}{\phi} \frac{d\phi}{du^2} = -a, \text{ where } a \text{ is a constant} \right|$$

Therefore

$$\frac{dF(u_x)}{du_x} = -2a u_x F(u_x)$$

Integration of this yields

$$\ln F(u_x) = -a u_x^2 + \ln C$$

so that

$$F(u_x) = C_x \exp(-a u_x^2)$$

Similarly

$$F(u_y) = C_y \exp(-au_y^2); \quad F(u_z) = C_z \exp(-au_z^2)$$

so that

$$f(u) = C \exp[-a(u_x^2 + u_y^2 + u_z^2)] = C \exp(-au^2) \quad (2)$$

The problem is now to determine the constant C and a under particular conditions.

It may be noted here that the assumption that the distribution of the three velocity components among the molecules is independent of the values of the others is not completely true, and thus the distribution function is not always of the form $f = C \exp(-au^2)$. However, for many practical purposes, this representation gives reasonably satisfactory results and Boltzmann and Lorentz showed the above to be approximately true by more rigorous methods.

The constants C and a may be evaluated in terms of the number density n and the absolute temperature T .

By definition

$$n = \int f(u) du$$

also

$$\frac{1}{2} m \bar{u}^2 = \frac{3}{2} kT \text{ and } \bar{u}^2 = \frac{\int u^2 f(u) du}{\int f(u) du}$$

Therefore

$$\frac{1}{2} n m \bar{u}^2 = \frac{3}{2} n kT = \frac{m}{2} \frac{\int u^2 f(u) du}{\int f(u) du}$$

where

m = mass of the particle, k = Boltzmann's constant, $f(u)$ = the velocity distribution function.

But

$$n = \int f(u) du = \int_0^\infty C \exp(-au^2) 4\pi u^2 du \quad (3)$$

The quantity $4\pi u^2 du$ is a volume element in spherical coordinate system.

Therefore

$$\frac{3}{2} nkT = 4\pi \frac{m}{2} \int_0^\infty C \exp(-au^2) u^4 du \quad (4)$$

One notes

$$\int_0^\infty u^{2n} \exp(-au^2) du = \frac{(2n-1)!}{(n-1)! 2^{(2n-1)} a^n} \int_0^\infty \exp(-au^2) du$$

also

$$\begin{aligned} \int_0^\infty \exp(-au^2) du &= \frac{1}{2} \sqrt{\frac{\pi}{a}} \\ \int_0^\infty u^2 \exp(-au^2) du &= \frac{1}{4} \frac{\sqrt{\pi}}{a^{3/2}} \\ \int_0^\infty u^4 \exp(-au^2) du &= \frac{3}{8} \frac{\sqrt{\pi}}{a^{5/2}} \end{aligned} \quad (5)$$

Combining these, one obtains the value of n .

$$n = 4\pi C \int_0^\infty u^2 \exp(-au^2) du = \frac{\pi C \sqrt{\pi}}{a^{3/2}} \quad (6)$$

$$\frac{3}{2} nkT = 2\pi C m \int_0^\infty u^4 \exp(-au^2) du = \frac{3}{4} \frac{\pi C m \sqrt{\pi}}{a^{5/2}} \quad (7)$$

Therefore

$$\frac{3}{2} kT = \frac{3/2 nkT}{n} = \frac{3}{4} \frac{m}{a}$$

so that

$$a = \frac{m}{2kT} \quad (8)$$

Also from (6) and (8)

$$C = \frac{n a^{3/2}}{\pi^{3/2}} = n \left[\frac{m}{2\pi kT} \right]^{3/2} \quad (9)$$

Finally

$$f(u) = n \left[\frac{m}{2\pi kT} \right]^{3/2} \exp \left[-\frac{m}{2kT} (\vec{v} - \vec{v}_0)^2 \right] \quad (10)$$

where $\vec{v} - \vec{v}_0 = \vec{u}$, $\vec{v}$ being the total velocity, $\vec{v}_0$ the drift velocity and $\vec{u}$ the random velocity.

Eq. (10) is the usual form of Maxwell's velocity distribution function.

It has thus been shown that for a given set of the properties such as the density, mean velocity and temperature of a uniform gas, there is only one possible permanent mode of distribution of the velocities of the molecules (namely, the Maxwellian distribution function). If the actual mode happens to be different from this mode, it will tend to approach this state. It is now obvious that since the derivation of the Maxwellian velocity distribution function assumes spherical symmetry and the absence of external forces, then the velocity distribution function of the ions at the plasma sheath will not be in general Maxwellian due to the presence of a potential gradient.

1.1.2 Properties of the Maxwellian State

The number of molecules per unit volume with velocities in the range u , du is $f du$ or $f du_x du_y du_z$. Hence the number of molecules per unit volume, in the Maxwellian state, whose component velocities lie between the limits u_x and $u_x + du_x$, u_y and $u_y + du_y$, and u_z and $u_z + du_z$ may be written as (using the value of f from (10))

$$n \left[\frac{m}{2\pi kT} \right]^{3/2} \exp \left[-\frac{m}{2kT} (u_x - u_{x0})^2 \right] du_x \exp \left[-\frac{m}{2kT} (u_y - u_{y0})^2 \right] du_y \exp \left[-\frac{m}{2kT} (u_z - u_{z0})^2 \right] du_z \quad (11)$$

This means that the distribution of u_x is independent of u_y and u_z and so on.

The mean value of any function ϕ of the molecular velocity of a gas in the Maxwellian state can be found from the equation

$$n \bar{\phi} = \int \phi f du = n \left[\frac{m}{2\pi kT} \right]^{3/2} \int \phi \exp \left[-\frac{m}{2kT} u^2 \right] du \quad (12)$$

For example, the mean of the peculiar speed u will be

$$\bar{u} = \left[\frac{m}{2\pi kT} \right]^{3/2} \int u \exp \left[-\frac{m}{2kT} u^2 \right] du$$

where the integration extends through all velocity space. Then

$$\bar{u} = 4\pi \left[\frac{m}{2\pi kT} \right]^{3/2} \int_0^\infty u^3 \exp \left[-\frac{m}{2kT} u^2 \right] du \quad (13)$$

One may now use the formula

$$\int_0^\infty \exp[-au^2] u^{2n+1} du = \frac{n!}{2a^{n+1}}, \quad n = 0, 1, 2, \dots \quad (14)$$

so that

$$\int_0^\infty \exp[-au^2] u^3 du = \frac{1}{2 \left[\frac{m}{2kT} \right]^2}.$$

Combining (13), and (14), one obtains

$$\bar{u} = \left[\frac{8kT}{\pi m} \right]^{1/2} \quad (15)$$

Another mean value often needed is that of the z -component of the peculiar velocity, averaged over those molecules at a given point for which this component is positive; this is denoted by $\bar{u}_{z+}$. Since the number density of such molecules is $\frac{1}{2}n$, from (14) one obtains

$$\frac{1}{2}n\bar{u}_{z+} = \int_+ f u_z du$$

The integration on the right extends over all values of u for which u_z is positive.

Hence, using (11)

$$\bar{u}_{z+} = 2 \left[\frac{m}{2\pi kT} \right]^{3/2} \int_{-\infty}^{\infty} \exp \left[-\frac{m}{2kT} u_x^2 \right] du_x \int_{-\infty}^{\infty} \exp \left[-\frac{m}{2kT} u_y^2 \right] du_y \int_0^\infty u_z \exp \left[-\frac{m}{2kT} u_z^2 \right] du_z$$

Using (5), (8), (14), and (15) this reduces to

$$\bar{u}_{z+} = 2 \left[\frac{m}{2\pi kT} \right]^{3/2} \left[\frac{2\pi kT}{m} \right]^{1/2} \left[\frac{2\pi kT}{m} \right]^{1/2} \left[\frac{kT}{m} \right]$$

$$= \sqrt{\frac{2kT}{\pi m}} = \frac{1}{2} \bar{u} \quad (16)$$

Obviously, the number of particles crossing unit area in one second along the positive z-direction will be

$$\frac{1}{2} n \cdot \frac{1}{2} \bar{u} = \frac{n\bar{u}}{4} \quad (17)$$

Relationship (17) is important in many problems. For example, in probe theory, when a probe is at space potential, all electric fields vanish and the rate of collection of positive ions will be determined solely by the random straight-line motion of the ions in the neighborhood. From an elementary point of view, it may be assumed that the probe is struck by the same current that would strike an equal element of area in the undisturbed gas. Thus if n is the gas density, $\bar{u}$ the mean velocity, and A the probe area, the current J will be given by

(using (17))⁵

$$J = e \frac{n\bar{u}A}{4} \quad (18)$$

Obviously (18) is valid only when the probe does not cause any appreciable disturbance in the distribution, because then the distribution function will not be Maxwellian.

1.2 Distribution of Ion Concentration in a Uniform Electrostatic Field⁴

In order to understand the physical principles involved in the phenomenon, it is suggestive to consider the variation of density of a gas in thermal equilibrium and under the influence of a gravitational field. This variation arises due to the interaction of the gravitational forces and the thermal agitation of the molecules.

A column of air, 'A' square meter in cross-section, maintained at a constant temperature and free from all disturbing forces will be considered. Let x be the distance in a direction opposite to that of G .

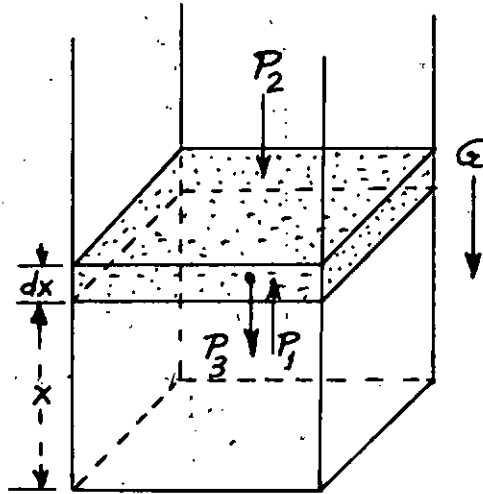


Figure 1. Section of a Gas column in equilibrium in gravitational field

In the steady state, only the upper and the lower surfaces need to be considered. The upward force P_1 due to gas pressure is $A p(x)$. The downward force P_2 on the upper surface is $A p(x+dx)$. The downward force P_3 due to gravitational force is $A \rho g dx$. Then, for equilibrium condition,

$$A p(x) - A p(x+dx) - A \rho g dx = 0$$

But $p(x+dx) = p(x) + \frac{\partial p}{\partial x} dx$. Therefore

$$\frac{\partial p}{\partial x} dx + \rho g dx = 0$$

Let n = number of molecules per cubic meter of the gas at any given temperature and pressure, m = mass of the molecules

Therefore $\rho = nm$

From the ideal gas laws

$$p = nkT$$

$$\frac{\partial p}{\partial x} = kT \frac{\partial n}{\partial x}$$

Then

$$\frac{dn}{n} = - \frac{gm}{kT} dx$$

$$n = n_0 \exp \left[- \frac{gm x}{kT} \right]$$

The quantity gm_x represents the potential energy of a molecule at the point x .

If the two surfaces are at x_1 and x_2 then

$$\int_{n_2}^{n_1} \frac{dn}{n} = - \int_{x_2}^{x_1} \frac{gm}{kT} dx$$

Therefore

$$\frac{n_1}{n_2} = \exp \left[- \frac{gm}{kT} (x_1 - x_2) \right] = \exp \left[\frac{u_{12}}{kT} \right] \quad (19)$$

The sign of u_{12} must be such that the concentration increases in the direction of the field.

Following the same principles, it is possible to show that the average steady state concentration of ions at two points in a uniform electrostatic field is given by an equation similar to (19), the potential energy term being given by the difference of the electrostatic potential multiplied by the charge q carried by each ion. The equation is then

$$\frac{n_1^+}{n_2^+} = \exp \left[\frac{-q(V_1 - V_2)}{kT} \right] \quad (20)$$

where

V_1 = potential at x_1

and

V_2 = potential at x_2 .

1.3 Effect of Space Charge

The probe current depends on the probe potential and also on the precise form of the electric field in the surrounding space. The form of the electric field may be determined by solving the problem of space charge motion in a self-created electric field. The solution is possible through the following simplifying features: (1) the inability of the plasma to sustain a large potential

drop; (2) the inability of electrons to reach the highly negative probe. Since neither the electrons nor the ions have more than a few volts of kinetic energy, the two factors stated are interdependent. Thus, if an electrode is made 20 volts negative, very few electrons can be present over most of the region of potential drop, which therefore takes the form of a positive ion sheath surrounding the electrode and shielding the plasma from most of the potential drop.

The calculation for the size of the sheath may be done first by considering an one dimensional case. One may consider two infinite parallel-plane electrodes in a vacuum, with one of the electrodes emitting electrons thermo-ionically. The emitter will be assumed to be an equipotential surface. Poisson's equation is valid in the region between the plates.

$$\frac{d^2V}{dx^2} = -\rho/\epsilon \quad (21)$$

where V is the potential at the point x , ρ is the charge density and ϵ is the dielectric constant of the medium. The force on an electron is

$$m \frac{dv}{dt} = e \frac{dV}{dx} \quad (22)$$

where m , e and v are the mass, charge and velocity of an electron, respectively.

The kinetic energy is

$$\frac{mv^2}{2} = eV \quad (23)$$

Then

$$v = \left[\frac{2eV}{m} \right]^{1/2} \quad (24)$$

This assumes that the velocity of the electron at emission is zero.

The current density is given by $J = -\rho v = -nev$ where n = number density of electrons at x . Then

$$\rho = -\frac{J}{v} = -J \left[\frac{m}{2Ve} \right]^{1/2} \quad (25)$$

Substituting this value of ρ in Poisson's equation, one obtains

$$\frac{d^2V}{dx^2} = \frac{J}{\epsilon} \left(\frac{m}{2Ve} \right)^{1/2} = \frac{J}{\epsilon} \left(\frac{m}{2e} \right)^{1/2} V^{-1/2} \quad (26)$$

Multiplying both sides of (26) by $2 \frac{dV}{dx}$

$$2 \left(\frac{dV}{dx} \right) \frac{d^2V}{dx^2} = \frac{J}{\epsilon} \left(\frac{m}{2e} \right)^{1/2} V^{-1/2} \left[2 \frac{dV}{dx} \right]$$

Then

$$\left(\frac{dV}{dx} \right)^2 = 4 \frac{J}{\epsilon} \left(\frac{m}{2e} \right)^{1/2} V^{1/2} + C \quad (27)$$

If it is assumed that the current is limited by the field or that there are more electrons being emitted than are reaching the anode. The field at the cathode must be zero, for all lines of force from the anode end on electrons rather than on the cathode. As the cathode potential is assumed to be zero, the constant C is zero. Therefore

$$\begin{aligned} \frac{dV}{dx} &= \left[4 \frac{J}{\epsilon} \left(\frac{m}{2e} \right)^{1/2} \right]^{1/2} V^{1/4} \\ V^{-1/4} dV &= \left[4 \frac{J}{\epsilon} \left(\frac{m}{2e} \right)^{1/2} \right]^{1/2} dx \\ \frac{4}{3} V^{3/4} &= \left[4 \frac{J}{\epsilon} \left(\frac{m}{2e} \right)^{1/2} \right]^{1/2} x + C_1 \end{aligned}$$

But $V = 0$ at $x = 0$, so that $C_1 = 0$

Thus,

$$x = \frac{V^{3/4}}{\left[\frac{9}{4} \frac{J}{\epsilon} \sqrt{m/2e} \right]^{1/2}} \quad (28)$$

This is the familiar space charge equation or Child-Langmuir law. It can be shown that this law holds for any electrode configuration. Relationship (28) holds for the thickness of the sheath of positive ions around a negatively charged electrode if v is the probe potential, m is the ionic mass, J is the current density. This will be valid under the assumption that the sheath contains only those positive ions which diffuse or drift into it from the interior of the plasma. The electron penetration into the sheath is negligible and it is assumed that at the plasma edge the field is zero, for if it were not, electrons would be repelled until this condition became satisfied. These assumptions are approximate. More accurate formulation will be given in section 1.4.

Further simplification of (28) may be achieved by replacing the current density (in this case, the positive ion current) with suitable values. The number of particles that cross unit area in unit time along a particular direction is $\frac{n\bar{u}}{4}$ (Eq. (17))

$$\text{But } \bar{u} = \left[\frac{8kT}{\pi m} \right]^{1/2} \quad (\text{See Eq. (15)})$$

Therefore, the positive ion current density/unit probe area will be

$$J = \frac{ne\bar{u}}{4} = ne \sqrt{\frac{kT}{2\pi m}} \quad (29)$$

This is the standard formula for the random current in a gas. Substituting this in (28), one obtains for the thickness of the sheath

$$x = \left[\frac{8\epsilon}{9n} \sqrt{\frac{\pi}{kTe}} \right]^{1/2} v^{3/4} \quad (30)$$

Equation (30) may also be written in terms of the Debye length defined as .

$$\lambda_d = \sqrt{\frac{\epsilon k T}{n e^2}}$$

Then

$$x = \frac{\pi \lambda_d}{3} \left(\frac{4 e V}{\pi k T} \right)^{3/4} \quad (31)$$

It is apparent from (31) that if the sheath voltage v is approximately equal to $\frac{kT}{e}$, then the sheath thickness is nearly equal to the Debye length.

1.4 Plasma Sheath Equation and Bohm's Criteria for a Stable Sheath⁵

It was mentioned previously that if a probe is introduced into a neutral plasma, a plasma sheath will be formed around the probe to shield the plasma from the probe potential. However, if the electrons have several volts of kinetic energy, they can penetrate several volts of potential drop and thus permit correspondingly large residual potential to penetrate into the plasma. Although these potentials are not too great, they may be large enough to draw in ions much more rapidly than would be possible by the ionic thermal velocities alone. Thus, the assumption of zero voltage at the sheath edge is approximate and is true only to the extent that electron energies are small compared with the total potential drop across the sheath.

Since electrons are not absorbed by the probe, they remain in equilibrium in its neighborhood and their density is therefore given by the Maxwell-Boltzmann relationship (See Equation (20))

$$n_e = n_0 \exp \left[-\frac{e V}{k T_e} \right] \quad (32)$$

where n_0 is the electron density in the neutral plasma,

V = potential of the probe

T_e = Electron temperature

If n_e is the electron density and n_i the ion density, then Poisson's equation becomes

$$\nabla^2 V = -\frac{e}{\epsilon} (n_e - n_i)$$

or

$$\nabla^2 V + \frac{e}{\epsilon} n_0 \exp \left[-\frac{eV}{kT_e} \right] = \frac{en_i}{\epsilon} \quad (33)$$

Before (33) can be solved, one needs another relationship between n_i and V .

Langmuir and Tonks have investigated the situation under a wide variety of assumptions for different conditions and showed that in all cases, the regions may be divided into two parts; a plasma region and a sheath. In the plasma region, $n_i \simeq n_e$ (almost neutral plasma)

Here

$$\nabla^2 V = 0$$

The field is weak in this region. In the sheath region, V is large. Therefore $n_e = n_0 \exp \left[-\frac{eV}{kT_e} \right]$ is small. Only positive ions are present in an appreciable amount. According to Bohm,⁵ et al., the boundary between the two regions may be set at a point where $\frac{eV}{kT_e} = \frac{1}{2}$. Beyond this, the electron density becomes rapidly negligible (the derivation of this will be shown later.) This happens because only a few electrons have enough kinetic energy to reach higher potentials. Thus, contrary to the assumption of Langmuir and Tonks and others, the voltage at the sheath edge is not zero but should be given by $V = \frac{kT_e}{2e}$. Until the sheath forms, the plasma will be almost neutral i.e., $n_e \simeq n_i$.

In the sequel, it will be shown that the potential penetration must be such as to accelerate ions to a velocity corresponding to half the mean electron kinetic energy. This criterion is not exact, however, because it depends somewhat on the distribution of ionic velocities at the sheath edge. When a neutral atom is ionized, secondary electrons are liberated with many volts of kinetic energy and ions with practically none. Because of their rapid motion, electrons collect on the walls, charging them negatively. This negative charge repels the electrons and attracts the ions, which fall to the wall as a result of the electric fields in which they happen to find themselves when they are formed. Meanwhile, by frequent collision, the electrons attain an approximately Maxwellian distribution with a temperature of 2 to 4 volts. Within the plasma region a very gradual change in potential takes place. There is no precise point at which the sheath begins. There is a transition region between the plasma and the sheath regions. The plasma region is characterized by a state of zero electric field and the sheath region by negligible electron density.

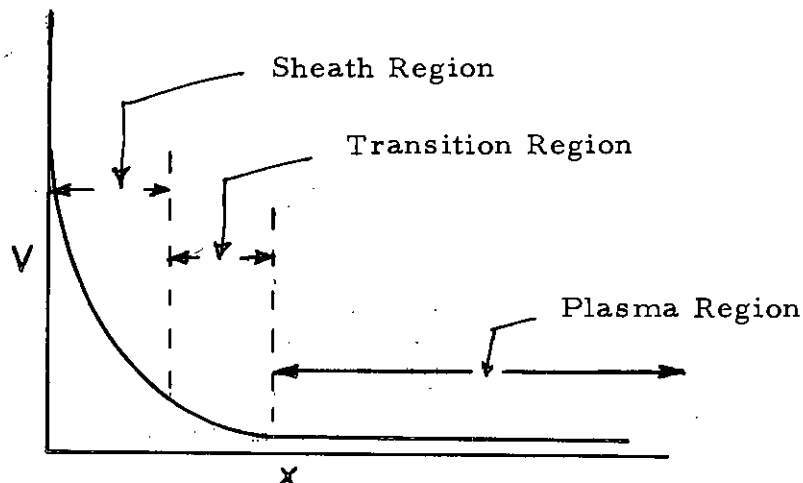


Figure 2. Potential Distribution about a probe (exact model)

Although the problem of potential penetration into the plasma can be treated exactly, such a treatment is always very complicated. To a first

approximation, it may be assumed that the plasma potential is constant, at least in so far as the processes involved in sheath formation are concerned. However, the plasma fields cannot be completely neglected, because over the long distances that they cover, they are able to accelerate positive ions up to appreciable energies, of the order of the plasma electron temperature. In the approximation that is used here, the role of these fields may be taken into account by assuming that they provide ions at the sheath edge with some mean energy eV_0 . Although there may actually be a distribution of energies, it can be shown that for quantitative purposes the distribution may be replaced by a stream of ions, all of which possess the mean energy of the distribution. There remains only a small uncertainty in V_0 arising from the lack of an exact definition of the sheath edge. This uncertainty is, however, quite unimportant, because of the smallness of the plasma fields. A simplified model of the situation is, thus, as shown in figure 3.

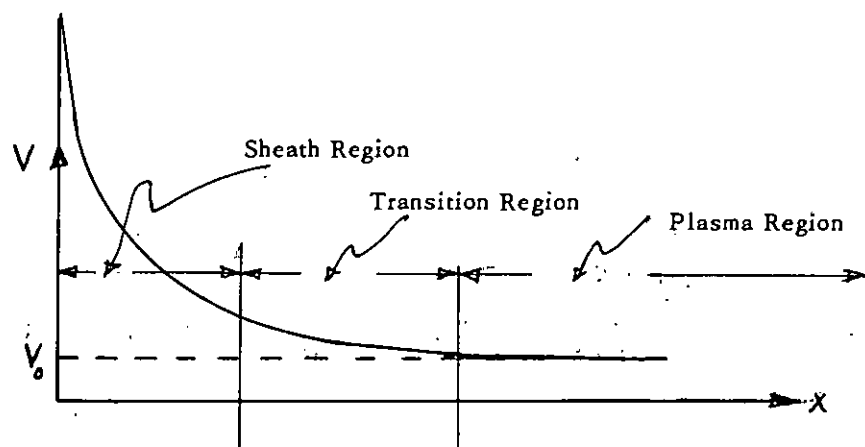


Figure 3. - Potential Distribution about a probe (approximate model)

To a first approximation, the change in plasma potential beyond the sheath edge may be neglected, assuming that far away from the electrode the potential approaches V_0 and that the electric field approaches zero.

For this simple case then, one may write Poisson's equation

$$\nabla^2 V = \frac{e}{\epsilon} (n_i - n_e)$$

For the thermal equilibrium condition, one may write the approximate equation

$$n_e = n_0 \exp \left[- \frac{e(V - V_0)}{kT_e} \right] \quad (34)$$

At the sheath edge the potential is $V = V_0$ and the electron density is n_0 . The ion current density

$$J_i = n_i v_i e$$

Since the kinetic energy of ions is eV ,

$$\frac{1}{2} m_i v_i^2 = eV$$

Therefore

$$J_i = n_i \sqrt{\frac{2eV}{m_i}} e = n_0 \sqrt{\frac{2eV_0}{m_i}} e \quad (35)$$

This is because the current density must be the same at all points (inside the sheath as well as at the sheath edge).

Therefore from (35)

$$n_i = n_0 \sqrt{\frac{V_0}{V}} \quad (36)$$

From (36), it is seen that the ionic density n_i drops as the ions are accelerated by increasing V . For the one dimensional case, Poisson's equation becomes (using (34) and (36))

$$\frac{\partial^2 V}{\partial x^2} = \frac{e}{\epsilon} (n_i - n_e) = \frac{en_0}{\epsilon} \left\{ \sqrt{\frac{V_0}{V}} - \exp \left[- \frac{e(V - V_0)}{kT_e} \right] \right\} \quad (37)$$

This is the so-called plasma-sheath equation.

1.4.1 Potential Distribution at the Sheath (Plane Electrode).

The potential distribution in the simple case of a plane probe (one dimensional case) is obtained from a solution of (37). For this purpose, multiply both sides of (37) by $2 \frac{\partial V}{\partial x}$.

$$2 \frac{\partial V}{\partial x} \frac{\partial^2 V}{\partial x^2} = \frac{en_0}{\epsilon} \left\{ \sqrt{\frac{V_0}{V}} - \exp \left[-\frac{e(V - V_0)}{kT_e} \right] \right\} 2 \frac{\partial V}{\partial x} \quad (38)$$

Eq. (38) may be reduced to

$$\frac{\partial}{\partial x} \left(\frac{\partial V}{\partial x} \right)^2 = \frac{\partial}{\partial x} \left\{ \frac{2en_0}{\epsilon} \left[2\sqrt{VV_0} + \frac{kT_e}{e} \exp \left[-\frac{e(V - V_0)}{kT_e} \right] \right] \right\} \quad (39)$$

That (39) is true can be easily seen if the differentiation is carried out explicitly. Now integrating (39), one obtains

$$\left(\frac{\partial V}{\partial x} \right)^2 = \frac{2en_0}{\epsilon} \left\{ 2\sqrt{VV_0} + \frac{kT_e}{e} \exp \left[-\frac{e(V - V_0)}{kT_e} \right] \right\} + C \quad (40)$$

where C is a constant of integration.

In order to evaluate the constant of integration, Bohm used the approximate boundary condition that at the sheath edge, the field disappears and the potential approaches V_0 . Although there is no precise point at which the sheath begins, it is shown that in the plasma near the sheath edge, the fields are so small that the potential remains close to V_0 over a distance several sheath thickness in extent. Thus, by setting $\frac{\partial V}{\partial x} = 0$ at $V = V_0$, one obtains from (40)

$$C = -\frac{2en_0}{\epsilon} \left[2V_0 + \frac{kT_e}{e} \right] \quad (41)$$

Substitute this in (40) to obtain

$$\left(\frac{\partial V}{\partial x}\right)^2 = \frac{2en_0}{\epsilon} \left\{ 2V_0 \left(\sqrt{\frac{V}{V_0}} - 1 \right) + \frac{kT_e}{e} \exp \left[-\frac{e(V - V_0)}{kT_e} \right] - \frac{kT_e}{e} \right\} \quad (42)$$

It is obvious that

$$\left(\frac{\partial V}{\partial x}\right)^2 \geq 0$$

This means that the right side of (42) must be positive for stable conditions.

For large V , this condition is obviously satisfied. In order to study the behavior when V is close to V_0 , one may expand the right side of (42) as a power series of $(V - V_0)$. To do this, it is noted that

$$V^{1/2} = (V_0 + V - V_0)^{1/2} = \sqrt{V_0} + \frac{1}{2} \frac{V - V_0}{\sqrt{V_0}} - \frac{1}{8} \frac{(V - V_0)^2}{(V_0)^{3/2}} + \dots$$

Also expanding the exponential function in a power-series, the right side of (42)

becomes equal to

$$\frac{2en_0}{\epsilon} \left[-\frac{1}{4} \frac{(V - V_0)^2}{V_0} + \frac{e}{2kT_e} (V - V_0)^2 - \dots \right]$$

Then equation (42) becomes

$$\left(\frac{\partial V}{\partial x}\right)^2 \simeq \frac{en_0}{\epsilon} \left[\frac{e}{kT_e} - \frac{1}{2V_0} \right] (V - V_0)^2 \quad (43)$$

Since $\left(\frac{\partial V}{\partial x}\right)^2$, $(V - V_0)^2$, e , n_0 and ϵ are all positive, Eq. (43) will be satisfied only if

$$V_0 \geq \frac{kT_e}{2e} \quad (44)$$

Equation (44) is Bohm's criteria for a stable sheath formation. This condition must be satisfied for the sheath to be stable. To put it in another way, "a stable sheath is possible only when ions reach the sheath with a kinetic energy at least half the electron temperature."

Referring now to equation (42), it may be observed that it will be very difficult to integrate this explicitly. Bohm used numerical techniques to evaluate this to find V . However, when only approximate result is sufficient, it may be assumed that throughout most of the sheath, the first term of the right side predominates. With this, equation (42) becomes

$$\left(\frac{dV}{dx}\right)^2 = \frac{4en_0V_0}{\epsilon} \sqrt{\frac{V}{V_0}}$$

or

$$\pm \frac{dV}{dx} = \sqrt{\frac{4en_0V_0}{\epsilon}} \left(\frac{V}{V_0}\right)^{1/4}$$

Only the negative sign seems to have physical meaning. Thus

$$-\frac{d(V/V_0)}{dx} = \sqrt{\frac{4en_0}{\epsilon V_0}} \left(\frac{V}{V_0}\right)^{1/4}$$

Integration of above yields

$$\frac{4}{3} \left(\frac{V}{V_0}\right)^{3/4} = -\sqrt{\frac{4en_0}{\epsilon V_0}} x + C$$

The condition that $V = V_0$ at $x = x_0$ gives the value of C to be

$$C = \frac{4}{3} + \sqrt{\frac{4en_0}{\epsilon V_0}} x_0$$

so that one obtains

$$\left(\frac{V}{V_0}\right)^{3/4} - 1 = \sqrt{\frac{9}{4} \frac{en_0}{\epsilon V_0}} (x_0 - x) \quad (45)$$

According to Hall,¹¹ the unity on the left of the above may be neglected in the region for which (45) is valid. The ion current density is

$$J_i = e n_i v_i = e n_o v_i = e n_o \sqrt{\frac{2 e V_o}{m_i}} \quad (46)$$

From (45) with the unity on the left neglected, one writes

$$\left(\frac{v}{V_o}\right)^{3/4} = \sqrt{\frac{9}{4} \frac{e n_o}{e V_o}} (x_o - x)$$

Therefore,

$$v_o^{1/2} = \frac{4 e V^{3/2}}{9 e n_o (x_o - x)^2} \quad (47)$$

Substituting (47) in (46), one obtains

$$J_i = \sqrt{\frac{2 e}{m_i}} \frac{4 e}{9} \frac{V^{3/2}}{(x_o - x)^2} \quad (48)$$

Here again, one obtains the familiar Child-Langmuir equation for space-charge limited current flow. This may be compared with equation (28) obtained previously. This shows that an approximate picture of the cathode sheath may be formed by considering the plasma to be simply an ion emitter. The emitted ions with an initial velocity of $\left[\frac{2 e V_o}{m_i}\right]^{1/2}$, move into the wall under the influence of a space-charge field in a manner similar to the behavior of electrons emitted from a hot cathode and moving into the plate of an electron tube, for which the Child-Langmuir equation was originally formulated.

1.4.2 Potential Distribution About a Cylindrical Probe¹³

As in the case of the plane probe, here also one starts with Poisson's equation

$$\nabla^2 V = \frac{e (n_i - n_e)}{\epsilon} \quad (49)$$

Since the electrons are approximately in thermal equilibrium,

$$n_e \approx n_o \exp \left[- \frac{e(V - V_o)}{k T_e} \right] \quad (50)$$

The value of n_i may be found from the current density.

$$J_i = n_i v_i e, \quad \frac{1}{2} m_i v_i^2 = eV$$

Therefore

$$v_i = \sqrt{\frac{2eV}{m_i}}$$

In above V is the potential at any radius r ,

v_i = velocity of ions where the potential is V

V_o = potential of sheath edge

n_o = electron density of sheath edge

From continuity equation

$$2\pi r n_i v_i e = 2\pi r_o n_o v_o e$$

one obtains

$$n_i = \frac{n_o r_o}{r} \sqrt{\frac{V_o}{V}} \quad (51)$$

Substitution of (51) into (49) yields

$$\nabla^2 V = \frac{e}{\epsilon} \left\{ \frac{n_o r_o}{r} \sqrt{\frac{V_o}{V}} - n_o \exp \left[- \frac{e(V - V_o)}{k T_e} \right] \right\} \quad (52)$$

Equation (52) is the plasma sheath equation. The exact solution of (52) will be very complicated, and only an approximate solution will be attempted.

In the sheath region V is large (not close to the sheath edge). For a solution to be valid in this region, then, the last term in (52) may be neglected,

so that

$$\nabla^2 V \approx \frac{e n_0 r_0}{\epsilon r} \sqrt{\frac{V_0}{V}}$$

In circular cylindrical coordinates, with circular symmetry, this becomes

$$r \frac{d^2 V}{dr^2} + \frac{dV}{dr} \approx \frac{e n_0 r_0}{\epsilon} \sqrt{\frac{V_0}{V}} \quad (53)$$

Solution of equations similar to (53) has been discussed by Langmuir and

Blodgett.¹⁰ Following their method, one lets

$$V = s r^{2/3} \beta^{4/3} \quad (54)$$

where s is a constant to be chosen later and β is a function of the ratio of the sheath radius to the radius at any point r . Substitution of (54) into (53) yields

$$r \frac{d^2 V}{dr^2} + \frac{dV}{dr} = \frac{4}{9} \frac{s}{\beta^{2/3} r^{1/3}} \left[3\beta r^2 \frac{d^2 \beta}{dr^2} + r^2 \left(\frac{d\beta}{dr} \right)^2 + 7\beta r \frac{d\beta}{dr} + \beta^2 \right]$$

also

$$\frac{e n_0 r_0}{\epsilon} \sqrt{\frac{V_0}{V}} = \frac{e n_0 r_0}{\epsilon} \frac{\sqrt{V_0}}{s^{1/2} \beta^{2/3} r^{1/3}}$$

Equating these two, one obtains

$$3\beta r^2 \frac{d^2 \beta}{dr^2} + r^2 \left(\frac{d\beta}{dr} \right)^2 + 7\beta r \frac{d\beta}{dr} + \beta^2 - \frac{9}{4} \frac{e n_0 r_0 \sqrt{V_0}}{\epsilon s^{3/2}} = 0$$

For convenience, one chooses

$$s = \left[\frac{9}{4} \frac{e n_0 r_0 \sqrt{V_0}}{\epsilon} \right]^{2/3} \quad (55)$$

Then

$$3\beta r^2 \frac{d^2 \beta}{dr^2} + r^2 \left(\frac{d\beta}{dr} \right)^2 + 7\beta r \frac{d\beta}{dr} + \beta^2 - 1 = 0 \quad (56)$$

Since the space charge-free potential is expressible in the form of a ratio of r/r_0 where r_0 is the sheath radius, it is convenient to make this substitution.

Defining a new variable v as $v = \frac{r}{r_0}$ Eq (56) becomes

$$3\beta v^2 \frac{d^2 \beta}{dv^2} + v^2 \left(\frac{d\beta}{dv} \right)^2 + 7\beta v \frac{d\beta}{dv} + \beta^2 - 1 = 0 \quad (57)$$

Now one makes the substitution

$$u = -\ln v \quad (58)$$

Then

$$\frac{du}{dv} = -\frac{1}{v}$$

Combining (57) and (58), the following is obtained.

$$3\beta \frac{d^2 \beta}{du^2} + \left(\frac{d\beta}{du} \right)^2 - 4\beta \frac{d\beta}{du} + \beta^2 - 1 = 0 \quad (59)$$

Equation (59) may now be solved as a series involving powers of u . Thus, let

$$\beta = a_1 u + a_2 u^2 + a_3 u^3 + \dots \quad (60)$$

Substituting (60) into (59) and equating coefficients of each power of u to zero, one obtains

$$a_1 = \pm 1, a_2 = \pm \frac{2}{5}, a_3 = \pm \frac{11}{120}, a_4 = \pm \frac{47}{3300}, \dots$$

Since β comes as β^2 in the expression for v , both $\pm$ values of the coefficient will yield the same β^2 . Therefore the solution is

$$\beta = u + \frac{2}{5} u^2 + \frac{11}{120} u^3 + \frac{47}{3300} u^4 + \dots \quad (61)$$

Values of β^2 are calculated and tabulated in reference 13 for various values of r_0/r . Combining (54) and (55), one obtains

$$v = M \left[\frac{\beta^2}{r_0/r} \right]^{2/3}$$

where M is a constant scale factor and is given by

$$M = \left[\frac{9 e n_o r_o^2 \sqrt{V_o}}{4 \epsilon} \right]^{2/3}$$

Therefore

$$\frac{V}{M} = [\beta^2 / (r_o / r)]^{2/3} \quad (62)$$

Knowing r_o / r , V may then be calculated.

1.4.3 Floating Probe in a Plasma

To derive a relationship between the voltage and current in a probe, one cannot use the Child-Langmuir formula, since it contains an unknown term, the sheath thickness x_o , the measurement of which is not always easily made. In such a case, one usually assumes that all of the attracted particles which start towards the probe reach it. In the absence of a retarding field the flux of ions and electrons to the surface of a probe is $\frac{n_i v_i}{4}$ and $\frac{n_e v_e}{4}$ respectively, where n is the number density and v is the mean thermal velocity. Since normally $n_i \approx n_e$ and $v_i \ll v_e$, the probe quickly acquires a negative charge and an electric field appears which equalizes the rates of ion and electron collection. The potential difference between the probe and the plasma V_f is mainly dropped across a region in front of the probe called the sheath which is approximately one Debye length thick. If one makes the simplified assumption that the shape and size of the probe is unimportant and that the ion current is unaffected by the potential difference between the probe and the plasma (neither of these assumptions is generally true) then one can equate the numbers of ions and electrons. The positive ions will be attracted and the current density will be

$$J_i = \frac{n_i v_i e}{4}$$

$$\text{But } v_i = \left[\frac{8kT_i}{\pi m_i} \right]^{1/2} = \text{mean thermal velocity}$$

Therefore

$$J_i = n_i e \sqrt{\frac{kT_i}{2\pi m_i}}$$

Only those repelled electrons whose energy is sufficient to get them up the potential hill will be collected, so that at the sheath edge

$$J_e = \frac{e n_e v_e}{4} = \frac{n_{eo} e}{4} \sqrt{\frac{8kT_e}{\pi m_e}} \exp \left[- \frac{e V_f}{kT_e} \right]$$

where

V_f = floating potential (true electrostatic potential of the probe with respect to the plasma at infinity)

T_e = electron temperature

n_{eo} = number density of electrons at the sheath edge

v_e = mean thermal velocity of electrons

J_e = electron current

Equating the two currents (with $n_{eo} \simeq n_i$), one obtains

$$\sqrt{\frac{kT_i}{2\pi m_i}} = \frac{1}{4} \sqrt{\frac{8kT_e}{\pi m_e}} \exp \left[- \frac{e V_f}{kT_e} \right]$$

Taking logarithm of both sides and rearranging

$$V_f = - \frac{kT_e}{2e} \ln \frac{m_e T_i}{m_i T_e} \quad (63)$$

V_f is generally negative with respect to the plasma potential.

1.5. Review of the theory of Electrostatic Probes¹

Dickson has presented a review of the theory of electrostatic probe. The following is a brief outline of the theory based on his report. In figures 4 and 5, typical characteristics of an electrostatic probe are shown.

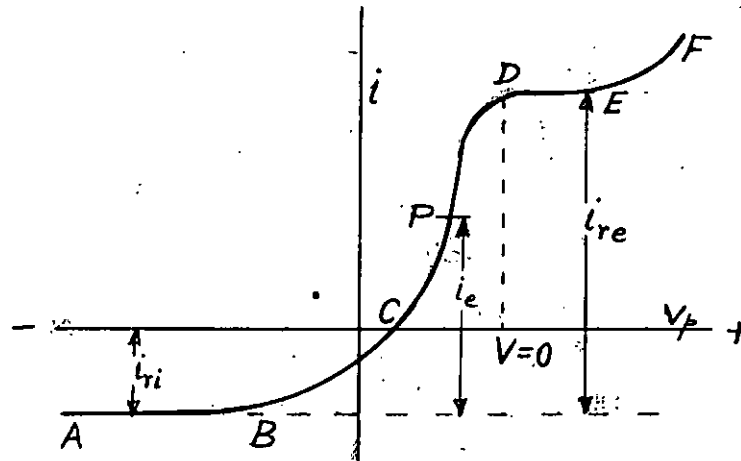


Figure 4 Single probe V-I characteristics. V_p is measured with respect to one of the discharge electrodes or with respect to a grounded conductor in the plasma.

The position of the curve along the horizontal axis is dependent upon the reference electrode chosen for the probe potential.

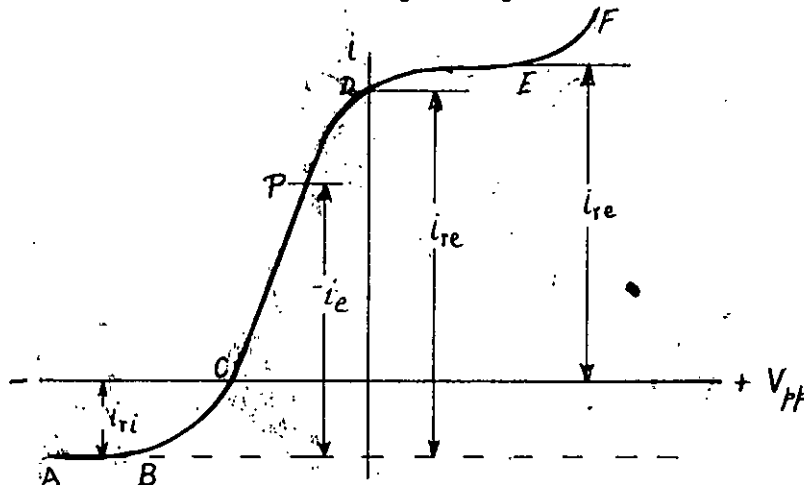


Figure 5 Single probe V-I characteristic (shifted)

i_{re} = random electron current

i_{ri} = random ion current

i_e = electron current

i = total probe current.

The different region in the volt-ampere characteristic curves may be explained as follows:

Region AB

In this region the probe is sufficiently negative relative to the plasma to repel all electrons. The probe is surrounded by a sheath of positive ions. The total probe to plasma potential drop (V_{pp}) appears across this sheath so that the outer edge of the sheath is at plasma potential. Making V_{pp} more negative only serves to increase the thickness of the sheath of positive ions about the probe. The current remains the same. The current density in the sheath J is proportional to $n_{pp} V_p e$ where n_{pp} is the density of ions in the plasma, V_p is the average velocity of the ions and the current is $I = JA$ where A is the probe area. In region AB the current is due to the random motion of positive ions in the sheath and is not affected by either the thickness of the sheath or the potential of the probe with respect to the plasma. Therefore, this current (i_{ri}) should lead to the determination of n_i , the positive ion density.

Region BC

As V_{pp} is made more positive, this positive ion space charge sheath becomes thinner so that some of the faster electrons are able to reach the probe. The electron current subtracts from the ion current so that the total probe current diminishes as shown in the region BC on the curve. At point C, the electron current

due to the more energetic electrons is equal to the ion current due to the random motion of the positive ions in the sheath and the overall current is zero.

Region CD

As the voltage is increased still further, the electron current exceeds the positive ion current so that the overall current is now in the opposite direction and equal to $i_e - i_{ri}$. However, the probe is still surrounded by a sheath of positive ions. This ion sheath remains until $V_{pp} = 0$, when the sheath thickness goes to zero. Then the sheath disappears and the plasma extends to the probe surface. The current at $V_{pp} = 0$ is due to the random ion and electron currents. When V_{pp} is made slightly positive, the positive ion current rapidly disappears resulting in a slight increase in overall current, after which the current remains essentially constant.

Region DE

In this region, the probe is surrounded by an electron sheath so that as in region AB, a change in voltage changes sheath thickness but does not affect the probe current. The probe current is determined by the random motion of the electrons in the sheath and is not affected by the thickness of the sheath, so that the determination of i_{re} should lead to n_e , the electron density. At E, the current increases rapidly. This is due to the fact that ionization by collision (secondary emission) with probe atoms produces positive ions within the sheath which neutralize the electron space charge and allow the current to increase indefinitely without further increase in voltage.

1.5.1 Determination of Electron Temperature (T_e) from Single Probe Measurements

The electrons reaching the probe in region BD will be those with sufficient energy to overcome the retarding potential V_{pp} in the ion sheath. Assuming a

Maxwellian velocity distribution, there will be n electrons reaching the probe per second where

$$n = n_e \exp \left[- \frac{e V_{pp}}{k T_e} \right] \quad (64)$$

n_e = number of electrons reaching outer edge of sheath per second.

The electron current density at any surface is equal to the number of electrons reaching unit area per unit time. Therefore, if one assumes a plane probe and neglects edge effects so that the sheath area is the same as the probe area, one obtains.

$$i_e = i_{re} \exp \left[- \frac{e V_{pp}}{k T_e} \right] \quad (65)$$

where

i_e = electron current at the probe

i_{re} = electron current at the outer edge of the sheath, which is the random electron current.

In an ordinary plasma, it is difficult to measure V_{pp} . In practice, the probe voltage is measured with respect to some electrode, with the result that the characteristic is shifted. The V_{pp} - I characteristic shifted by an amount V_d which is the potential difference between the plasma and the reference electrode. As shown in Figure 4, the plasma potential is positive with respect to the reference electrode as would be the case if the cathode were used as a reference electrode. If the anode were used as a reference electrode, the characteristic curve would have been shifted to the left.

In Figure 4

$$V_{pp} = V_p - V_d$$

so that

$$i_e = i_{re} \exp \left[- \frac{e (V_p - V_d)}{k T_e} \right]$$

Taking logarithm on both sides, one obtains

$$\ln i_e = \ln i_{re} - \frac{e V_p}{k T_e} + \frac{e V_d}{k T_e}$$

Assume i_{re} , V_d and T_e to be constants. Then

$$\ln i_e = \text{constant} - \frac{e V_p}{k T_e}$$

so that

$$\frac{d}{d V_p} (\ln i_e) = - \frac{e}{k T_e} = - \frac{1.602 \times 10^{-19}}{1.38 \times 10^{-23} T_e} = - \frac{11,600}{T_e} \quad (66)$$

where T_e is the electron temperature in degrees Kelvin.

If $\ln i_e$ is plotted against V_p , a straight line with slope m_0 is obtained where

$$m_0 = - \frac{e}{k T_e} = - \frac{11,600}{T_e} = \frac{d}{d V_p} (\ln i_e)$$

so that

$$T_e = - \frac{11,600}{m_0} \text{ } ^\circ\text{K}$$

If the plot of $\ln i_e$ against V_p is not a straight line, then the velocity distribution of the electrons may not be Maxwellian and the preceding theory does not apply. If this is the case, the probe readings will not reveal any information concerning the plasma temperature. At the present time, it is not possible to determine the ion temperature by the use of electrostatic probes. Present probe theory has yielded ion temperatures which are higher in order of magnitude than the actual ion temperature obtained by other means.

1.5.2 Determination of Electron Density

This again depends upon the assumption of a Maxwellian velocity distribution for the electrons in plasma.

It is first necessary to determine T_e by the above method. Next it is necessary to know the random current i_{re} . As is apparent from Figure 5, this is the value of i_e at $V_{pp} = 0$ or point D on the characteristic. On the usual characteristic (Fig 4) it is difficult to determine the exact location of this point. Therefore, the current value of the horizontal segment DE may be used since this consists entirely of random electron current. But this too may be difficult because DE may not be horizontal due to edge effects at the probe. However, by careful examination of the characteristic, a reasonably good value of i_{re} may be obtained.

But from the Kinetic Theory of gases, the random electron current is given by

$$J_r = e n_e \sqrt{\frac{k T_e}{2 \pi m_e}}$$

so that

$$i_{re} = A e n_e \sqrt{\frac{k T_e}{2 \pi m_e}} \quad \text{amperes} \quad (67)$$

where A is the area of the probe. Other terms were defined previously.

From (67)

$$n_e = \frac{i_{re}}{A e} \sqrt{\frac{2 \pi m_e}{k T_e}} = \text{electron density/meter}^3$$

Since T_e is already known, i_{re} is obtained from the characteristic curve. All the terms on the right side of above are known and hence n_e may be calculated.

1.5.3 Limitations

(1) In order to be able to obtain a V-I characteristic, the plasma current must be high enough for convenient measurement. Thus the probe method is successful when the electron density is of the order of 10^{10} electrons per cm^3 or greater.

(2) In the V-I characteristic, no account is taken of any secondary or thermoionic emission from the probe. Therefore the plasma temperature should not be so high as to make these effects appreciable. Thus the probe method is not useful where an extremely high temperature is encountered. An example of this type may occur in controlled fusion research. Apart from secondary emissions, the probe may itself be damaged by such high temperature.

(3) One basic assumption in the probe technique is that the size of the probe is small compared to the mean free path of the electrons. When the probe is large, it collects so many electrons that the surrounding space is depleted more rapidly than can be compensated for by diffusion from distant regions and electron collection is considerably reduced. In the absence of a magnetic field, no problem arises because the probe can be easily made small compared to the mean free path.

1.6 Characterization of the Plasma Sheath by Equivalent Complex Dielectric Properties

In order to study a problem in Electromagnetic theory involving a plasma sheath, it is convenient to characterize the electrical behavior of the sheath by an equivalent complex permittivity, which is dependent primarily on the electron density distribution inside the sheath. A relationship between the electron density distribution and the dielectric property may be derived starting from the equation of motion for an electron in an electric field:

$$m_e \frac{d^2 r}{dt^2} + \frac{m_e}{\tau} \frac{dr}{dt} = -e E_0 \exp(-i\omega t) \quad (68)$$

where τ is an average time between collision of electrons and neutral particles.

At steady state, the above equation reduces to

$$-\omega^2 m_e r - i\omega \frac{m_e}{r} r = -e E_0$$

so that

$$-i\omega r = \frac{-e E_0}{m_e (1/r - i\omega)} = v_e, \text{ velocity of the electron.}$$

The current density will then be

$$J = -e v_e n_e = \frac{e^2 n_e E_0}{m_e (1/r - i\omega)} \quad (69)$$

One may now define an equivalent conductivity and dielectric property by the relationship

$$J = (\sigma_p - i\omega \epsilon_p) E_0 \quad (70)$$

Equating (69) and (70), one obtains

$$(\sigma_p - i\omega \epsilon_p) E_0 = \frac{e^2 n_e E_0}{m_e (1/r - i\omega)}$$

So that

$$\begin{aligned} \sigma_p &= \frac{e^2 n_e r}{m_e [1 + (\omega r)^2]} \\ \epsilon_p &= - \frac{e^2 n_e r^2}{m_e [1 + (\omega r)^2]} \end{aligned} \quad (71)$$

Equation (71) gives the plasma contributions to the conductivity and the dielectric property. To this, the free space dielectric constant must be added to obtain the total dielectric property. Therefore, the total dielectric property is

$$\epsilon_1 = \epsilon_p + \epsilon = \epsilon \left[1 - \frac{e^2 n_e}{m_e \epsilon} \frac{r^2}{1 + (\omega r)^2} \right] \quad (72)$$

It may be observed here that the quantity $\frac{e^2 n_e}{m_e \epsilon}$ is dimensionally an angular frequency squared. In plasma physics, it is customary to define a plasma frequency ω_p as

$$\omega_p^2 = \frac{e^2 n_e}{m_e \epsilon}$$

The relaxation time ϵ may also be replaced by the equivalent collision frequency defined as $\nu = \frac{1}{\tau}$. Substituting these in (71) and (72), one obtains,

$$\begin{aligned} \sigma_p &= \frac{e^2 n_e \nu}{m_e (\nu^2 + \omega^2)} \\ \epsilon_1 &= \epsilon \left[1 - \frac{\omega_p^2}{\nu^2 + \omega^2} \right] \end{aligned} \quad (73)$$

For collisionless plasma, $\nu = 0$, so that $\sigma_p = 0$ and

$$\epsilon_1 = \epsilon \left[1 - \frac{\omega_p^2}{\omega^2} \right] \quad (74)$$

Equations corresponding to (68) to (74) may be derived for the case of the ions. But it is apparent from (71) that since $m_e \ll m_i$, σ_p and ϵ_p due to them will be much smaller compared to those due to the electrons. Therefore, the effect of the ions on the dielectric properties of the sheath and the plasma may be neglected. To find an expression for the dielectric properties of the sheath, it is therefore necessary to find the nature of electron distribution in the sheath and in the transition region.

Bohm has shown that the sheath consists mostly of ions and very few electrons. Therefore, the dielectric property of the region inside the sheath and close to the probe is approximately equal to the free space dielectric constant.

Under conditions of thermal equilibrium

$$n_e = n_0 \exp \left[- \frac{e(V - V_0)}{kT_e} \right] \approx n_0 \left[1 - \frac{e \Delta V}{kT_e} \right] \quad (75)$$

where $V - V_0 = \Delta V$ (in the sheath edge) and $\frac{e \Delta V}{kT_e}$ assumed to be small compared to unity.

But the differential equation which is valid in this region is (equation (43))

$$\frac{d(V - V_0)}{dx} = \sqrt{\frac{e n_0}{\epsilon} \left[\frac{e}{kT_e} - \frac{1}{2V_0} \right]} (V - V_0)$$

so that

$$V - V_0 = A \exp(-ax) \quad (76)$$

where

$$a = \sqrt{\frac{e n_0}{\epsilon} \left[\frac{e}{kT_e} - \frac{1}{2V_0} \right]}$$

and A is a suitable constant.

Substituting (76) into (75), one obtains

$$n_e \approx n_0 \left[1 - \frac{eA}{kT_e} \exp(-ax) \right] \quad (77)$$

The constant A may be evaluated from the condition that at $x = 0$, $n_e \approx 0$.

This gives

$$A = \frac{kT_e}{e}$$

Therefore

$$n_e \approx n_0 [1 - \exp(-ax)] \quad (78)$$

where a is given by (76) and x is the distance of the point of observation from the probe surface. For a cylindrical probe, the relationship will be obtained simply by replacing x with r . If one considers the fact that the electron

distribution is zero for a distance inside the sheath, say from $r = 0$ to $r = a$, then (78) should be written as

$$n_e \approx n_0 \{1 - \exp[-\alpha(r-a)]\} \quad (79)$$

Equation (79) substituted in (73) yields the value of the complex dielectric property of the plasma and the sheath. Thus,

$$\begin{aligned} \sigma_p &\approx \frac{e^2 \nu n_0}{m_e (\nu^2 + \omega^2)} \{1 - \exp[-\alpha(r-a)]\} \\ \epsilon_1 &\approx \epsilon \left[1 - \frac{e^2 n_0}{m_e \epsilon} \frac{1 - \exp[-\alpha(r-a)]}{\nu^2 + \omega^2} \right] \end{aligned} \quad (80)$$

It is therefore evident that when a probe is introduced into a plasma medium, the dielectric property of the sheath and the transition regions varies from free space value (near to the probe) to the final plasma value in accordance with (80).

Some pertinent data are given below for ready reference.

Mass of an electron, $m_e = 9.107 \times 10^{-31}$ kg

Charge of an electron, $e = 1.602 \times 10^{-19}$ coulomb

Boltzmann constant, $k = 1.38 \times 10^{-23}$ watt sec/°K

1 electron volt = 1.602×10^{-19} watt sec.

Apparently, kT_e is given in watt-sec and $\frac{kT_e}{e}$ in volts.

APPENDIX 2

WAVES WITHIN A RADially INHOMOGENEOUS

DIELECTRIC CYLINDER AND THE PROBLEM OF SCATTERING

The dielectric cylinder is assumed to be infinitely long and oriented along the z-direction. The radius of the cylinder is "a" and the material is characterized by $\mu = \mu_0$ and $\epsilon = \epsilon(r)$. Starting from Maxwell's equations, the following relationships are obtained (for a source free region).

$$\nabla^2 \vec{E} + k^2(r) \vec{E} + \nabla \left[\vec{E} \cdot \frac{\nabla \epsilon(r)}{\epsilon(r)} \right] = 0 \quad (1)$$

$$\nabla^2 \vec{H} + k^2(r) \vec{H} + \left[\frac{\nabla \epsilon(r)}{\epsilon(r)} \times (\nabla \times \vec{H}) \right] = 0 \quad (2)$$

$$\nabla^2 \vec{A} + k^2(r) \vec{A} - \frac{2}{k} (\nabla \cdot \vec{A}) \nabla k = 0 \quad (3)$$

These equations and their solutions under specialized cases have been discussed by several authors.¹⁻³ It is sufficient to indicate here that no general solution of (1) - (3) is available. If, however, one makes the simplifying assumption that $\epsilon(r)$ is sufficiently slowly varying such that the last terms in (1) - (3) are negligible compared to the other terms, then some solutions may be obtained. These solutions may later be used in (1) - (3) to obtain more accurate solutions by iteration techniques. The simplified equations may be written as

$$[\nabla^2 + k^2(r)] \begin{matrix} \vec{A} \\ \vec{E} \\ \vec{H} \end{matrix} = 0 \quad (4)$$

2.1 The Solution of the Wave Equation

The z-component of $\vec{A}$, $\vec{E}$ and $\vec{H}$ in (4) will satisfy an equation of the form

$$[\nabla^2 + k^2(r)] \Phi = 0 \quad (5)$$

The solution is

$$\Phi \sim \exp[ih z + i n \phi] Z_n(r) \quad (6)$$

where $Z_n(r)$ is a solution of the equation

$$\frac{1}{r} \frac{d}{dr} \left[r \frac{dZ_n}{dr} \right] + \left[k^2(r) - h^2 - \frac{n^2}{r^2} \right] Z_n = 0 \quad (7)$$

The function Z_n will depend on the function $k^2(r)$. For example, if $k^2(r)$ is a constant, Z_n is the solution of the Bessel's equation. When $k^2(r)$ is not a constant, one approximate solution may be obtained by letting

$$Z_n(r) \doteq r^{-1/2} \psi_n \quad (8)$$

This substituted in (7) will yield

$$\frac{d^2 \psi_n}{dr^2} + \left[k^2(r) - h^2 + \frac{\frac{1}{4} - n^2}{r^2} \right] \psi_n = 0 \quad (9)$$

The case to be considered here is that of the collisionless plasma sheath. It was shown in Appendix 1, that the variation of the dielectric property is given by (Equation (80) in Appendix 1)

$$\epsilon \simeq \epsilon_0 \left\{ 1 - \frac{e^2 n_0}{m_e \epsilon_0 \omega^2} [1 - \exp -a(r-a)] \right\} \quad (10)$$

where for a collisionless plasma, ν has been assumed to be zero.

From (10),

$$k^2(r) = \omega^2 \mu_0 \epsilon \simeq k_0^2 - \frac{e^2 n_0 \mu_0}{m_e \omega^2} + \frac{e^2 n_0 \mu_0}{m_e} \exp(aa) \exp(-ar) \quad (11)$$

The solution of (9) with (11) is known only for the special case when $n = \pm \frac{1}{2}$

because then (9) reduces to

$$\frac{d^2 \psi_n}{dr^2} + [-B^2 + c^2 \exp(-ar)] \psi_n = 0 \quad (12)$$

where

$$-B^2 = k_0^2 - h^2 - \frac{e^2 n_0 \mu_0}{m_e \omega^2} \quad (13)$$

$$c^2 = \frac{e^2 n_0 \mu_0}{m_e} \exp(aa) \quad (14)$$

To solve (12), one makes a change of variable from r to u such that

$$u = \exp \left[-\frac{ar}{2} \right] \quad (15)$$

With (15) substituted into (12), one obtains

$$\frac{d^2 \psi_n}{du^2} + \frac{1}{u} \frac{d\psi_n}{du} + \left[\frac{4c^2}{a^2} - \frac{4B^2}{a^2 u^2} \right] \psi_n = 0 \quad (16)$$

Equation (16) is of the form of Bessel's equation and the solutions are

$$\psi_n \sim J_m(v), N_m(v)$$

where

$$m = \frac{2B}{a}, \quad v = \frac{2c}{a} u \quad (17)$$

It is apparent that the solutions in (17) are not useful in boundary value problems where one seeks a solution for arbitrary n . A more general solution of (9) with (11) is possible only through some approximations for the electron density distribution. Among the many possibilities, two are important in the sense that they permit formal solutions of (7). These are the electron density distribution according to the functions (a) $c + \frac{d}{r}$ and (b) $\left[\frac{r}{\ell} \right]^2$ where c , d and ℓ are suitable constant to be chosen in order to fit closely the actual distribution under specific conditions. The solution for (7) under these two conditions will be discussed in the following.

2.2 Electron Distribution According to the Function $c + \frac{d}{r}$

In this case

$$k^2(r) = k_o^2 - \frac{e^2 n_o \mu_o}{m_e} \left[c + \frac{d}{r} \right] \quad (18)$$

This substituted into (7) gives

$$\frac{d^2 Z_n}{dr^2} + \frac{1}{r} \frac{dZ_n}{dr} + \left[-g^2 + \frac{2\beta}{r} - \frac{n^2}{r^2} \right] Z_n = 0 \quad (19)$$

where

$$k_o^2 = h^2 - \frac{e^2 n_o \mu_o c}{m_e} = -g^2; \quad -\frac{e^2 n_o \mu_o d}{m_e} = 2\beta \quad (20)$$

Equation (19) is of the form of the wave equation in parabolic coordinates. Solutions of this equation have been discussed by many authors. The equation has one regular singular point at $r = 0$ and an irregular one at $r = \infty$. A regular singular point is defined to be one where the general solution has a pole or a branch point. An irregular singular point is one where the general solution has an essential singularity ('a' is an essential singular point of 'f' if $f(z) (z-a)^n \rightarrow \infty$ as $z \rightarrow a$ for all finite values of n). The singular points of a differential equation

$$\frac{d^2\psi}{dz^2} + p(z) \frac{d\psi}{dz} + q(z)\psi = 0$$

are the singular points of $p(z)$ or $q(z)$ or both. All other values of z where p and q are analytic functions are the ordinary points of the equation.

The solution of (19) is obtained by making the following substitutions:

$$Z_n^* = r^n \exp(-gr) F(r); \quad r = \frac{x}{2g} \quad (21)$$

These substitutions give the following equation that F must satisfy

$$x \frac{d^2F}{dx^2} + (t-x) \frac{dF}{dx} - sF = 0 \quad (22)$$

where $t = \frac{1}{2} + 2n$; $s = \frac{1}{2} (1 + 2n) - \frac{\beta}{g}$. Equation (22) is the confluent hypergeometric equation. Its solution is obtained by letting

$$F = \sum_{m=0}^{\infty} a_m x^m$$

This, when substituted in (22), yields the following recursion formula for the coefficients.

$$a_{m+1} (t+m) (m+1) - (s+m) a_m = 0$$

Thus the solution of F which is analytic at $r = 0$ is given by

$$F(s|t|x) = 1 + \frac{s}{t} x + \frac{s(s+1)}{2! t(t+1)} x^2 + \dots + \frac{s(s+1) \dots (s+m-1)}{m! t(t+1) \dots (t+m-1)} x^m + \dots \quad (23)$$

The series (23) is known as the confluent hypergeometric series and it converges in the range $-\infty < x < \infty$. Various properties of this series have been discussed in the literature.⁴

In the solution (21), it was tacitly assumed that $n \gg 0$, i.e. $\text{Re } t \gg 1$ because otherwise the solution is not analytic at $r = 0$. If this is not the case, then the solution becomes (for n negative)

$$Z_n = r^{-n} \exp(gr) F(s/t/x) \quad (24)$$

where

$$s = \frac{1}{2}(1 - 2n) + \frac{\beta}{g}$$

$$t = 1 - 2n$$

$$x = -2gr$$

and F is given by the series (23).

2.3 Electron density distribution according to the function $(r/l)^2$

In this case, $k^2(r) \simeq k_0^2 - (r/l)^2$ where l is a constant. Equation (8) now becomes

$$\frac{d^2 Z_n}{dr^2} + \frac{1}{r} \frac{dZ_n}{dr} + \left[k_0^2 - h^2 - \left(\frac{r}{l}\right)^2 - \frac{n^2}{r^2} \right] Z_n = 0 \quad (25)$$

The solution of (25) may be obtained in the following manner:

$$\text{Let } Z_n = r^n \exp\left(-\frac{r^2}{2l}\right) F(r) \quad (26)$$

This substituted into (25) yields the following equation that F must satisfy.

$$\frac{d^2 F}{dr^2} + \left[\frac{1+2n}{r} - \frac{2r}{l} \right] \frac{dF}{dr} + \left[k_0^2 - h^2 - \frac{2(1+n)}{l} \right] F = 0 \quad (27)$$

If one changes the variable from r to x defined as $x = r^2/(2l)$, then Equation

(27) reduces to

$$x \frac{d^2 F}{dx^2} + (1+n-x) \frac{dF}{dx} - \left[\frac{1+n}{2} + \frac{(k_0^2 - h^2)l}{4} \right] F = 0 \quad (28)$$

This is the same form as (22) and the solution is given by the series (23), with the following s and t .

$$s = \frac{1+n}{2} - \frac{(k_o^2 - h^2)l}{4} \quad (29)$$

$$t = 1+n$$

As in section 2.2, (26) is valid for $n > 0$ or $\text{Re } t \geq 1$. For $\text{Re } t < 1$, the solution is

$$Z_n = r^{-n} \exp\left(\frac{r^2}{2l}\right) F(s/t/x)$$

where

$$t = 1+n$$

$$s = \frac{1+n}{2} - \frac{(k_o^2 - h^2)l}{4}$$

$$x = -r^2/l \quad (30)$$

$F(s/t/x)$ is given by the series in (23).

2.4 Scattering of Electromagnetic waves by a Radially Inhomogeneous dielectric cylinder.

Referring to Figure 1 in Chapter 1, and with the same type of incident field, it is fairly straightforward to solve the problem of a single dielectric cylinder, once the solution of Equation (7) is known. If it is assumed that the function Z_n (which is finite at $r = 0$) for a given electron density distribution is known, then following the procedure of Chapter 1, one writes the field components for the incident and scattered field, E_z^i , E_z^s , H_z^i , H_z^s , E_ϕ^i , E_ϕ^s , H_ϕ^i , and H_ϕ^s exactly as in Chapter 1.

For the field components inside the cylinder, one writes, however,

$$\begin{aligned} E_z &= \sum_{n=-\infty}^{\infty} a_n Z_n(r) F_n \\ H_z &= \sum_{n=-\infty}^{\infty} b_n Z_n(r) F_n \end{aligned} \quad (31)$$

where a_n and b_n are constants and F_n is the same as in Chapter 1.

Applying the boundary conditions as in Chapter 1, and after some algebraic manipulations, one obtains

$$\begin{aligned}
 \hat{a}_n &= [(-i)^n E_o \sin \theta J_n(v) + a_n^s H_n^{(1)}(v)] \frac{1}{Z_n(a)} \\
 a_n^s &= E_o (-i)^n \sin \theta \left[-\frac{J_n(v)}{H_n^{(1)}(v)} + \frac{2i \left[\frac{H_n^{(1)'}(v)}{v H_n^{(1)}(v)} - \frac{Z_n'(a)}{\lambda^2 a Z_n(a)} \right]}{\pi v^2 [H_n^{(1)}(v)]^2 D} \right] \\
 b_n &= \frac{k_o}{\omega \mu_o} E_o \sin \theta (-i)^n \left[\frac{2}{\pi v^2} \left[\frac{1}{u^2} - \frac{1}{v^2} \right] \frac{n \cos \theta}{[H_n^{(1)}(v)]^2 D} \right] \\
 b_n^s &= b_n \frac{Z_n(a)}{H_n^{(1)}(v)} \tag{32}
 \end{aligned}$$

where

$$D = \left[\frac{H_n^{(1)'}(v)}{v H_n^{(1)}(v)} - \frac{Z_n'(a)}{\lambda^2 a Z_n(a)} \right] \left[\frac{H_n^{(1)'}(v)}{v H_n^{(1)}(v)} - \frac{N^2 Z_n'(a)}{\lambda^2 a Z_n(a)} \right] - n^2 \cos^2 \theta \left[\frac{1}{v^2} - \frac{1}{u^2} \right]^2$$

$$v = \lambda_o a, \quad u = \lambda a, \quad N^2 = k^2/k_o^2, \quad \lambda = \sqrt{k^2 - k_o^2 \cos^2 \theta}$$

$\lambda_o = k_o \sin \theta$ and a = radius of the dielectric cylinder.

APPENDIX 3

METHOD OF STEEPEST DESCENT

It is desired to evaluate an integral of the form (35) in chapter 2 approximately by the method of steepest descent. The integral may be written as

$$I = \int_{-\infty}^{\infty} f(a) e^{i[az + \sqrt{k_0^2 - a^2} z]} da \quad (1)$$

where $f(a)$ is in general a complicated function of a .

With the substitution in (36), in chapter 2, Eq. (1) reduces to

$$I = \int_c f(k_0 \sin \phi) e^{ik_0 R \cos(\theta - \phi)} k_0 \cos \phi d\phi \quad (2)$$

where c is a suitable contour.

The method of steepest descent (originally introduced by Debye for obtaining asymptotic expansions of the Hankel functions) is discussed in detail in various literature, and only the salient points will be outlined here.

The transformation of integral (1) from the a plane to the ϕ plane changes the contour from along the real axis between $-\infty$ to $+\infty$ to ABC as shown in figure 1. It is desired to deform this contour along which the imaginary part of $ik_0 R \cos(\theta - \phi)$ is constant in the region where its real part is largest. At the maximum point of $ik_0 R \cos(\theta - \phi)$, namely at $\phi = \theta - i0$, $\text{Re}[ik_0 R \cos(\theta - \phi)]$ is largest. This is the so called "saddle point." The equation to the deformed contour is obtained by evaluating $\text{Im}[ik_0 R \cos(\theta - \phi)]$ at $\phi = \theta - i0$.

This gives

$$\cos(\theta - \phi_1) \cosh \phi_2 = 1 \quad (3)$$

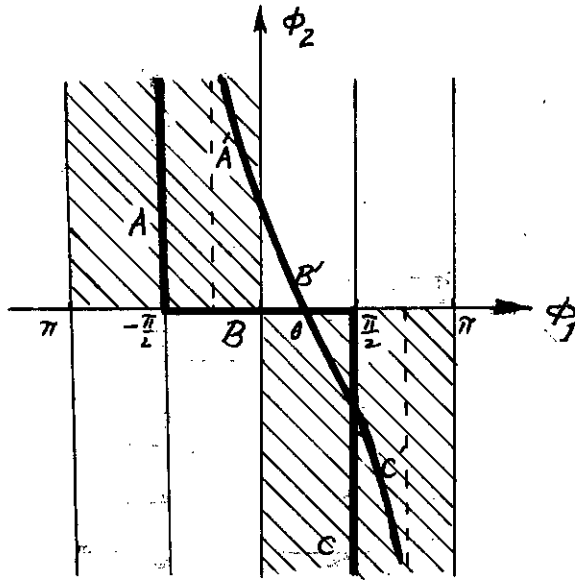


Figure 1. Contours of Integration in the Complex ϕ -plane.

The plot of (3) within the allowed Riemann sheet is A'B'C' shown in Figure 1. Along this contour, (i. e. ϕ_1 and ϕ_2 satisfying Equation (3))

$$e^{ik_0 R \cos(\theta - \phi)} = e^{ik_0 R} e^{-k_0 R \frac{\sinh^2 \phi_2}{\cosh \phi_2}} \quad (4)$$

As stipulated, the imaginary part of the exponent in (4) is constant, along the new contour. For large R , the quantity has a maximum at $\phi_2 = 0$ and rapidly decreases on both sides of the saddle point.

Before evaluating the integral at the saddle point, it is necessary to examine the nature of poles the integrand has and if there is any significant contribution from such singularities. The poles on the proper Riemann sheet are the surface wave poles and those on the improper sheet are the leaky-wave poles. Without explicitly evaluating the residues at these poles, it may be observed that from physical point of view, these contributions must be very small if the field far away from the surface of the cylinder is desired, because all such contributions will decay exponentially. Therefore, if the field far away from the surface is desired, the significant contribution will be from the saddle point.

About the saddle point

$$\frac{\sinh^2 \phi_2}{\cosh \phi_2} \simeq \phi_2^2 \quad (5)$$

and it may be shown easily that along the new contour

$$d\phi = \sqrt{2} e^{-i\frac{\pi}{4}} d\phi_2 \quad (6)$$

Combining (2) to (6), the following is obtained

$$I = \int_{-u}^u \sqrt{2} f(k_0 \sin \phi) e^{ik_0 R} e^{-k_0 R \phi_2^2} e^{-i\frac{\pi}{4}} k_0 \cos \phi d\phi_2 \quad (7)$$

where u is a point on the contour close to the saddle point. Within this region $f(k_0 \sin \phi) \cos \phi \simeq f(k_0 \sin \theta) \cos \theta$, independent of ϕ_2 . Also, if the limit of the integration is changed to $-\infty$ to ∞ , very little is added to the original value. Then

$$\begin{aligned} I &\simeq \sqrt{2} f(k_0 \sin \theta) k_0 \cos \theta e^{-i\frac{\pi}{4}} e^{ik_0 R} \int_{-\infty}^{\infty} e^{-k_0 R \phi_2^2} d\phi_2 \\ &\simeq \sqrt{\frac{2\pi}{k_0 R}} f(k_0 \sin \theta) k_0 \cos \theta e^{-i\frac{\pi}{4}} e^{ik_0 R} \end{aligned} \quad (8)$$

Equation (8) is the result of the integration in (1) by the method of steepest descent.

APPENDIX 4

GREEN'S FUNCTIONS INVOLVING SOURCES AND BOUNDARY SURFACES

The three dimensional Green's function for an unbounded medium having the properties μ, ϵ, k is given by the formula

$$G_k(r/r_o) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{ikR}}{R} d\phi \quad (1)$$

where

$$R = \sqrt{r^2 + r_o^2 - 2rr_o \cos\phi + (z - z_o)^2}$$

Relationship (1) is a solution to the equation

$$\nabla^2 G_k(r/r_o) + k^2 G_k(r/r_o) = -4\pi \delta(r - r_o) \quad (2)$$

It will be shown that the inhomogeneous equation

$$(\nabla^2 + k^2) A_z = -\mu J_z(r) \quad (3)$$

subject to arbitrary boundary conditions on a closed boundary surface may be expressed in terms of $G_k(r/r_o)$. To do this, one multiplies (2) by A_z and (3) by $G_k(r/r_o)$, and subtracts the results to obtain

$$G_k(r/r_o) \nabla_o^2 A_z(r_o) - A_z(r_o) \nabla_o^2 G_k(r/r_o) = -\mu J_z(r_o) G_k(r/r_o) + 4\pi A_z(r_o) \delta(r - r_o)$$

Integrating this over all source coordinates r_o inside s_o , one obtains

$$\iiint [G_k(r/r_o) \nabla_o^2 A_z(r_o) - A_z(r_o) \nabla_o^2 G_k(r/r_o)] dv_o = -\mu \iiint J_z(r_o) G_k(r/r_o) dv_o + 4\pi \iiint A_z(r_o) \delta(r - r_o) dv_o$$

Using the property of the delta function and Green's Theorem, the above reduces to

$$\begin{aligned} & \frac{1}{4\pi} \iint_{s_0} [G_k(r/r_0^s) \nabla_0 A_z(r_0^s) - A_z(r_0^s) \nabla_0 G_k(r/r_0^s)] \cdot d\vec{s}_0 \\ & + \frac{\mu}{4\pi} \iiint J_z(r_0) G_k(r/r_0) dv_0 = \begin{cases} A_z(r); & r \text{ within and on } s \\ 0; & r \text{ outside } s \end{cases} \end{aligned} \quad (4)$$

The object is to apply (4) to the case of a coated antenna. The cylinder, covering the antenna, will be assumed to be of infinite length for simplicity. Normally, it will be necessary to write two integral equations and solve them simultaneously to obtain $A_z(r)$. This is, however, very complicated. Since the aim is only to check the formulation of Chapter 3, all is needed is to apply the solution obtained in Chapter 3 to the equations in (4) and see if they are satisfied.

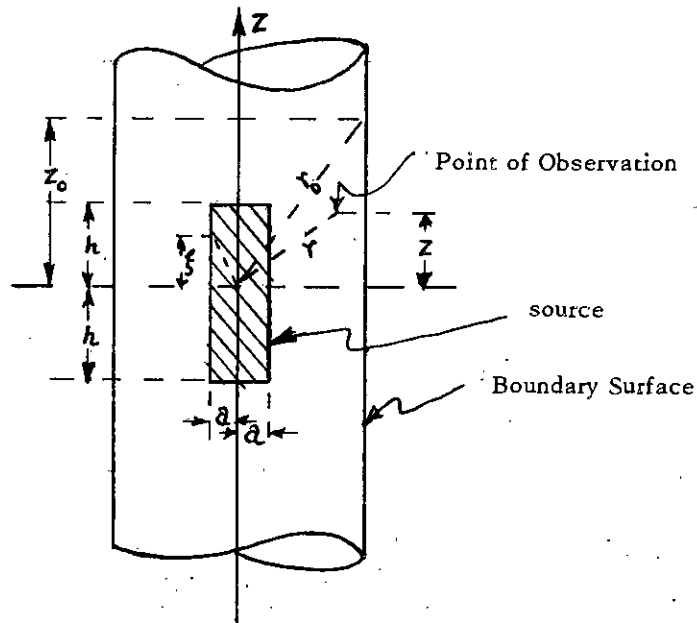


Figure 2. Source point, observation point and boundary surface for formulating Green's function

It was shown in Chapter 3 that the vector potential inside the coating is given by

$$A_z(r) = \frac{i\mu}{8\pi} \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} J_0(la) H_0(lr) e^{-ia(z-\xi)} da + \int_{-\infty}^{\infty} a(a) J_0(lr) e^{-iaz} da \quad (5)$$

where $a(a)$ was given in Eq. (6), Chapter 3.

The first part of (5) is simply the contribution from the antenna in an unbounded medium, since this is equal to

$$\frac{\mu}{4\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_{-h}^h J_z(r) \frac{e^{ikR_1}}{R_1} d\xi; \quad R_1 = \sqrt{r^2 + a^2 - 2ra \cos\phi + (z-\xi)^2} \quad (6)$$

Comparing (4), (5), and (6), one concludes that in order to show the equivalence between (4) and (5), it is only necessary to show the following to be true:

$$\begin{aligned} & \iint [G_k(r/r_0^S) \nabla_0 A_z(r_0^S) - A_z(r_0^S) \nabla_0 G_k(r/r_0^S)] \cdot d\vec{s} \\ &= 4\pi \int_{-\infty}^{\infty} a(a) J_0(lr) e^{-iaz} da \end{aligned} \quad (7)$$

To prove (7), one starts with the formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{ikR}}{R} d\phi = \begin{cases} \frac{i}{2} \int_{-\infty}^{\infty} J_0(lr_0) H_0(lr) e^{-ia(z-z_0)} da; & r > r_0 \\ \frac{i}{2} \int_{-\infty}^{\infty} J_0(lr) H_0(lr_0) e^{-ia(z-z_0)} da; & r < r_0 \end{cases} \quad (8)$$

In order to prove the first part of (4), it is noted that for this case $r < r_0$, so that one chooses the second expansion of (8).

$$G_k(r/r_0^S) = \frac{i}{2} \int_{-\infty}^{\infty} J_0(lr) H_0(lb) e^{-ia(z-z_0)} da; \quad r_0^S = b \quad (9)$$

$$\nabla_0 G_k(r/r_0^S) = \frac{i}{2} \int_{-\infty}^{\infty} J_0(lr) H_0'(lb) e^{-ia(z-z_0)} da; \quad r_0^S = b \quad (10)$$

Also from (5), with $r = r_0^S = b$

$$A_z(r_0^S) = \frac{i\mu}{8\pi} \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} J_0(la) [H_0(lb) + f(a) J_0(lb)] e^{-ia(z_0-\xi)} da \quad (11)$$

$$\nabla_0 A_z(r_0^S) = \frac{i\mu}{8\pi} \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} J_0(la) [H_0'(lb) + f(a) J_0'(lb)] e^{-ia(z_0-\xi)} da \quad (12)$$

It is convenient to use the following abbreviated notations:

$$G(a) = H_0(lb) + f(a) J_0(lb) \quad (13)$$

$$\begin{aligned} &= \frac{2i H_0(l_0 b)}{\pi l b \left[\frac{l}{l_0} \frac{K_m}{N^2} J_0(lb) H_0'(l_0 b) - J_0'(lb) H_0(l_0 b) \right]} \\ E(a) = H_0'(lb) + f(a) J_0'(lb) &= \frac{2i \frac{l}{l_0} \frac{K_m}{N^2} H_0'(l_0 b)}{\pi l b \left[\frac{l}{l_0} \frac{K_m}{N^2} J_0(lb) H_0'(l_0 b) - J_0'(lb) H_0(l_0 b) \right]} \end{aligned} \quad (14)$$

In the above the following wronskian relationship has been used,

$$J_0(v) H_0'(v) - J_0'(v) H_0(v) = \frac{2i}{\pi v}$$

Combining (11) and (12) with (13) and (14), one obtains

$$\begin{aligned} A_z(r_0^S) &= \frac{i\mu}{8\pi} \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} G(a) J_0(la) e^{-ia(z_0-\xi)} da \\ \nabla_0 A_z(r_0^S) &= \frac{i\mu}{8\pi} \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} E(a) J_0(la) e^{-ia(z_0-\xi)} da \end{aligned} \quad (15)$$

One substitutes (9), (10), and (15) into the left side of (7) to obtain

$$\begin{aligned} &\iint_{s_0} [G_k(r/r_0^S) \nabla_0 A_z(r_0^S) - A_z(r_0^S) \nabla_0 G_k(r/r_0^S)] \cdot d\vec{s}_0 \\ &= \frac{i}{2} \frac{i\mu}{8\pi} \iint_{s_0} ds_0 \left[\int_{-\infty}^{\infty} J_0(lr) H_0(lb) e^{-ia(z-z_0)} da \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} da E(a) J_0(la) l e^{-ia(z_0-\xi)} \right. \\ &\quad \left. - \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} G(a) J_0(la) e^{-ia(z_0-\xi)} da \int_{-\infty}^{\infty} l J_0(lr) H_0'(lb) e^{-ia(z-z_0)} da \right] \end{aligned} \quad (16)$$

The surface integration around the infinite cylinder of radius b may be carried out in the following manner. With cylindrical symmetry

$$\iint_{s_0} ds_0 = 2\pi b \int_{-\infty}^{\infty} dz_0$$

By interchanging the order of the integration, one may carry out the integration with z_0 first, and then one of the a -integrations, using the following properties of the Dirac Delta function:

$$\int_{-\infty}^{\infty} e^{-i(\beta-a)z_0} dz_0 = 2\pi \delta(\beta-a) \quad (17)$$

$$\int_{-\infty}^{\infty} g(\beta) \delta(\beta-a) d\beta = g(a)$$

Carrying out the two integrations in the right of (16) in the above indicated steps and substituting the values of $G(a)$ and $E(a)$ from (13) and (14), one obtains after some straightforward manipulations,

$$\iint_{S_0} [G_k(r/r_0^S) \nabla_0 A_z(r_0^S) - A_z(r_0^S) \nabla_0 G_k(r/r_0^S)] \cdot d\vec{s}_0$$

$$= \frac{i\mu}{2} \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} J_0(la) \frac{H'_0(lb) H_0(l_0b) - \frac{K_m}{N^2} \frac{l}{l_0} H_0(lb) H'_0(l_0b)}{\frac{K_m}{N^2} \frac{l}{l_0} J_0(lb) H'_0(l_0b) - J'_0(lb) H_0(l_0b)} J_0(lr) e^{-ia(z-\xi)} da$$

Using the value of $a(a)$ from Equation (6) in Chapter 3, this reduces to

$$4\pi \int_{-\infty}^{\infty} a(a) J_0(lr) e^{-iaz} da$$

This is exactly what was to be proved in (7). Thus it is shown that the $A_z(r)$ given in Chapter 3, Equation (5), is indeed a solution for the first part of the integral equation (4).

It is noted that in (4), r appears only on the Green's function $G_k(r/r_0)$. In order to prove the second part of (4), one chooses the Green's function for $r > r_0$ (r outside s) in (8) so that

$$G_k(r/r_0^s) = \frac{i}{2} \int_{-\infty}^{\infty} J_0(lb) H_0(lr) e^{-ia(z-z_0)} da \quad (18)$$

$$\nabla_0 G_k(r/r_0^s) = \frac{i}{2} \int_{-\infty}^{\infty} l J'_0(lb) H_0(lr) e^{-ia(z-z_0)} da$$

Using (15) and (18) together into (4), one obtains

$$\begin{aligned} & \iint_{s_0} [G_k(r/r_0^s) \nabla_0 A_z(r_0^s) - A_z(r_0^s) \nabla_0 G_k(r/r_0^s)] \cdot d\vec{s}_0 \\ &= \frac{i}{2} \frac{\mu}{8\pi} \iint_{s_0} ds_0 \left[\int_{-\infty}^{\infty} J_0(lb) H_0(lr) e^{-ia(z-z_0)} da \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} J_0(la) l E(a) e^{-ia(z_0-\xi)} da \right. \\ & \quad \left. - \int_{-\infty}^{\infty} d\xi I(\xi) \int_{-\infty}^{\infty} G(a) J_0(la) e^{-ia(z_0-\xi)} da \int_{-\infty}^{\infty} l J'_0(lb) H_0(lr) e^{-ia(z-z_0)} da \right] \end{aligned}$$

Carrying out the integration as in the previous case, the above reduces to

$$- \frac{\mu i}{2} \int_{-h}^h d\xi I(\xi) \int_{-\infty}^{\infty} J_0(la) H_0(lr) e^{-ia(z-\xi)} da$$

and this is exactly equal to

$$-\mu \iiint J_z(r_0) G_k(r/r_0) dv_0$$

Therefore the second part of (4) is proved, that is

$$\iint_{S_0} [G_k(r/r_0^S) \nabla_0 A_z(r_0^S) - A_z(r_0^S) \nabla_0 G_k(r/r_0^S)] \cdot d\vec{s}_0$$

$$+ \mu \iiint J_z(r_0) G_k(r/r_0^S) dv_0 = 0; r > r_0$$

It is thus proved that the value of $A_z(r)$ obtained in Chapter 3 satisfies both the integral equations in (4), and therefore is the correct value of the vector potential.

APPENDIX 5

EVALUATION OF THE CURRENT DISTRIBUTION FOR LARGE $|z|$ BY THE METHOD OF STEEPEST DESCENT

It is the desire to evaluate asymptotically an integral of the following form (Eq. (13), Chapter 3).

$$I(z) = \int_{-\infty}^{\infty} \frac{e^{-iaz}}{l^2 G(a)} da = \int_{-\infty}^{\infty} \frac{e^{-ia|z|}}{l^2 G(a)} da \quad (1)$$

where $l^2 = k^2 - a^2$ and $G(a)$ is an even function of a . An integral of this type has been evaluated by the method of steepest descent by various authors.^{1,2} A similar approach will be adopted here.

The integral in (1) is transformed from the a -plane to a complex ψ -plane defined by the following relationships

$$a = k \sin \psi, \quad \psi = \psi_1 + i\psi_2$$

Therefore

$$I(z) = \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} - i\infty} \frac{e^{-ik|z|\sin\psi}}{k \cos\psi G(k \sin\psi)} d\psi \quad (2)$$

The path of integration in the a -plane is along the real axis from $-\infty$ to $+\infty$.

If k happens to be real, then the contour is chosen with an upward indentation at $a = -k$ and a downward indentation at $a = +k$. The path of integration in the ψ -plane is from $-\frac{\pi}{2} + i\infty$ to $\frac{\pi}{2} - i\infty$, and is the same as shown in Appendix 3.

In order to find the path of steepest descent, it is convenient to write (2) in the following form:

$$I(z) = \frac{1}{k} \int_{-\frac{\pi}{2} + i\infty}^{\frac{\pi}{2} - i\infty} \frac{e^{k|z| \left[-\frac{1}{k|z|} \ln(\cos\psi) + i \sin\psi \right]}}{G(k \sin\psi)} d\psi \quad (3)$$

One defines

$$f(\psi) = -\frac{1}{k|z|} \ln(\cos\psi) + i \sin\psi \quad (4)$$

It is desired to deform the contour in the complex ψ -plane in such a way that along the new path, the imaginary part of $f(\psi)$ is constant in the region where its real part is largest. From here on, the method is exactly as shown in Appendix 3. The saddle point is at

$$f'(\psi_0) = \frac{1}{k|z|} \tan \psi_0 + i \cos \psi_0 = 0$$

This gives

$$\sin \psi_0 = \frac{1}{2ik|z|} \pm \sqrt{1 - \frac{1}{4k^2 z^2}} \simeq \pm 1 + \frac{1}{2ik|z|} \quad (5)$$

where term of the order of $\frac{1}{4k^2 z^2}$ has been assumed negligible compared to unity, $k|z|$ being quite large.

It is apparent from (5) that the saddle point is close to the point $\psi_0 = \pm \frac{\pi}{2}$.

Near $+\frac{\pi}{2}$,

$$\sin \psi_0 = \cos \left[\psi_0 - \frac{\pi}{2} \right] \simeq 1 - \frac{1}{2} \left[\psi_0 - \frac{\pi}{2} \right]^2 \quad (6)$$

Near $-\frac{\pi}{2}$,

$$\sin \psi_0 = -\cos \left[\psi_0 + \frac{\pi}{2} \right] \simeq -1 + \frac{1}{2} \left[\psi_0 + \frac{\pi}{2} \right]^2 \quad (7)$$

In (6) and (7), terms of order higher than $(\psi_0 \mp \frac{\pi}{2})^2$ have been neglected.

Comparing (6) and (7) with (5), one obtains

$$\psi_0 \approx \frac{\pi}{2} \pm \sqrt{\frac{i}{k|z|}} \text{ for } \psi_0 \text{ near } \frac{\pi}{2}$$

$$\psi_0 \approx -\frac{\pi}{2} \pm \sqrt{\frac{1}{ik|z|}} \text{ for } \psi_0 \text{ near } -\frac{\pi}{2}$$

There are thus four saddle points and it is necessary to examine $f(\psi)$ closely to choose the correct path of integration. Kuehl¹ has shown that the saddle point which contributes to the integration is at

$$\psi_0 \approx \frac{\pi}{2} - \sqrt{\frac{i}{k|z|}} \quad (8)$$

The deformed contour crosses the saddle point at an angle of $-\frac{\pi}{4}$. The path (taken from Reference 1) is shown in figure 1.

Once the correct saddle point is obtained, one can directly write down the result of integration as an infinite series.³ The first term of the series is

$$I(z) \approx \frac{1}{k} \frac{e^{k|z|f(\psi_0)}}{G(k \sin \psi_0)} \sqrt{\frac{2\pi}{k|z|e^{i\pi} f''(\psi_0)}} \quad (9)$$

It is now necessary to evaluate $f(\psi_0)$ and $f''(\psi_0)$.

From (4), (6) to (8), one obtains

$$f(\psi_0) \approx -\frac{1}{k|z|} \ln \left[\sqrt{\frac{i}{k|z|}} \right] + i + \frac{1}{2k|z|}$$

$$f''(\psi_0) \approx -2i$$

Therefore

$$I(z) \simeq -\frac{i}{k} \sqrt{e\pi} \frac{e^{ik|z|}}{G(k \sin \psi_0)} \quad (10)$$

with

$$\sin \psi_0 \simeq 1 + \frac{1}{2ik|z|}$$

$$\cos \psi_0 \simeq \sqrt{\frac{i}{k|z|}}$$

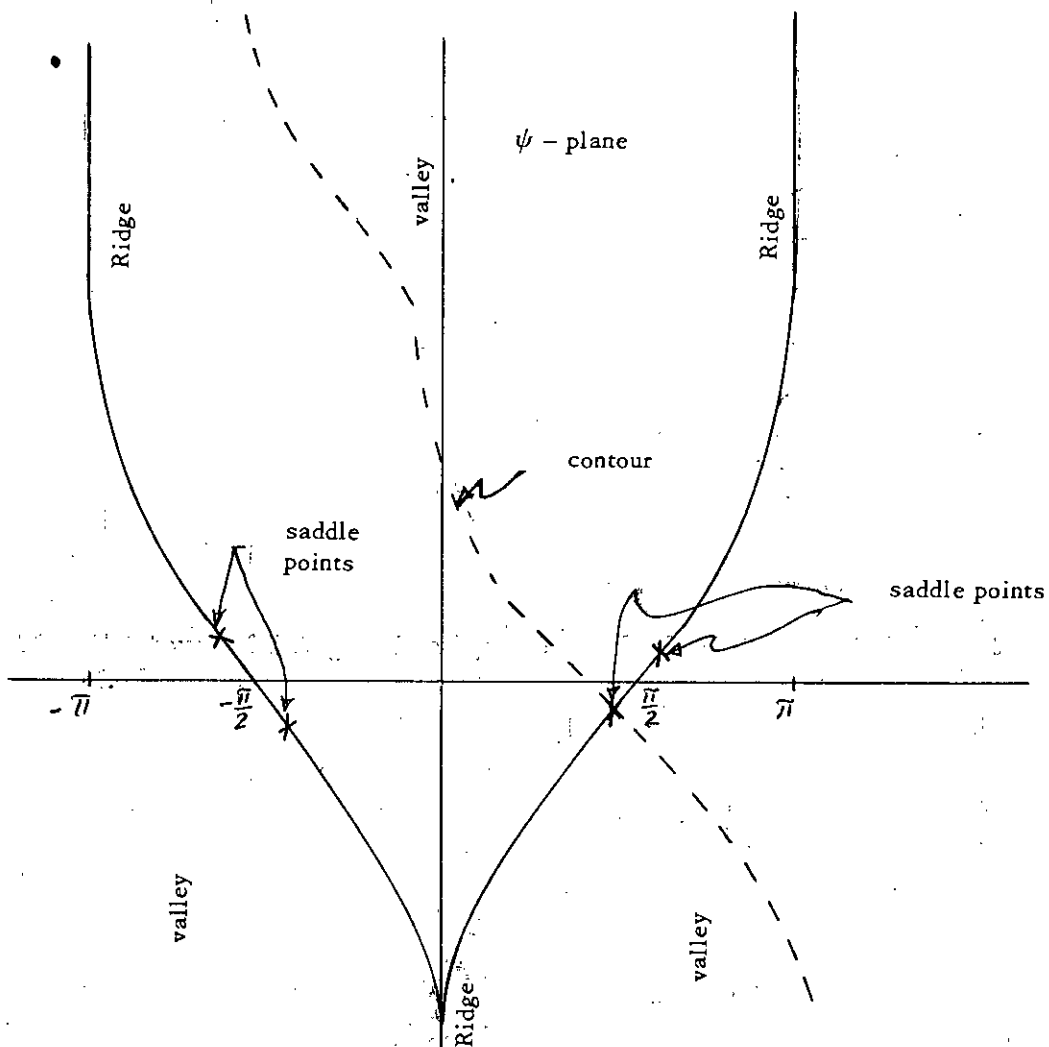


Figure 1 The Contour of Integration by the Method of Steepest Descent (taken from Ref. 1)

APPENDIX 6

THE SOLUTION OF INTEGRAL EQUATION BY THE WIENER-HOPF TECHNIQUE

It is desired to solve an integral equation of the type given in (39) along with the condition in (40) in chapter 3.

$$\int_{-\infty}^{\infty} I(\zeta) d\zeta \int_{-\infty}^{\infty} F(a) e^{-ia(z-\zeta)} da = \begin{cases} -\frac{i8\pi k}{\omega\mu} V e^{ikz}, & z > 0 \\ m(z), & z < 0 \end{cases} \quad (1)$$

with the condition

$$I(z) = I(0) e^{-ikz}, \quad z < 0 \quad (2)$$

In order to solve equation (1), one may use the method commonly utilized to solve the Wiener-Hopf type of integral equation. Levine and Schwinger¹ used this method to determine the radiation of acoustic waves from the open, unflanged end of a circular pipe. Hallén² used it to calculate the reflected current wave on a cylindrical uncoated antenna. Wu³ used it in the problem of semi-infinite coated and uncoated antennas. Noble⁴ has compiled in his book a great number of problems with solutions based on this method. The same method will be used for the present problem. A set of Fourier Transforms will first be defined.

$$I(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(a) e^{-ia\zeta} da$$
$$G(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(\zeta) e^{ia\zeta} d\zeta \quad (3)$$

It is assumed that the function $G(a)$ consists of two parts, one part being analytic in the upper half plane and the other part in the lower half plane. Thus

$$G(a) = G_+(a) + G_-(a)$$

where

$$\begin{aligned} G_+(a) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} I(u) e^{iau} du \\ G_-(a) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 I(u) e^{iau} du \end{aligned} \quad (4)$$

Since $I(u)$ is known from (2) for negative values of u , $G_-(a)$ may be calculated. This will be

$$G_-(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 I(u) e^{-iku} e^{iau} du = \frac{1}{\sqrt{2\pi}} \frac{I(0)}{i(a-k)} \quad (5)$$

Thus $G_-(a)$ is analytic in a , in the whole upper half of the complex a -plane and also in a strip in the lower half plane in which $|\operatorname{Im}(a)| < \epsilon$ where ϵ is equal to the imaginary part of k . The function $G_+(a)$ is regular in the whole lower half of the a plane. It must vanish as $a \rightarrow i\infty$. The limit value $G_+(a)$ with $a \rightarrow i\infty$ may obviously be obtained from the value of $I(u)$ in the vicinity of $u=0$.

Thus

$$\lim_{a \rightarrow i\infty} G_+(a) = \frac{1}{\sqrt{2\pi}} I(0) \int_0^{\infty} e^{iau} du = \frac{1}{\sqrt{2\pi}} \frac{I(0)}{(-ia)} \quad (6)$$

The function $m(z)$ defined in (1) is given by

$$m(z) = \begin{cases} \frac{-i8\pi k}{\omega\mu} \gamma e^{ikz}, & z > 0 \\ \text{an unknown}, & z < 0 \end{cases}$$

A set of Fourier Transforms of this function corresponding to Eq. (3) may be defined as

$$m(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} M(a) e^{-iaz} da \quad (7)$$

with

$$M(a) = M_+(a) + M_-(a)$$

where $M_+(a)$ and $M_-(a)$ have definitions similar to those in (4).

Apparently

$$M_+(a) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{-i8\pi k}{\omega\mu} V e^{i(k+a)z} dz = \frac{1}{\sqrt{2\pi}} \frac{8\pi k V}{\omega\mu(k+a)} \quad (8)$$

$M_+(a)$ is regular in the whole lower a -plane and in a strip of the upper half plane in which $|\operatorname{Im} \{a\}| < \epsilon$.

$$M_-(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 m(z) e^{iaz} dz \quad (9)$$

$M_-(a)$ is unknown. It is regular in the whole upper half plane and vanishes at infinity essentially as $(\frac{1}{ia})$ in this plane.

Substituting (3), (4), and (7) into (1), one obtains

$$\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\beta [G_+(\beta) + G_-(\beta)] e^{-i\beta\zeta} \int_{-\infty}^{\infty} da F(a) e^{-ia(z-\zeta)} = \int_{-\infty}^{\infty} [M_+(a) + M_-(a)] e^{-iaz} da$$

This may be reduced to

$$\int_{-\infty}^{\infty} \{2\pi[G_+(a) + G_-(a)] F(a) - M_+(a) - M_-(a)\} e^{-iaz} da = 0 \quad (10)$$

It is now necessary to examine the location of the poles of the function $F(a)$.

The denominator of $F(a)$ under consideration is

$$\frac{l}{l_0} \frac{1}{\epsilon_r} J_0(lb) H_0'(l_0 b) - J_0'(lb) H_0(l_0 b) \quad (11)$$

where

$$l = \sqrt{k^2 - a^2}, \quad l_0 = \sqrt{k_0^2 - a^2}$$

This function has been examined by Adler.⁵ He has shown that if the dielectric cylinder is of a material whose dielectric constant is complex or has at least a non-vanishing imaginary part, then all zeros of (11) will be away from the real a -axis. In a similar manner, if k_0 is assumed to have nonvanishing imaginary part ϵ_0 , then $F(a)$ is regular and non vanishing in the strip of $2\epsilon_0$ which extends on both sides of the real axis. Therefore $\ln F(a)$ is regular in the strip $|\operatorname{Im} a| < \operatorname{Im} k_0$. Consequently it may be represented by a Cauchy integral, with a closed contour which is deformed to fit the narrow strip of regularity.

Thus

$$\begin{aligned} \ln F(a) = & \frac{1}{2\pi i} \int_{-\infty - i\delta}^{\infty - i\delta} \frac{\ln F(\gamma)}{\gamma - a} d\gamma \\ & - \frac{1}{2\pi i} \int_{-\infty + i\delta}^{\infty + i\delta} \frac{\ln F(\gamma)}{\gamma - a} d\gamma \end{aligned}$$

where

$$0 < |\delta| < \operatorname{Im} k_0$$

If $F(a)$ is written as

$$F(a) = \phi_1(a) / \phi_2(a) \quad (12)$$

then

$$\ln F(a) = \ln \phi_1(a) - \ln \phi_2(a)$$

so that one may write

$$\phi_1(a) = \exp \frac{1}{2\pi i} \int_{-\infty - i\delta}^{\infty - i\delta} \frac{\ln F(\gamma)}{\gamma - a} d\gamma, \quad \phi_2(a) = \exp \frac{1}{2\pi i} \int_{-\infty + i\delta}^{\infty + i\delta} \frac{\ln F(\gamma)}{\gamma - a} d\gamma \quad (13)$$

The function $\phi_1(a)$ is regular and non-vanishing in the whole upper half a -plane, and $\phi_2(a)$ in the lower half plane, and both these regions of analyticity can be extended to the other side of the real axis provided that the border line is within $\text{Im } k_0$.

One observes that

$$\begin{aligned}
 \phi_1(-a) &= \exp \frac{1}{2\pi i} \int_{-\infty-i\delta}^{\infty-i\delta} \frac{\ln F(\gamma)}{\gamma+a} d\gamma \\
 &= \exp \frac{1}{2\pi i} \int_{\infty+i\delta}^{-\infty+i\delta} \frac{\ln F(-\gamma)}{\gamma-a} d\gamma \\
 &= \exp \frac{1}{2\pi i} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{\ln F(\gamma)}{a-\gamma} d\gamma
 \end{aligned} \tag{14}$$

This is true because $F(\gamma) = F(-\gamma)$. Comparing (13) and (14) it is obvious that

$$\phi_1(-a) \phi_2(a) = 1 \tag{15}$$

Equation (10) now becomes

$$\begin{aligned}
 &\int_{-\infty}^{\infty} \left\{ 2\pi [G_+(a) + G_-(a)] \frac{\phi_1(a)}{\phi_2(a)} - M_+(a) - M_-(a) \right\} e^{-iaz} = 0 \\
 \text{or} \\
 &\int_{-\infty}^{\infty} e^{-iaz} \frac{\phi_1(a)}{a^2 - k^2} \left\{ 2\pi [G_+(a) + G_-(a)] \frac{a^2 - k^2}{\phi_2(a)} - [M_+(a) + M_-(a)] \frac{a^2 - k^2}{\phi_1(a)} \right\} da = 0
 \end{aligned} \tag{16}$$

The functions $\phi_1(a)/(a^2 - k^2)$, $(a^2 - k^2)/\phi_2(a)$, $G_-(a)$ and $M_+(a)$ are all regular within the strip $\pm\delta$. By assuming k_0 to have a non-vanishing imaginary part, Hallén has shown that both $G_+(a)$ and $M_-(a)$ are also regular in the strip and in order that (16) must be satisfied, the integrand must vanish. Then

$$2\pi [G_+(a) + G_-(a)] \frac{a^2 - k^2}{\phi_1(a)} = [M_+(a) + M_-(a)] \frac{a^2 - k^2}{\phi_2(a)}$$

Using (5) and (8), one obtains

$$2\pi \frac{a^2 - k^2}{\phi_2(a)} G_+(a) + \frac{\sqrt{2\pi}}{i} \frac{k+a}{\phi_2(a)} I(0) = \frac{1}{\sqrt{2\pi}} \frac{8\pi kV}{\omega\mu} \frac{a-k}{\phi_1(a)} + M_-(a) \frac{a^2 - k^2}{\phi_1(a)} \quad (17)$$

= a single polynomial, $P(a)$

The left side of (17) is regular in the lower half plane, the right side in the upper half plane and both coincide within the strip $|\text{Im } a| < \delta$. Therefore they must be equal to a single polynomial $P(a)$. Since both $G_+(a)$ and $M_-(a)$ are bounded, $P(a)$ must be a polynomial of no higher than the second degree. But since $G_+(a)$ and $M_-(a)$ both vanish at infinity, $P(a)$ can at best be a polynomial of first degree.

Thus

$$P(a) = C_1 + C_2(a-k)$$

where C_1 and C_2 are constants.

At $a = k$, $P(a) = C_1$, so that from (17)

$$C_1 = \sqrt{2\pi} \frac{2k}{i \phi_2(k)} I(0) \quad (18)$$

Also dividing the left side of (17) by $a - k$ and letting a approach $i\infty$, one obtains, by using the result in (6),

$$C_2 = \frac{\sqrt{2\pi} (a+k)}{\phi_2(a)} \frac{I(0)}{(-ia)} + \frac{\sqrt{2\pi} (a+k)}{i(a-k)} \frac{I(0)}{\phi_2(a)} = 0$$

Therefore $P(a)$ is a constant and is equal to C_1 . Thus

$$P(a) = \sqrt{2\pi} \frac{2k}{i \phi_2(k)} I(0) \quad (19)$$

Using the right side of (17) and letting $a = -k$, one obtains

$$P = \frac{1}{\sqrt{2\pi}} \frac{8\pi kV}{\omega\mu} \frac{(-2k)}{\phi_1(-k)}$$

so that

$$V = \frac{i\omega\mu}{4k} \frac{I(0)}{\phi_2^2(k)} \quad (20)$$

In order to find the Fourier Transform of $G(a)$, one combines (17), (19) and (5) to obtain

$$G(a) = G_+(a) + G_-(a) = \frac{2k i \phi_2(a)}{\sqrt{2\pi} \phi_2(k) (k^2 - a^2)} I(0) \quad (21)$$

Combining (3), (13), (20), and (21), one obtains finally

$$I(z) = \frac{4k^2 V \phi_2(k)}{\omega\mu\pi} \int_{-\infty}^{\infty} \frac{\phi_1(a) e^{-iaz}}{l^2 F(a)} da \quad (22)$$

Equation (22) is the formal solution for $I(z)$.

APPENDIX 7

AN ELECTRIC DIPOLE WITHIN A SPHERE AND A CYLINDER OF INHOMOGENEOUS MATERIAL

In Chapter 2, the electromagnetic fields of an electric dipole surrounded by a sphere and a cylinder of homogeneous material were obtained. In this appendix, the corresponding problem with an inhomogeneous material will be discussed briefly. The case of the sphere, being simpler, will be treated first, and then the problem of the cylinder will be taken up. Azimuthal symmetry will be assumed in both cases.

7.1 An Electric Dipole Within an Inhomogeneous Sphere

The theory of a biconical antenna within an unbounded radially inhomogeneous spherical medium has been given by Fikioris.¹ A similar approach is applicable to the present case. The permeability is assumed to be a constant and the permittivity to vary with the radial distance only. Thus $\epsilon = \epsilon(R)$, $\mu = \text{constant}$, $\frac{\partial}{\partial \phi} = 0$. With a z-oriented dipole, the field components will be (as in chapter 2) H_ϕ, E_R, E_θ . The relationships among the various components are provided by Maxwells' equations.

$$\begin{aligned} \frac{1}{R} \left[\frac{\partial}{\partial R} (R E_\theta) - \frac{\partial E_R}{\partial \theta} \right] &= i\omega\mu H_\phi \\ \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (H_\phi \sin \theta) &= -i\omega\epsilon E_R \\ -\frac{1}{R} \frac{\partial}{\partial R} (R H_\phi) &= -i\omega\epsilon E_\theta \end{aligned} \quad (1)$$

Combining the three equations in (1), one obtains an equation for H_ϕ . This is

$$\frac{\partial}{\partial R} \left[\frac{1}{\epsilon} \frac{\partial}{\partial R} (R H_\phi) \right] + \frac{\partial}{\partial \theta} \left[\frac{1}{\epsilon R \sin \theta} \frac{\partial}{\partial \theta} (H_\phi \sin \theta) \right] + \omega^2 \mu R H_\phi = 0 \quad (2)$$

Using the separation of variables techniques, (2) yields the following equations:

$$H_\phi = G(R) T(\theta)$$

$$\frac{\epsilon}{R} \frac{d}{dR} \left[\frac{1}{\epsilon} \frac{d}{dR} (RG) \right] + \left[\omega^2 \mu \epsilon - \frac{a}{R^2} \right] G = 0$$

$$\frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} (T \sin \theta) \right] + aT = 0 \quad (3)$$

where a is the separation constant. If a is chosen to be $n(n+1)$, then the third equation in (3) is a well-known equation whose solutions are given by the associated Legendre functions $P_n^1(\cos \theta)$, $Q_n^1(\cos \theta)$. Since only $P_n^1(\cos \theta)$ remains finite at $\cos \theta = \pm 1$, $P_n^1(\cos \theta)$ is the only allowed solution. It is also well known that $P_n^1(\cos \theta) = \frac{d}{d\theta} P_n(\cos \theta)$. With $a = n(n+1)$, the equation for G in (3) becomes

$$\frac{\epsilon}{R} \frac{d}{dR} \left[\frac{1}{\epsilon} \frac{d}{dR} (RG) \right] + \left[k^2 - \frac{n(n+1)}{R^2} \right] G = 0 \quad (4)$$

The solution for G in (4) will in general depend on the particular choice of ϵ . Fikioris¹, for example, treats the case for $\epsilon = \frac{R+a}{R+b} \epsilon_0$ where a and b are two arbitrary constants. Without going into any detail about the specific choice of ϵ , it may be assumed that Eq. (4) will have two solutions, $G_n^{(1)}$ and $G_n^{(2)}$. $G_n^{(1)}$ will be finite at $R = 0$. The function $G_n^{(2)}$ is expected to behave as an outgoing wave at $R = \infty$.

If ϵ is a constant then Equation (4) reduces to the well known equation

$$\frac{d^2 G}{dR^2} + \frac{2}{R} \frac{dG}{dR} + \left[k^2 - \frac{n(n+1)}{R^2} \right] G = 0 \quad (5)$$

Two independent solutions of (5) are $j_n(kR)$, $h_n^{(1)}(kR)$. These are the spherical Bessel and Hankel functions. A comparison of (4) and (5) suggests that the functions $G_n^{(1)}$ and $G_n^{(2)}$ should be normalized such that the asymptotic behaviors and the limiting solutions for R approaching zero are the same as those of $j_n(kR)$ and $h_n^{(1)}(kR)$.

That is

$$\begin{aligned} G_n^{(1)} &\underset{R \rightarrow 0}{\sim} j_n(k_1 R) \\ G_n^{(2)} &\underset{R \rightarrow 0}{\sim} h_n^{(1)}(k_1 R) \end{aligned} \quad (6)$$

where k_1 is the value of $k(R)$ at $R = 0$. Using the solution for H_ϕ obtained and combining with Eq. (1), one then writes

$$\begin{aligned} H_{\phi n} &= a_n G_n \frac{d}{d\theta} P_n(\cos \theta) \\ E_{\theta n} &= \frac{a_n}{i\omega\epsilon R} \frac{d}{dR} (R G_n) \frac{d}{d\theta} P_n(\cos \theta) \\ E_{Rn} &= \frac{a_n}{i\omega\epsilon R} G_n n(n+1) P_n(\cos \theta) \end{aligned} \quad (7)$$

In the last equation, the following property of $P_n(\cos \theta)$ has been used.

$$\frac{d}{d\theta} \left[\sin \theta \frac{d}{d\theta} P_n(\cos \theta) \right] = -n(n+1) \sin \theta P_n(\cos \theta) \quad (8)$$

In (7) and (8), $H_{\phi n}$, $E_{\theta n}$ etc. represent the n -th mode, G_n stands for either $G_n^{(1)}$ or $G_n^{(2)}$ or a linear combination of both (depending on the region of validity of the solutions), a_n is an arbitrary constant which is to be determined from the proper boundary conditions.

It is desired to construct the field components for an electric dipole in an unbounded radially inhomogeneous medium using the elementary solutions in (7), and (8). Comparing (7) and (8) with the response of an electric dipole in an unbounded homogeneous medium (see Eq. (5), Chapter 2) and noting the normalization of the G_n function in (6), it is apparent that (7) and (8) will be the response of a dipole if $n = 1$, $G_n = G_n^{(2)}$ and $a_n = \frac{Ih k_1^2}{i 4\pi}$. Therefore, the primary excitation due to the dipole at the origin of an unbounded radially inhomogeneous medium is given by

$$\begin{aligned}
H_{\phi P} &= \frac{I h k_1^2}{i 4 \pi} G_1^{(2)} \frac{d}{d\theta} P_1(\cos \theta) \\
E_{\theta P} &= - \frac{I h k_1^2}{4 \pi \omega \epsilon R} \frac{d}{dR} [R G_1^{(2)}] \frac{d}{d\theta} P_1(\cos \theta) \\
E_{RP} &= - \frac{I h k_1^2}{2 \pi \omega \epsilon R} G_1^{(2)} P_1(\cos \theta)
\end{aligned} \tag{9}$$

Once the primary excitation is known, it is straightforward to set up the total field components due to an electric dipole within a radially inhomogeneous sphere of radius 'a', following the principle described in chapter 2. Thus, (within the sphere)

$$\begin{aligned}
H_{\phi}^{(1)} &= \left[\frac{I h k_1^2}{i 4 \pi} G_1^{(2)} + \epsilon_1 G_1^{(1)} \right] \frac{d}{d\theta} P_1(\cos \theta) \\
E_{\theta}^{(1)} &= \left\{ - \frac{I h k_1^2}{4 \pi \omega \epsilon R} \frac{d}{dR} [R G_1^{(2)}] + \frac{\epsilon_1}{i \omega \epsilon R} \frac{d}{dR} [R G_1^{(1)}] \right\} \frac{d}{d\theta} P_1(\cos \theta)
\end{aligned} \tag{10}$$

In region 2 (outside the sphere)

$$\begin{aligned}
H_{\phi}^{(2)} &= f_1 h_1^{(1)}(k_o R) \frac{d}{d\theta} P_1(\cos \theta) \\
E_{\theta}^{(2)} &= \frac{1}{i \omega \epsilon_o R} f_1 \frac{d}{dR} [R h_1^{(1)}(k_o R)] \frac{d}{d\theta} P_1(\cos \theta)
\end{aligned} \tag{11}$$

In (10) and (11), ϵ_1 and f_1 , are suitable constants to be determined from the boundary conditions which are

$$\begin{aligned}
E_{\theta}^{(1)} &= E_{\theta}^{(2)} \\
H_{\phi}^{(1)} &= H_{\phi}^{(2)}
\end{aligned} \quad \text{at } R = a \tag{12}$$

Substitution of (12) into (10) and (11) immediately yields (exactly as in chapter 2)

$$f_1 = \frac{I h k_1^2}{i 4 \pi} \left\{ \frac{G_1^{(2)} \frac{d}{dR} [R G_1^{(1)}] - G_1^{(1)} \frac{d}{dR} [R G_1^{(2)}]}{h_1^{(1)}(k_o R) \frac{d}{dR} [R G_1^{(1)}] - \epsilon_r G_1^{(1)} \frac{d}{dR} [R h_1^{(1)}(k_o R)]} \right\}_{R=a}$$

The constant ϵ_1 may also be evaluated in a similar manner. Any further discussion about the field components would require a knowledge about the exact nature of the function $\epsilon(R)$. For the purpose of this appendix, the discussion for the case of the sphere will be concluded here.

7.2 An Electric Dipole Within a Radially Inhomogeneous Cylinder

In chapter 2, the problem of an electric dipole within an infinitely long homogeneous cylinder was treated. In this section, the corresponding problem with an inhomogeneous cylinder will be discussed. For simplicity, the cylinder is assumed to be only radially inhomogeneous and the dipole located symmetrically at the origin.

With $\mu = \mu_0$, $\epsilon = \epsilon(r)$ and $\frac{\partial}{\partial \phi} = 0$, Maxwell's equations yield

$$\begin{aligned} \frac{\partial H_\phi}{\partial z} &= -i\omega\epsilon E_r \\ \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= -i\omega\epsilon E_\phi \\ \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) &= -i\omega\epsilon E_z \end{aligned} \quad (13)$$

Combining the three equations in (13), one obtains

$$\epsilon \frac{\partial}{\partial r} \left[\frac{1}{\epsilon r} \frac{\partial}{\partial r} (r H_\phi) \right] + \frac{\partial^2 H_\phi}{\partial z^2} + k^2 H_\phi = 0 \quad (14)$$

It is desired to solve Eq. (3) for H_ϕ . The function $\epsilon(r)$ will be assumed to be such that the resulting $H_\phi(r, z)$ belongs to the Lebesgue class L^2 in the interval of z from $-\infty$ to $+\infty$. A function $f(z)$ is said to be in the Lebesgue class L^p in the interval (a, b) if $|f(z)|^p$ ($p > 0$) is integrable over (a, b) . When $H_\phi(r, z)$ is in the Lebesgue class L^2 in the interval $-\infty < z < \infty$, then a function $h_\phi(r, a)$ defined as

$$h_\phi(r, a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_\phi(r, z) e^{iaz} dz \quad (15)$$

exists. $h_\phi(r, a)$ is the Fourier Transform of $H_\phi(r, z)$. The inverse transform is

$$H_\phi(r, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_\phi(r, a) e^{-iaz} da$$

Excluding the cases when $H_\phi(r, z)$ does not satisfy the above condition, one may solve Eq. (14) by the Fourier Transform method. To do this, one takes the Fourier Transform of (14). Applying the usual condition of the function and its derivative vanishing at ∞ , one obtains after carrying out the integration,

$$\epsilon \frac{d}{dr} \left[\frac{1}{\epsilon r} \frac{d}{dr} [r h_\phi(r, a)] \right] + (k^2 - a^2) h_\phi(r, a) = 0 \quad (16)$$

The function $h_\phi(r, a)$ which satisfies Eq. (16) will in general depend on the function $\epsilon(r)$. For example, if $\epsilon(r)$ is a constant, (16) reduces to the Bessel Equation. Then two possible solutions are:

$$h_\phi(r, a) \sim a(a) J_1(\sqrt{k^2 - a^2} r), \quad b(a) H_1^{(1)}(\sqrt{k^2 - a^2} r)$$

so that

$$H_\phi(r, z) = \int_{-\infty}^{\infty} \frac{a(a) J_1(\sqrt{k^2 - a^2} r)}{b(a) H_1^{(1)}(\sqrt{k^2 - a^2} r)} e^{iaz} da \quad (17)$$

where $a(a)$ and $b(a)$ are suitable coefficients to be determined from boundary conditions. For certain special choices of $\epsilon(r)$, methods of solving equations of the type (16) were discussed in Appendix 2. Samaddar² solved an equation of the type (16) with $\epsilon(r) = \beta + \gamma r^2$. Eq. (16) then reduces to

$$\frac{d^2}{dy^2} (\sqrt{y} h_\phi) - \frac{\sigma}{1+\sigma y} \frac{d}{dy} (\sqrt{y} h_\phi) + \left[\frac{1}{y} \left(\frac{\eta}{2} \right)^2 + \left(\frac{\xi}{2} \right)^2 \right] \sqrt{y} h_\phi = 0 \quad (17)$$

where

$$y = r^2, \quad \sigma = \gamma/\beta$$

$$\eta^2 = k_0^2 \beta - a^2, \quad \xi^2 = k_0^2 \gamma$$

If $\epsilon(r)$ does not vanish on $0 \leq r \leq a$, the solution of (17) which is finite at $r = 0$ has been shown to be

$$h_\phi \rightarrow J_1(\eta r) + \sum_{n=1}^{\infty} d_n r^{2n+1} \quad (18)$$

where d_n 's are functions of η, σ, ξ . The values of the different d_n s have been shown in Reference 2.

Without going into any further detail about the specific choice of $\epsilon(r)$ and the corresponding solution of (16), it may be said, in general, that h_ϕ will have typical solutions $R^{(1)}(r, a)$ and $R^{(2)}(r, a)$, the first of which remains finite at $r = 0$ and the other behaving as an outgoing wave at $r = \infty$.

Then

$$H_\phi(r, z) = \int_{-\infty}^{\infty} \frac{c(a)R^{(2)}(r, a) - d(a)R^{(1)}(r, a)}{e^{iaz}} da \quad (19)$$

As long as $\epsilon(r)$ does not become zero or infinity at $r = 0$, the R functions may conveniently be normalized such that their asymptotic behaviors as r approaches zero are

$$\begin{aligned} R^{(1)} &\underset{r \rightarrow 0}{\sim} J_1(\sqrt{k_1^2 - a^2} r) \\ R^{(2)} &\underset{r \rightarrow 0}{\sim} H_1^{(1)}(\sqrt{k_1^2 - a^2} r) \end{aligned} \quad (20)$$

where k_1 is the value of $k(r)$ at $r = 0$.

It is now possible to write down the field components inside and outside the cylinder.

In region 1 (inside the cylinder)

$$\begin{aligned} H_\phi^{(1)} &= \int_{-\infty}^{\infty} [c(a)R^{(2)}(r, a) + d(a)R^{(1)}(r, a)] e^{iaz} da \\ E_z^{(1)} &= \frac{i}{\omega\epsilon} \int_{-\infty}^{\infty} \left[c(a) \frac{1}{r} \frac{d}{dr} [rR^{(2)}(r, a)] + d(a) \frac{1}{r} \frac{d}{dr} [rR^{(1)}(r, a)] \right] e^{iaz} da \end{aligned} \quad (21)$$

In region 2 (outside the cylinder)

$$\begin{aligned}
 H_{\phi}^{(2)} &= \int_{-\infty}^{\infty} e(a) H_1^{(1)}(\sqrt{k_o^2 - a^2} r) e^{iaz} da \\
 E_z^{(2)} &= \frac{i}{\omega \epsilon_o} \int_{-\infty}^{\infty} e(a) \frac{1}{r} \frac{d}{dr} [r H_1^{(1)}(\sqrt{k_o^2 - a^2} r)] e^{iaz} da
 \end{aligned} \tag{22}$$

The functions $d(a)$ and $e(a)$ will be determined from the conditions that

$$\begin{aligned}
 H_{\phi}^{(1)} &= H_{\phi}^{(2)} \\
 E_z^{(1)} &= E_z^{(2)}
 \end{aligned} \quad \text{at } r = a \tag{23}$$

Substitution of (23) into (21) and (22) yields the following value of $e(a)$.

$$e(a) = c(a) \left\{ \frac{R^{(2)} \frac{1}{r} \frac{d}{dr} [r R^{(1)}] - R^{(1)} \frac{1}{r} \frac{d}{dr} [r R^{(2)}]}{H_1^{(1)}(l_o r) \frac{1}{r} \frac{d}{dr} [r R^{(1)}] - \epsilon_r R^{(1)} \frac{1}{r} \frac{d}{dr} [r H_1^{(1)}(l_o r)]} \right\}_{r=a}$$

where

$$l_o = \sqrt{k_o^2 - a^2}$$

The value of $c(a)$ will have to be chosen from the excitation condition at $r = 0$. It will depend largely on the exact nature of the function $R^{(2)}(r, a)$, which again depends on the nature of $\epsilon(r)$.

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